On the Poloidal Beta Evolution of Fixed-q Heating in Tokamaks

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ABSTRACT:

For typical tokamak safety factor, q, profiles and aspect ratio, A, fixed-q heating leads to a limit for equilibrium code convergence near a poloidal beta value of 1+A. Also presented is a simple tanh functional fit that saturates *just* above the 1+A limit. This poloidal beta limit does not depend on the q-profile, but the total beta achieved at this critical poloidal beta does depend strongly on q. This paper confirms very early computations of the flux-conserving tokamak that routinely achieved a poloidal beta > A.

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I. Introduction:

In the quest for an economical future tokamak fusion reactor, a quantity beta that represents the plasma pressure to the expensive magnetic field pressure containing the plasma should be maximized. If this beta and the beta measured with respect to the part of the magnetic field created by just the plasma current, the poloidal beta, are both large, then the plasma can become so diamagnetic that the applied field can be almost completely cancelled out [1]. That fully diamagnetic state would be of great economic benefit.

In order to study the stability properties of a given ideal MHD (magnetohydrodynamic) equilibrium, one must first produce (numerically) such an equilibrium. In this study, an attempt is made to numerically produce a fully diamagnetic plasma while keeping the safety factor, q, profile fixed along with the first derivative of the pressure profile and other quantities being held fixed as the relative pressure magnitude, beta, is increased. This scheme for heating the tokamak was called the "flux-conserving tokamak" [2,3,4,5]. In this scheme, an equal amount of toroidal flux as poloidal flux is added within each flux surface as the beta is increased, thereby holding q fixed. The poloidal beta and plasma current both adjust to keep q fixed, and this paper presents our findings particularly on the poloidal beta evolution in this fixed-q heating scheme. Based on our findings, a limit in poloidal beta exists very near to where the saturation level in the poloidal beta occurs. It is still unclear whether a reasonable q profile can lead to a fully diamagnetic equilibrium or how close one can come to that goal.

In the next section, the codes that produce the equilibrium and calculate all the related quantities that we used are described. In Section III, the results are displayed and outlined. In effect, we were *not* able to get the fully diamagnetic plasma equilibrium that we wanted, but instead, we learned a great deal about this fixed-q method of trying to get to near unity beta. The discussion of Section IV puts these results in historical context. This subject stretches back to the beginnings of tokamak research. However, there are still some uncertainties about the equilibrium beta limit and the path that the poloidal beta takes during fixed-q heating. Because recent thrusts in the fusion program are focussing on exploring unity beta tokamaks, these equilibrium beta limit issues are once again important to understand.

II. Code Descriptions and Input Parameters:

Two well-known toroidal equilibrium codes were used for this study. Both codes are capable of fixed-q solutions in which the current was allowed to vary in order for the codes to converge on a solution to the Grad-Shafronov equation [3,6,7]. The main code used is the "inverse-type solver," TOQ [8] (TOroidal Q-solver). This code was designed for just this type of study of fixed-q evolution, and it solves for positions in terms of flux-surface variables. Fixed-q solutions are often more difficult computationally, and at higher betas, we needed to continuously feed in the previous lower beta solution as an initial guess as we moved upwards in beta (keeping the same pressure profile shape in flux coordinates). The user-specified pressure profile slope as a function of flux coordinates was $dp(\psi)/d\psi = 1-\psi$ throughout (Note that dp(r)/dr will still be zero at r = 0 after flux coordinates are converted to spatial coordinates). All of the runs had a plasma minor radius, a = 0.70m, and the plasma major radius, R, that varied to provide the aspect ratio, A=R/a, desired (A varied from 3 to 7). The code uses a fixed outer flux surface shape: circular for all cases reported here. We did vary the shape in earlier stages of our study, but the overall the story remained the same. The mesh size, (flux surface=ψ) X (poloidal angle=θ), varied from 19 X 17 to 131 X 129 in TOQ with no significant effect on the solution. As mentioned, the safety factor, $q(\psi)$ defined below, was specified and held fixed (see Fig. 1). When the overall magnitude of q was altered, strong differences in the poloidal beta evolution were observed and are reported in the next section. The peak beta relative to the volume average vacuum field was then varied and the poloidal beta (defined below) was calculated and noted as well as other parameters discussed in the next section. The definitions of some important quantities are as follows:

