Optimal design of 2-D and 3-D shaping for linear ITG stability*

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Results

i. Design targets for linear ion temperature gradient (ITG) modes
   - Rigorous analytic proxy/cost functions
   - Understanding of 2-D and 3-D shaping coefficients

ii. Mechanisms for target optimization
   - Analytic geometry calculations (local 3-D equilibrium theory)
   - Understanding & manipulation of typical geometric symmetry properties

iii. Calculations justifying targets & mechanisms
   - Analytic fluid limit & numerical gyrokinetics (GENE) ITG calculations
   - Focus on most unstable growth rate at high gradients
   - Good analytic and numeric agreement achieved
     - Important result because geometry analytics usually intractable
Optimization of turbulent transport is an intriguing prospect for stellarator design theory

- Past 20-30 years, substantial progress in ...
  - neoclassical transport theory
  - neoclassical optimization using shaping -- e.g. quasisymmetry
  - turbulent transport theory -- gyrokinetics theory & simulation
- Turbulence optimization using shaping now being explored
  - Relies on proxy/cost functions for $Q_{turb}$ in optimization loop -- e.g. in STELLOPT
- Proof of concept has been shown
  - Design strategy based on quasilinear transport minimization
    - Minimization of linear growth rates -- e.g. for ITG modes
  - Simulations (GENE) confirm reduced $Q_{turb}$ at selective flux-tubes & gradients
Local 3-D equilibrium theory and linear analytic ITG theory are applied to reach a detailed understanding of optimal shaping

- **Theory elements existed for some time -- instability & MHD equilibrium**
  - Typically relies on advanced codes -- e.g. VMEC
  - Detailed understanding has proven elusive
    - Why some shapes better than others (“bottle” vs. “bread slice”)?
    - Why crescent/concave-inboard shapes always seem to appear?
    - Which coefficients & shaping are most important?

- **Consider the sub-problem of the maximum linear ITG growth rate**
  - electrostatics, low-beta, adiabatic electrons, no flows

- **Concentrate on equilibrium modeling of a single surface**
  - analytic local 3-D equilibrium theory (Hegna PoP 2000)
  - assume straight field line angles equal to geometric angles
Analytic ITG theory yields eigenmode ODE along a field line with three principal geometric coefficients

- Consider ballooning, flux-tube limit along field line (\(\eta\) coordinate)
- Consider fluid limit \(\omega/k_{||} \gg v_{th}\)
- Gyrokinetics theory (Romanelli 1989) yields ODE, cubic in eigenfrequency

\[
\hat{\omega}^3 (1 + \frac{b}{L_k^2} \hat{k}_\perp^2) \hat{\phi} - \hat{\omega}^2 \frac{\sqrt{b}}{L_n} \left( 1 + 2 \frac{\rho L_n}{L_k^2} \hat{\omega}_d \right) - W_K \frac{b}{L_k^2} \hat{k}_\perp^2 \hat{\phi} \\
+ \hat{\omega} \left( -2 W_K \frac{b}{L_d L_n} \rho \hat{\omega}_d - \frac{1}{2} \frac{\rho^2}{L_{||}^2} \hat{k}_{||}^2 \right) \hat{\phi} - W_K \frac{\sqrt{b}}{L_n} \frac{1}{2} \frac{\rho^2}{L_{||}^2} \hat{k}_{||}^2 \hat{\phi} = 0
\]

where

\(\hat{\phi} = \hat{\phi}(\eta)\) & \(\hat{\omega}\)
\(\hat{\omega}_d(\eta)\) & \(\hat{k}_\perp^2(\eta)\) & \(\hat{k}_{||}^2(\eta)\)
\(W_K = \frac{1}{\tau} (1 + \eta_i)\) & \(\eta_i = (T_i'/T_i)/(n_i'/n_i)\)
\(b = (k_\alpha \rho_{th})^2\)
\(\tau = T_e/T_i\)
\(L_n = -B_0 \rho n'/n\)
\(B_0, L_k, L_d, \rho, L_{||}\)

\(\hat{\omega}\) & \(\hat{\phi}\)
\(\hat{\omega}_d\)
\(\hat{k}_\perp^2\)
\(\hat{k}_{||}^2\)
eigenfunction & eigenfrequency
geometric coefficients
drive parameters
FLR constant
temperature ratio
density scale length
normalization constants
General solution is found which rigorously identifies influence of geometric coefficients

