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Time-Dependent Neoclassical Viscosity*

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Abstract

Fluid moment equations including viscous stresses in low collisionality regimes require kinetic closures for viscous forces. The parallel viscous force affects the momentum balance equation. Thus, two important effects are introduced. A viscous term in the parallel Ohm's law yields a correction in the electrical conductivity due to the interaction between trapped and circulating particles. Also, the parallel flow evolution is affected by the viscous drag which induces poloidal flow damping and transport in axisymmetric geometries.

For time scales $t \sim 1/\nu$ a static assumption is not valid and a dynamic closure is needed. Previous works calculated damping rates for $\langle \mathbf{B} \cdot \nabla \cdot \Pi \rangle$ and their results agree up to a numerical factor that depends on some power of the aspect ratio. Here, a time dependent closure is developed and calculated. Thus, evolution equations for the electrical conductivity and the parallel flow are possible and obtained. Having proper time-dependent evolution equations makes possible the description of the short and long time limits as well as the transient stages for various neoclassical phenomena.

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1 Introduction

Most plasma confinement devices operate in the low collisionality banana or neoclassical regime where the relevant closure moment is the parallel stress tensor. In particular, the viscous drag in the parallel Ohm's law introduces a modification of the electrical conductivity. The closure for the parallel viscous force and thus the correction to the conductivity have been calculated assuming a steady state [1]-[4]. Using the closures obtained, the damping rate of the poloidal flow in a toroidal plasma becomes comparable with the ion-ion collision frequency which violates the static assumption used to derive the closure [5].

Various authors have addressed the time dependent problem [6]-[10] both by a variational principle and using an expansion in Cordey [11] eigenfunctions of the pitch-angle scattering operator. Damping rates are estimated to be of the order of the ion collision frequency ν_{ii} times some power of the inverse aspect ratio.

In this work, the time dynamic case is approached in a different way. The main objective is to describe the *time evolution* of the parallel viscous force and thus the poloidal flow. That is, we are interested in an expression for the closure with an explicit time dependence. This approach responds to two concerns. First, the search for a complete picture of the transition to the steady state and the determination of the time scale on which an equilibrium assumption is formally valid. The relaxation of the poloidal flow occurs on fast time scales but is still relevant to some experiments, for example after a sawtooth crash [12]. Most neoclassical effects are modeled either in steady state or with an estimated damping rate for the flow. The time evolution calculated here will permit a more rigorous description and analysis of the transient stages. A second motivation for the calculation on the following pages is the need for time dependent closure relations, in particular for the electrical conductivity and poloidal flow damping, to introduce these effects in numerical codes like NIMROD [13].

The procedure presented here is based on a Chapman-Enskog-like approach [14] which is described in Section 2 together with the model used in the calculation. In Section 3 the closure for the parallel viscous force and the electrical conductivity are obtained. The reduction to the static result as well as the small-field-modulations approximation are also adressed here. In Section 4 the effect of the obtained closure on the flow evolution is studied. A summary and concluding remarks are included in Section 5. In Appendices A and B the calculations leading to the key results in Section 3 are shown. Details of the Laplace-transform inversions are shown in Appendix C.

2 Chapman-Enskog-Like Procedure

In a Chapman-Enskog-like approach, the distribution function of a system is assumed to be a dynamic, flow-shifted Maxwellian plus a small deviation. In this work, heat flux effects will be neglected for simplicity but in order to include them, one should also consider a heat flow shift in the lowest order distribution. The distortion is considered small (in a small gyroradius approximation) compared with the equilibrium Maxwellian solution.

The first three moments of the kinetic distortion vanish by the Chapman-Enskog hypothesis. That is, the time dependence of the distribution function is given through the variations in the thermodynamic variables in the Maxwellian. Thus, the Chapman-Enskog Ansatz is given by

$$f = f_M + F \ , \tag{1}$$

where f_M is

$$f_M\left(\mathbf{v}', \mathbf{x}, t\right) = n\left(\mathbf{x}, t\right) \left(\frac{m}{2\pi kT\left(\mathbf{x}, t\right)}\right)^{3/2} exp\left[-\frac{m\left(\mathbf{v}' - \mathbf{V}\right)^2}{2kT\left(\mathbf{x}, t\right)}\right] \ , \tag{2}$$

and for the distortion F we have the constraint conditions

$$\int \mathrm{d}^3 v \, F = 0, \quad \int \mathrm{d}^3 v \, \mathbf{v} F = 0, \quad \int \mathrm{d}^3 v \, \frac{mv^2}{2} F = 0 \ , \tag{3}$$