Aspect ratio: $A \equiv R/a,$ inverse aspect ratio: $\varepsilon \equiv 1/A,$ (1)

Poloidal beta: $$\beta_p \equiv \frac{2\mu_0 \bar{p}}{\langle B_p^2 \rangle_{\text{edge}}},\qquad (2)$$

Peak beta: $$\beta_{\text{peak}} \equiv \frac{2\mu_0 p_{\text{peak}}}{\overline{B_0^2}},\qquad (3)$$

Safety factor: $$q(\psi) \equiv \frac{1}{2\pi}\oint \frac{d\theta}{\vec{B}\bullet\nabla\theta}\frac{F(\psi,\theta)}{R^2(\psi,\theta)}.\qquad (4)$$

In the above formulae, a bar on top of a quantity represents the plasma volume average, and the angular brackets represent a flux-surface average of the plasma edge (last closed flux surface). The quantity $F=RB_T$, and B_T and B_0 are the toroidal and vacuum total magnetic field intensity respectively.

The code used to check the TOQ results was the "direct-type solver," RSTEQ [2] (Resistive Stability Toroidal EQuilibrium). The code RSTEQ is computationally different from TOQ since it solves for flux surfaces in terms of positions, but it was also designed to run with fixed-q. The very same a, R, $dp(\psi)/d\psi$, $q(\psi)$, and peak betas were used, but the resolution of 81 X 81 on a cartesian grid in the R (radius), Z (height) plane was primarily used for RSTEQ. This resolution was also varied in RSTEQ with the same negligible effects on our results reported here. The same circular boundary is also used for RSTEQ. In the next section, one can notice that these two codes matched each other very closely as expected, but this agreement is a good check since these codes solve the same equation in two different ways.