- Define an average

\[ \langle f \rangle = \frac{\int_{\eta_i}^{\eta_f} d\eta \hat{\phi}^* f \hat{\phi}}{\int_{\eta_i}^{\eta_f} d\eta |\hat{\phi}|^2}, \]

- Should converge as limits go to \( \pm \infty \) since modes are localized

- Apply average, neglect slab drive (last) term, solve quadratic, get

\[
\hat{\omega} = -\frac{\sqrt{b} \left( -1 - 2\frac{\rho L_n}{L_d} \langle \hat{\omega}_d \rangle + W_K \frac{b}{L_k^2} \langle \hat{k}_\perp^2 \rangle \right)}{2(1 + \frac{b}{L_k^2} \langle \hat{k}_\perp^2 \rangle)} \pm \sqrt{\frac{\sqrt{b}}{L_n} \left( -1 - 2\frac{\rho L_n}{L_d} \langle \hat{\omega}_d \rangle + W_K \frac{b}{L_k^2} \langle \hat{k}_\perp^2 \rangle \right)^2 + 4(1 + \frac{b}{L_k^2} \langle \hat{k}_\perp^2 \rangle) \cdot (2W_K \frac{b_p}{L_d L_n} \langle \hat{\omega}_d \rangle + \frac{b^2}{2L_{||}^2} \langle \hat{k}_\parallel^2 \rangle)}{2(1 + \frac{b}{L_k^2} \langle \hat{k}_\perp^2 \rangle)}.
\]

- Suggests the ITG cost function should scale roughly as

\[
C_{ITG}^2 = \frac{\langle \hat{\omega}_d \rangle}{\langle \hat{k}_\perp^2 \rangle}.
\]

Only get \( \gamma = \text{Im}(\omega) > 0 \) for \( \langle \omega_d \rangle < 0 \) make this less negative
The geometric inputs decompose into five primitive equilibrium and geometry coefficients

- Formal parameter definitions
  \[
  \hat{\omega}_d = \frac{L_d^2 \omega_{di}}{k_\alpha \hat{B}} \quad \text{drift parameter}
  \]
  \[
  \hat{k}_\perp = \frac{L_k^2 k_\perp^2}{k_\alpha^2 \hat{B}^2} \quad \text{shear parameter}
  \]
  \[
  \hat{k}_\parallel = -L_\parallel^2 (\sqrt{g^2} B^2)^{-1} d_\eta \quad \text{parallel derivative}
  \]

- Geometric coefficients simplify to

  Normal curvature (1) \quad Geodesic curvature (2) \quad Normal torsion (3)

  \[
  \hat{\omega}_d = 2L_d^2 B_0 \frac{\kappa_n - \Lambda \kappa_g}{|\nabla \psi|} \quad \text{Flux surface proximity (4)}
  \]

  \[
  \hat{k}_\perp = L_k^2 B_0^2 \frac{1 + \Lambda^2}{|\nabla \psi|^2} \quad \text{Mod B (5)}
  \]

- Integrated local shear

  \[
  \Lambda = \frac{\nabla S \cdot \nabla \psi}{B} = -\frac{|\nabla \psi|^2}{B} \int_{\eta_k}^{\eta} d\eta \sqrt{g} \frac{B^2}{|\nabla \psi|^2} s = -\frac{|\nabla \psi|^2}{B} \int_{\eta_k}^{\eta} d\eta \sqrt{g} \frac{B^2}{|\nabla \psi|^2} (\sigma - 2\tau_n)
  \]

\[
\begin{align*}
B &= \nabla \psi \times \nabla S \quad \text{magnetic field} \\
S &= \theta - \iota \zeta \quad \text{field line label} \\
k &= k_\alpha \nabla S \quad \text{wave-vector} \\
\omega_{di} &= \mathbf{v}_d \cdot \mathbf{k} \quad \text{drift coefficient} \\
\hat{B} &= B / B_0 \quad \text{normalized field}
\end{align*}
\]
The linear instability targets therefore naturally separate into three distinct goals for optimal shaping

### Goals for instability

1. Shift & maximize \( \left\langle \frac{\kappa_n}{|\nabla \psi|} \right\rangle \to +\infty \)
2. Shift & maximize \( \left\langle \frac{-\Lambda \kappa_g}{|\nabla \psi|} \right\rangle \to +\infty \)
3. Maximize \( \left\langle \frac{\Lambda^2}{|\nabla \psi|^2} \right\rangle \to \infty \)
   - a. Minimize \( |\nabla \psi|^2 \)
   - b. Similarly, maximize \( B \) and \( \sqrt{g} \)