By including the first three moments of the distribution function in the Maxwellian part, the kinetic distortion does not add terms to the density, momentum and energy balance equations. Introducing the distribution given in Eq. (1) in the kinetic equation

$$\frac{df}{dt} = C\left(f\right) \ , \tag{4}$$

and taking moments of it one can readly obtain evolution equations for the macroscopic variables of the system. The basic fluid moment balance equations are then

$$\frac{\partial n_j}{\partial t} + \nabla \cdot n_j \mathbf{V}_j = 0 \ , \tag{5}$$

$$\frac{3}{2} n_j \frac{dT_j}{dt} + n_j T_j \left(\nabla \cdot \mathbf{V}_j\right) = -\Pi_j{:}\nabla \mathbf{V}_j \ , \tag{6}$$

$$m_j n_j \frac{d\mathbf{V}_j}{dt} = -n_j q_j (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \nabla p_j - \nabla \cdot \Pi_j + \mathbf{F_{0j}} \ . \tag{7}$$

Here n is the number density, $\mathbf{V}$ the flow velocity, T the temperature, m the particle mass, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The index j indicates the plasma species, electrons or ions. The collisional friction forces and stress tensor are defined by

$$\mathbf{F}_{0j} = \int d^3 v \, m \mathbf{v} \, C\left(f_j\right) \ , \tag{8}$$

$$\Pi = \int d^3 v \, m \left(\mathbf{v}\mathbf{v} - \frac{1}{3} v^2 \mathbf{I}\right) f_j \ . \tag{9}$$

The variable $\mathbf{v}$ is the relative velocity $\mathbf{v} = \mathbf{v}' - \mathbf{V}$, i.e., the velocity of individual particles in the flow's ($\mathbf{V}$) rest frame. The higher order moment of the distribution function in Eq. (9) is usually ignored in MHD formulations and gives rise to the neoclassical effects and thus the interactions of trapped and circulating particles that are of concern in this work. The set of equations (5) to (7) is not complete even though Maxwell equations relate electromagnetic fields to the current J and charge densities ρ_q. In order to describe the evolution of the thermodynamic variables one has to specify Π (in this case) through a closure in terms of the lower order moments n, T, $\mathbf{V}$.

Kinetic theory provides a framework for calculating these closures from the distribution function f which evolves according to the plasma kinetic equation

(4). For a system of charged particles and independent variables $\mathbf{v}$, $\mathbf{x}$, and t, the total time derivative in the left side of the plasma kinetic equation is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \left[\frac{q}{m} \left(\mathbf{E} + \frac{1}{c} \left(\mathbf{V} + \mathbf{v} \right) \times \mathbf{B} \right) - \frac{\partial \mathbf{V}}{\partial t} - \left(\mathbf{V} + \mathbf{v} \right) \cdot \nabla \mathbf{V} \right] \cdot \frac{\partial f}{\partial \mathbf{v}} \ . \tag{10}$$

The distribution function given by Eqs. (1)-(3) can be introduced in Eq. (10). Then, using the balance equations (5)-(7), the total time derivative of the distribution function can be written in terms of F. Using a model collision operator that separates the effects on f_M and F, Wang and Callen [14] recast the kinetic equation in a formal gyroaveraged drift kinetic equation (DKE) for the kinetic distortion F. The full equation is not written here since only a simplified version of it will be considered. This is, neglecting all heat flux terms and higher (than $L_1^{1/2}$) moments, Eq. (127) in Reference [14] reduces to

$$\frac{\partial F}{\partial t} + \left(v_{\|} \mathbf{b} + \mathbf{V} \right) \cdot \nabla \mathbf{F} - C_R \left(F \right) = \left\{ - \left(v_{\|}^2 - \frac{v_{\perp}^2}{2} \right) G + \frac{v_{\|}}{p} \mathbf{b} \cdot \nabla \cdot \Pi \right\} f_m \ , \tag{11}$$

where the "flow drive" term is

$$G = \frac{2}{v_t^2} \left[\mathbf{V} \cdot \nabla \ln \mathbf{B} - \frac{1}{B} \mathbf{b} \cdot \nabla \times \left(\mathbf{B} \times \mathbf{V} \right) + \frac{2}{3} \nabla \cdot \mathbf{V} \right] \ , \tag{12}$$

and the approximate collision operator we will consider is

$$C_R \left(F \right) = \frac{1}{2} \nu_{\perp} \left(v \right) \pounds \left(F \right) \ , \tag{13}$$

which represents pitch angle scattering effects. The full collision operator takes care of momentum conservation. Because the calculations are performed in the laboratory frame and from (3) $\int \mathrm{d}^3 v \, \mathbf{v} F = 0$, no momentum restoring term is needed. Also, since the kinetic distortion F represents the effects of the magnetic field modulations, only perpendicular diffusion in velocity space (pitch-angle scattering) will be considered in the collision operator.

2.1 Magnetic field model.

In toroidal geometries, the magnetic field lines have a helical twist. Field modulations along field lines create potential wells into which charged particles get trapped. In the spirit of restricting the calculation to the effect of interactions between these trapped particles and those that flow freely along field lines, we consider a cylinder magnetic field with periodic bumps. The functional form of the field is given by

$$B(l) = B_{\min} + \Delta B \sin^2 \frac{\pi l}{L} \; . \tag{14}$$

This geometry allows a simple way of retaining only the $\nabla_{\|} B$ effects. As will be seen in Section 4, the model can be extended to axisymmetric geometries in a simple way. Within the scale length of interest, the periodicity of the cylinder L, only the magnetic field will be allowed to vary. In Eq. (14) $\Delta B = B_{\max} - B_{\min}$ and the spatial variable l follows the field line as it curves.