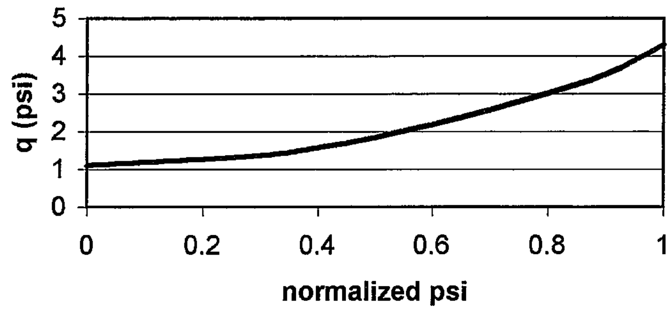


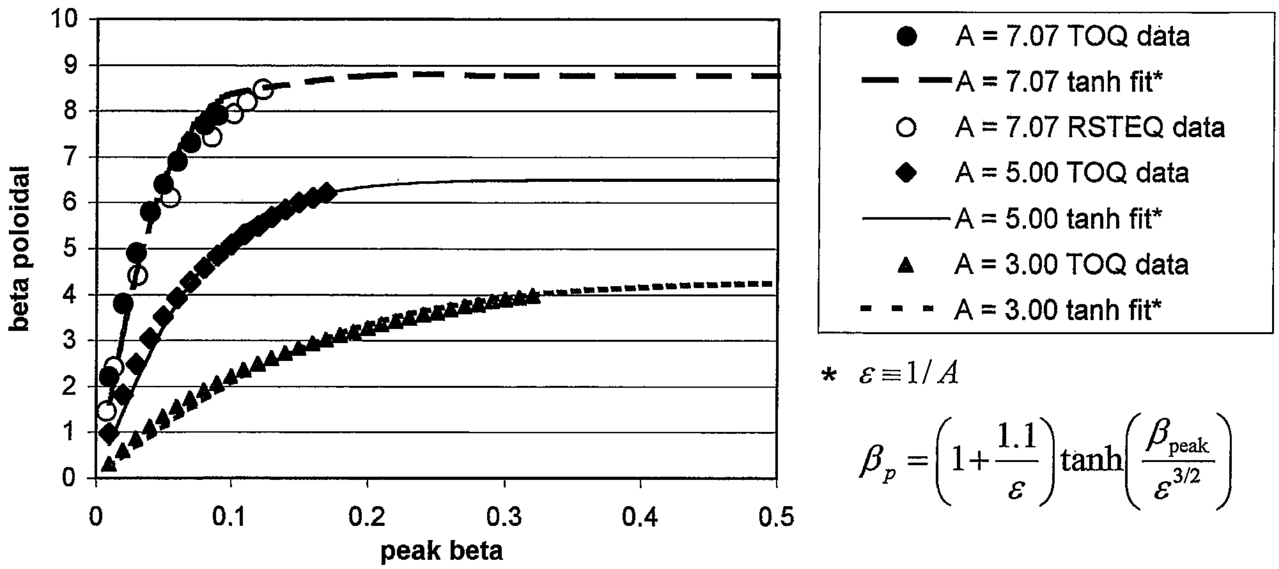
FIGURE 1:

The safety factor profile across flux surfaces (psi = Ψ) that was held fixed in this study.

The goal for this study was to use standard and easy input so that the results would be as generic as possible. Another goal was to use relatively high aspect ratio in order to explore any implications for the UCLA (University of California – Los Angeles) experiment: the Electric Tokamak (ET) [9,10] that has a similar a and R as used here and will attempt unity beta, fully diamagnetic operation. We chose these codes because of their well-tested reputation in the community. Neither code was modified from the release version beyond input and output routines. All calculations reported here ran in double precision on a variety of platforms.

III. Fixed-q Evolution Results:

The original goal of this project was to produce a "fully diamagnetic" tokamak plasma equilibrium such as presented in Fig. 4 of [1] or originally in Fig. 10 of reference [6]. Therefore, the main result of this study is that a fixed-q evolution (with $q > 1$) does not lead us to this fully diamagnetic state in practice, but it might be very close. For the q profile chosen here, both codes reached their limits in poloidal beta before becoming very diamagnetic. This main result further subdivides into two sub-results: 1) There is an equilibrium poloidal beta limit (defined below) at about 1+A. This limit does not seem to be at very high beta or high diamagnetism. 2) There is also a distinctive functional shape in the poloidal beta versus peak beta for the fixed-q increases in pressure and current, and this shape seems to be well approximated by a simple tanh function fit described below. Then, other minor results include strong dependencies of the arguments of this tanh function to both the q profile and the (inverse) aspect ratio. The poloidal beta as a function of peak beta for fixed q but varying aspect ratio obtained from TOQ and RSTEQ is plotted in Fig. 2, with the poloidal beta calculated according to Eq. 2.



* $\varepsilon \equiv 1/A$

$$\beta_p = \left(1 + \frac{1.1}{\varepsilon}\right) \tanh\left(\frac{\beta_{\text{peak}}}{\varepsilon^{3/2}}\right)$$

FIGURE 2:

Beta poloidal evolution in the codes (points) compared to a functional tanh fit (curves) for three values of the aspect ratio, A. There are no points beyond a certain value of poloidal beta because the codes cannot converge there. The pressure profile, plasma shape, and q profile are held fixed for all points here: only A and beta values are varied.

One can notice from Fig. 2 that the codes do not make it to unity beta, but instead cannot converge beyond $\beta_p \sim 1 + A$.