### Goals for shaping

- Goal (1) -- minimize bad curvature
- Goal (3) -- maximize shear, averaged torsion (in 2-D), and/or local torsion
- Assuming stellarators are net current free
  - Second goal implies optimal symmetry design rules:
    \[ \kappa_g \cdot \int_{\eta_k}^{\eta} d\eta (\tilde{\tau}_n - \tau_n) > 0 \text{ for } \kappa_n < 0 \text{ & } \hat{s} < 0 \text{ } \text{ 2-D} \]
    \[ \kappa_g \cdot \int_{\eta_k}^{\eta} d\eta (\tau_n) < 0 \text{ for } \kappa_n < 0 \text{ } \text{ 3-D} \]
- Goal (2) -- dephase the geodesic curvature and normal torsion

### Physical mechanisms

- Rayleigh-Taylor (RT) like drive
- Twisting-buckling RT drive
- Parallel/perpendicular dissipation
- Shear/FLR/polarization effects
- Enhancement and/or prerequisite for (3)
Curvature drive optimization relies on understanding and manipulation of the geometric symmetry distributions.

### 2-D Symmetry Lines
- **Normal Curvature** ($\kappa_n R_0$)
- **Geodesic Curvature** ($\kappa_g R_0$)
- **Normal Torsion** ($\tau_n R_0$)

In 2-D, symmetry lines are typically left/right or up/down.

### 3-D Symmetry Lines
- **Normal Curvature** ($\kappa_n R_0$)
- **Geodesic Curvature** ($\kappa_g R_0$)
- **Normal Torsion** ($\tau_n R_0$)

In 3-D, the symmetry lines follow corners/edges/cusps.
Consider a sequence of 2-D geometries with modifications to the standard surface convexity. Colors indicate geodesic curvature \( \kappa_g \).

- **t0**: Typical up/down symmetry.
- **t1** and **t2**: Outboard altered northeast/southwest and northwest/southeast symmetry.

<table>
<thead>
<tr>
<th>Case</th>
<th>( L_\perp / R_0 )</th>
<th>( \iota )</th>
<th>( \hat{s} )</th>
<th>Cross section type</th>
</tr>
</thead>
<tbody>
<tr>
<td>t0</td>
<td>.2</td>
<td>.67</td>
<td>1.34</td>
<td>Circular, ( B_p(\theta) ) from VMEC</td>
</tr>
<tr>
<td>t1</td>
<td>.2</td>
<td>.67</td>
<td>1.34</td>
<td>Circular</td>
</tr>
<tr>
<td>t2</td>
<td>.32</td>
<td>.67</td>
<td>1.34</td>
<td>Elongation and triangularity</td>
</tr>
<tr>
<td>t3</td>
<td>.2</td>
<td>.67</td>
<td>1.34</td>
<td>Concave outboard</td>
</tr>
<tr>
<td>t4</td>
<td>.2</td>
<td>.67</td>
<td>1.34</td>
<td>Concave inboard</td>
</tr>
</tbody>
</table>
Maximum linear growth rates on 2-D cases show concave outboard & inboard shaping is highly influential

- Compare GENE and analytic maximum linear growth rates
- $\eta_i = 10$
- $L_n^{-1} = 2$
- $L_T^{-1} = 20$
- Assume $\phi^2 = \exp(-\eta^2/2\sigma^2)$ with $\sigma = (\eta_f - \eta_i)/6$
- Domain $\theta \in [-\pi, \pi]$

Figure: Maximum growth rates over scan of $k_\alpha \in [0.1, 1]$ in steps of 0.1 for GENE, and steps of 0.01 in analytic model.

→ Analytic model captures the trends fairly accurately
Analytics provide detailed explanation (perhaps excruciating detail) of how the coefficient behaviors have been improved.

- Excellent curvature stabilization for case $t_4$.
- Normal curvatures comparable.
- Geodesic curvature flips sign for $t_3$.
- Big local torsional bump at cusps.
- Circ. $t_0$, circ. $t_1$, elong. $t_2$, conc. out $t_3$, conc. in $t_4$.