For the evolution to be at fixed-q, the equilibrium departs from the low beta *linear* regime, and it would stay linear if the current was fixed instead of q. The necessary linear increase in total plasma current, together with flux surface changes and Shafranov shift increases, causes the poloidal beta to roll-over. The unfortunate result is that this roll-over is not pronounced enough for these values of q to keep the poloidal beta under the limit. Eventually, the poloidal magnetic field on the inboard side of the torus moves towards zero [3,4,6,7]. The code is unable to converge at the point where that inner surface poloidal magnetic field is zero. This is expected to happen (*if no large-aspect ratio approximation taken*) [6] at the "critical beta poloidal" of $\beta_{p\_crit} \geq 0.5 + A$.

In fact, we find it happens mostly at $1+A$. This value is expected since if we start from the edge poloidal magnetic field expression at the inner surface (poloidal angle theta =π) [6,7]:

$$B_p(a,\pi)=\langle B_p(a)\rangle(1+\varepsilon\Lambda\cos(\pi)), \tag{5}$$

where the quantities,

coefficient of asymmetry: $$\Lambda \equiv \beta_p + \mathrm{li}/2 - 1, \tag{6}$$

internal inductance: $$\mathrm{li} \equiv \frac{\overline{B_p^2}}{\langle B_p^2 \rangle_{\mathrm{edge}}}. \tag{7}$$

At the critical point, we can assume that the plasma is very diamagnetic and the current is mostly on the outer parts of the plasma. This assumption is similar to the skin-current models [6,7,11], but with *no large aspect ratio approximation*) such that $\mathrm{li}/2 << 1 << \beta_p$, so that when we apply the critical condition for the field to remain positive on the inboard side of the torus:

$$\varepsilon\Lambda \approx 1, \tag{8}$$

it then leads to

$$\beta_{p\_crit} \sim 1+A, \tag{9}$$

if highly diamagnetic such that li/2 is very small. This critical point is near where both codes fail to converge.

Also notice in Fig. 2 that the best, simple fit we could find without much complexity is the following hyperbolic tangent function (all else being equal):

$$\beta_p = \left(1+\frac{1.1}{\varepsilon}\right)\tanh\left(\frac{\beta_{(\mathrm{peak})}}{(c\varepsilon)^{3/2}}\right). \tag{10}$$

This works with the value c = 1 for a typical q profile that we used (see Fig. 1) and generic pressure profiles, etc. The factor c can change especially if the q-profile is varied and possibly other factors that we have not explored yet, such as shaping for instance. We made factor of two changes to the q values and found that the c values change strongly: the total betas achievable greatly change (see Fig. 3). For the q/2 profile: c = 3.3, for the q profile: c = 1, and for the 2*q profile: c = 0.67 in the tanh fit (see Eq. 10) for Fig. 3.

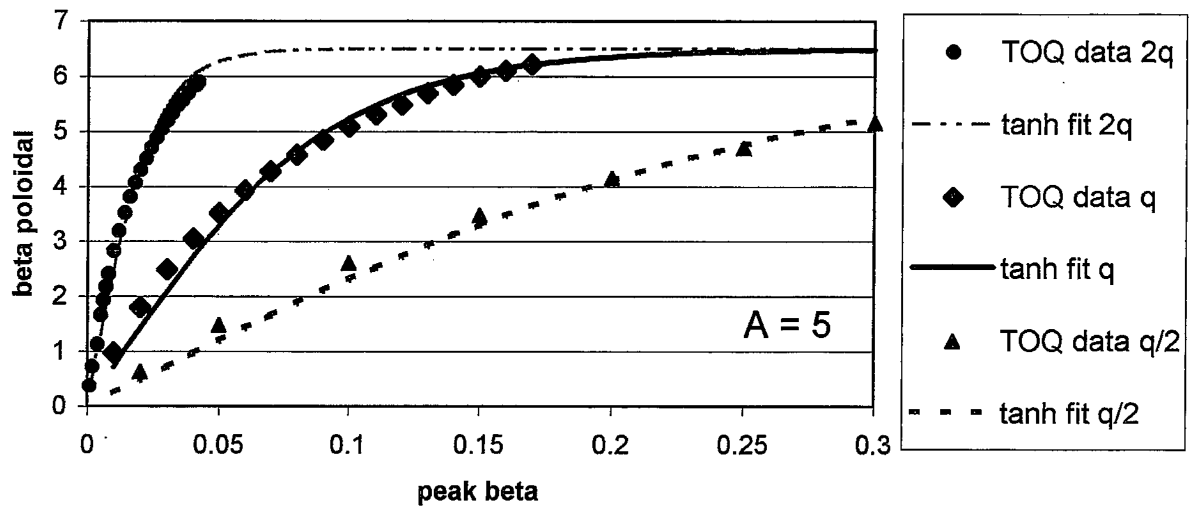


FIGURE 3:

The A = 5 case of TOQ points from Fig. 2 is presented here again along with cases where the q profile is uniformly halved and doubled. The tanh fit (curves) are also provided: see the text for details.

It is also interesting that the rate of toroidal current increase needed to hold q fixed as beta increases is not a (strong) function of aspect ratio (see Fig. 4) and is linear as expected [3,4].

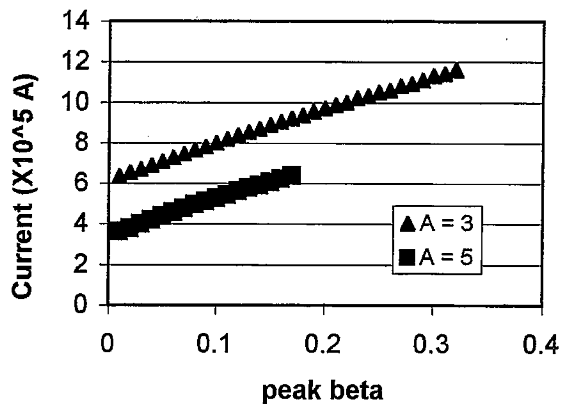


FIGURE 4:

The total plasma current that the TOQ code chose in order to fix q as beta was increased for two aspect ratios. Note that the same linear increases occur for two different aspect ratios.

For a typical plasma with a standard q-profile (see Fig. 1), usually considered safe for ideal stability, the equilibrium flux surface contours near the highest point in beta obtained are presented for the A = 5 case in TOQ in Fig. 5. The Shafranov shift in Fig. 5 is near a/2, and $\beta_p = 6.0$.

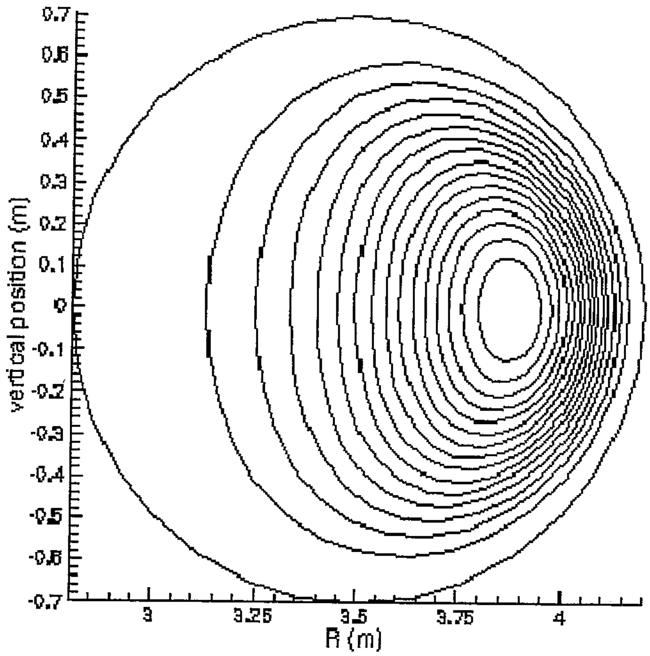


FIGURE 5:

Poloidal flux contours for nearly the highest beta and diamagnetism achieved for A = 5 fixed-q heating with TOQ in this study. The poloidal beta here is 6.0 and Fig. 1 is the q profile.