- Excellent FLR stabilization for $t_3$ & $t_4$.
- Good.
- Bad.
- $|\Lambda|$ much larger for $t_2$, $t_3$, $t_4$ due to larger net currents / averaged torsion.
Next consider a sequence of 3-D geometries with purely rotational shaping harmonics

- Colors indicate normal torsion
- $\theta=0$ at outboard mid-plane oblate side
- shear changes with shape to keep net currents small

<table>
<thead>
<tr>
<th>Case</th>
<th>$L_\perp/R_0$</th>
<th>$\iota$</th>
<th>$\hat{s}$</th>
<th>$\rho_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sa0</td>
<td>.14</td>
<td>.9</td>
<td>-.27</td>
<td>$\rho_2/10$</td>
</tr>
<tr>
<td>sa1</td>
<td>.14</td>
<td>.9</td>
<td>-.11</td>
<td>$0$</td>
</tr>
<tr>
<td>sa2</td>
<td>.14</td>
<td>.9</td>
<td>-.27</td>
<td>$-\rho_2/10$</td>
</tr>
<tr>
<td>sa3</td>
<td>.14</td>
<td>.9</td>
<td>-.77</td>
<td>$-2\rho_2/10$</td>
</tr>
</tbody>
</table>

$m=1$ harmonic

$$R = R_0 + \rho_0 \cos(\bar{\theta}) + \rho_2 \cos(\bar{\theta} - N\bar{\zeta}) + \rho_3 \cos(2\bar{\theta} - N_2\bar{\zeta} - \omega_2)$$

$$Z = \rho_0 \sin(\bar{\theta}) + \rho_2 \sin(\bar{\theta} - N\bar{\zeta}) + \rho_3 \sin(2\bar{\theta} - N_2\bar{\zeta} - \omega_2)$$

$N = 6$, $N_2 = 9$, $\rho_0 = 1.05$, $\rho_2 = \rho_0/3$, $\omega_1 = \pi/N$, $\omega_2 = \pi/2$, $\bar{\zeta} \rightarrow \bar{\zeta} - \omega_1$
Fairly good agreement achieved for gaussian eigenfunction even in stellarator geometry

- Analysis indicates improvement due to shear $|\Lambda|$ boosting
- Symmetries (nodal positions) basically unchanged
Fully 3-D shaping may be employed to manipulate the symmetry distributions, for example, by deforming concave inboard to ellipse.

- Introduce deformational harmonics e.g.

\[ R = R_0 + \rho_0 \cos(\tilde{\theta}) + \rho_2 \cos(\tilde{\theta} - N\tilde{\zeta}) \cdot (1 - \rho_4 \cos^2(\frac{N}{2}\tilde{\zeta})) + \cdots \]

- Goal (1): minimize bad curvature
  - deformation helps shorten connection length

- Goal (2): dephase geodesic curvature and torsion
  - geodesic curvature made tokamak-like
    - positive on top, negative on bottom
  - torsion stays negative on edges
    - field line follows almost along edge at \( \iota \approx 0.9 \)
  - Goal (2) considerably improved
Three cases are considered with rotational and deformational harmonics and small global shear

- Colors indicate geodesic curvature

si0b

θ=0 at outboard midplane, oblate side

In reality, must scan:

- Multiple locations on surface
- Multiple parallel flux-tube lengths -- until maximum growth rate clearly established
Instability calculations demonstrate efficacy of dephasing mechanism, in this case, mainly via $k_{\text{perp}}^2$. 

Better normal curvature is controlled with much larger values for $\theta$. 

[Graph showing curves and bars with labels si0b, si1b, si2b.]
Flux-tubes centered at edge have maximum growth rate that is a little lower and isn't as significantly affected by the shaping.

Thus maximum growth rate on surface appears to be reduced.
Conclusions

i. Design targets for linear ion temperature gradient (ITG) modes
   - Minimization of curvature drive
   - Shear/FLR/polarization stabilization
   - Dephasing of geodesic curvature and torsion

ii. Mechanisms for target optimization
    - Use of 2-D shaping
    - Use of 3-D shaping including rotation + deformation

iii. Calculations justifying targets & mechanisms
    - Gaussian eigenfunction model appears workable
Future work

i. Implementation of detailed ITG proxy and eigenfunction model with STELLOPT (Mynick, Xanthopoulos)

ii. Apply geometry analysis to HSX optimized geometries (Talmadge, Mynick)

iii. Determine analytic proxies for TEM, kinetic effects (Hegna)

iv. Propose nonlinear proxies (Hegna)
   • Develop understanding of coupling and/or interaction between linear modes
   • Turbulence is a nonlinear phenomena.