It might be possible to go beyond this level of diamagnetism for this large an aspect ratio, but one should expect sacrifices in the q profile. A flat or even partially reversed q profile and/or parts of it below unity may be required to allow the code to produce a fully diamagnetic plasma with a Shrafranov shift on the order of the minor radius [12].

IV. Discussion and Implications:

In spite of the long history of tokamak equilibrium studies, there still remains some uncertainties related to the principal issues raised in this paper. First and foremost, is the issue of exactly where the equilibrium beta limit is. The critical poloidal beta, β_{p_crit}, is defined as the upper most poloidal beta possible because of the fundamental equilibrium limit that results from the inboard side poloidal magnetic field approaching zero. We find this critical poloidal beta to definitely be at $\beta_{p\_crit} > A$, and mostly it seems to be very near $\beta_{p\_crit} \approx 1 + A$. A very good original review paper on equilibrium from work in the Soviet Union in the 1960's [6] put the limit $\beta_{p\_crit} \approx 0.5 + A$. This is also reported in [7].

Later investigations in the 1970's with equilibrium codes routinely found $\beta_{p\_crit} > A$ [5,13,15]. In the 1980's, however, Freidberg [3] arrived at a somewhat more pessimistic estimate for the critical poloidal beta of $\beta_{p\_crit} < A$. Our finding of $\beta_{p\_crit} \approx 1 + A$ does nevertheless agree with computational work in the 1970's and 1980's on flux conserving tokamaks [5]. Note that in reference [4], the value of the aspect ratio, A=4, from reference [5] is missing from the reporting of that reference in Fig. 23 in [4] in which we can see that $\beta_p \approx 5$ is reached.

Perhaps the most Shafranov-shifted and the most diamagnetic tokamak equilibrium we could find in the published literature comes from Ling and Jardin [15] in their Fig. 14. Pictures of extremely high diamagnetism with a Shafranov-shift almost equal to the minor radius such as Fig. 4 in [1] and Fig. 10 in [6] have so far eluded the equilibrium codes used here. A possible explanation might be the following. In [1], Eq. 31, one can observe that the assumption of 1/A << 1 and a pure skin current was taken prior to finding the condition that poloidal magnetic field goes to zero on the inboard side of the torus. The same $\varepsilon\beta_{p\_crit} = \pi^2/16$ is derived in [6,7] and can be found in similar forms elsewhere [4] using a skin-current model [6,11]. Note that the poloidal beta defined as β_I in [1] is the same as the typical poloidal beta used by most computational works. These skin-current models that also implicitly assume 1/A <<1 lead to the often quoted $\varepsilon\beta_{p\_crit} \approx 0.5$. They are certainly not as accurate as using the $\varepsilon\Lambda \approx 1$ condition for β_{p_crit} described in the previous section, presented in [7], and originally presented in [6] which does NOT assume 1/A << 1 a priori. Therefore, a major result of our study is to reconfirm some of the earlier work of the 1960's and 1970's that routinely reported $\beta_{p\_crit} > A$.

The only two other fits for the poloidal beta versus peak beta relation that we could find in the literature are as follows: 1) the fit of [3,4] that saturates too low, and 2) the power fit $\beta_p \sim (c\beta_{\text{peak}})^{1/3}$ (c=1000 used here) from [14] that doesn't saturate at all. Neither of these previous fits do as well as our suggested tanh fit (Eq. 10), see Fig. 6.

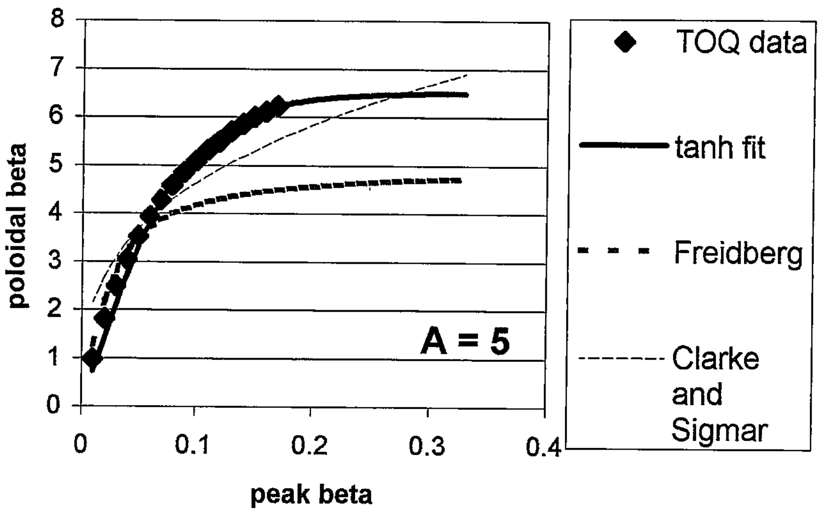


FIGURE 6:

Our tanh functional fit for beta poloidal is compared with the only two others available in the literature for fixed-q heating of tokamaks.

Unfortunately, despite many attempts to derive this tanh function (Eq. 10), we are still unable to do so. Moreover, there are functions related to the q-profile (and other effects such as shaping) that we have not fully explored yet. Future work will concentrate on the derivation of Eq. 10. Future work will also concentrate on pushing the critical beta poloidal limit of the codes by modifying them to run with yet higher precision. This may allow the codes to reach a higher beta than has been possible so far. The inboard poloidal magnetic field is still expected to become zero at some point and still produce the equilibrium beta limit. Yet, it is unclear if the fixed-q evolution will allow the beta poloidal to just barely stay under this limit.

Another important point that should be made from Fig. 2 is that lower aspect ratio provides real benefits in the beta that can be obtained from just equilibrium calculations alone. One can also exceed $\beta_p \sim A$ to a larger degree at lower A, but the diamagnetism is not necessarily larger. Nevertheless, the principal economic metric is the total beta, and a lower aspect ratio provides that benefit from the perspective of this study. From Fig. 3, one can see that if one is willing to sacrifice the stability benefits of a q > 1 (and enough shear as well), then even higher betas are possible from just an equilibrium standpoint.

V. Conclusions:

The results presented here do not discount the possibility that a fully diamagnetic plasma equilibrium is possible. In fact, if the proposed tanh fit is correct, then we are actually very close: the poloidal beta saturates *just* above our highest beta poloidal calculations. Therefore, this tanh fit is very optimistic relative to the vision of a unity beta tokamak [1]. It is very interesting that this tanh fit is so simple and the saturation level seems so close to the critical poloidal beta, β_{p_crit}. Future work will need to concentrate on attempting an analytical derivation of this tanh functional fit. We might need to do extensive rewrites in TOQ in order to run with 16-byte precision calculations that may take us further. Yet, it is very unclear if such efforts in higher precision calculations would provide for any significant advances towards a fully diamagnetic equilibrium.

This paper does shed light on the uncertainties that have existed within the subject of fixed-q evolutions. High beta concepts such as ET and spherical tokamaks have been built and the need to be more precise about what the equilibrium codes can do in terms of unity beta is once again a significant issue. The codes easily give $\beta_p > A$, but do not easily give $\beta_p > 1 + A$, a limit predicted early in tokamak history [6]. Subsequent work [1,3,4] often imposed an *a priori* high aspect ratio approximation to arrive at a more pessimistic beta limit. Our work points to an optimistic $\beta_{p\_crit} \approx 1 + A$ that seems to argue well for high beta tokamak operations.

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