Some simplifications can be made at this point. The particle continuity equation (5) for bounce time scales is

$$0 \simeq \frac{\partial n}{\partial t} = -n \nabla \cdot \mathbf{V} - \mathbf{V} \cdot \nabla n \tag{15}$$

which leads to an incompressible flow to lowest order for density constant along the magnetic field. As a consequence, by noting that

$$\nabla \cdot \mathbf{V} = (\mathbf{B} \cdot \nabla) \left(V_{\|} / B \right) = 0 \; , \tag{16}$$

the incompressibilty constraint can then be satisfied by defining a parallel flow variable $V_{\|} (l, t) / B (l) = U (t)$. Also, in this simple one-dimensional configuration

$$\nabla \times (\mathbf{B} \times \mathbf{V}) = 0 \ . \tag{17}$$

Introducing these simplifications in Eq. (12), the flow drive term reduces to $G = \frac{2}{v_t^2}\mathbf{V} \cdot \nabla \ln \mathbf{B}$ and using the relation $v_\parallel \mathbf{b} \cdot \nabla \left(v_\parallel B\right) = \left(v_\parallel^2 - v_\perp^2/2\right) \mathbf{B} \cdot \nabla \ln \mathbf{B}$ we have

$$\left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) G = \frac{2}{v_t^2} v_\parallel \mathbf{b} \cdot \nabla \left(v_\parallel B\right) U \ . \tag{18}$$

With the previous simplifications, the drift kinetic equation becomes

$$\frac{dF}{dt} + v_\parallel \mathbf{b} \cdot \nabla \left[F + \frac{2}{v_t^2} v_\parallel B U f_M\right] - C_R(F) = \frac{v_\parallel}{p} \left(\mathbf{b} \cdot \nabla \cdot \Pi_\parallel\right) f_M \ . \tag{19}$$

The first term is the dynamic evolution of the kinetic distortion while the second term reflects spatial variations of both the kinetic distortion and the free streaming flow of the circulating particles. Collisions in this case can drive particles through a perpendicular diffusion process (in velocity space) into or out of the trapped region of velocity space. The parallel viscous force is a source for the evolution of the unknown distribution F and hence will be present in the solution. However, the Chapman-Enskog constraints given in Eq. (3) will allow an expression for this term without an explicit dependence on F. Equation (19) will be solved by a perturbation technique in the next section in the low collisionality banana regime.

3 Solution of the Drift Kinetic Equation for F

In the banana collisionality regime, trapped particles can complete their orbits before scattering. Then, the parameter $\nu_* = \frac{\nu}{\epsilon^{3/2}\omega_b}$ is small and will be used as an expansion parameter for determining F. In defining ν_*, ν is the collision frequency, $\epsilon \equiv \Delta B/2B$ and ω_b is the bounce frequency.

The kinetic distortion is then expanded as follows

$$F = F_0 + \nu_* F_1 + \dots \; . \tag{20}$$

If the time derivative is assumed to be order ν_*, the lowest order DKE is

$$v_\parallel \mathbf{b} \cdot \nabla \left[F + \frac{2}{v_t^2} v_\parallel B U f_M \right] = 0 \; . \tag{21}$$

Since for the model magnetic field considered $\mathbf{b} \cdot \nabla = \frac{\partial}{\partial l}$, the terms in square brackets cannot depend on l. Thus, the lowest order distortion is given by

$$F_0 = -\frac{2}{v_t^2} v_\parallel B U f_m + g(v, \lambda, \varsigma, t) \; . \tag{22}$$

Here, g is an integration "constant," which is a function of all the variables of the system other than the spatial l over which the integration was performed. The first term in Eq. (22) represents the free streaming part of F while the second term is a collisional correction. The variable ς depends on the direction in which the particles circulate and is defined as $\varsigma = \left|v_\parallel\right| / v_\parallel$. Note that the only difference up to here with a static calculation is in the time dependence of the function g since the only variables left after the integration in this case would only be energy (or speed) v and pitch angle variable $\lambda = \frac{2\mu}{v^2}$.

To next order in ν_* the DKE equation is

$$\frac{dF_0}{dt} + v_{||}\mathbf{b}\cdot\nabla F_1 - C_R(F_0) = v_{||}\frac{1}{p}\mathbf{b}\cdot\nabla\cdot\Pi_{||}f_M \ . \tag{23}$$

Since $v_{\|}\mathbf{b}\cdot\nabla F_1 = v_{\|}\frac{\partial F_1}{\partial l}$, taking a bounce average defined by

$$\oint\frac{dl}{v_{\|}} = \begin{cases} \int_0^L \frac{dl}{v_{\|}}, & \textit{untrapped particles} \ , \\ \sum_{\varsigma}\int_{-l_c}^{l_c}\frac{dl}{|v_{\|}|}, & \textit{trapped particles} \ , \end{cases} \tag{24}$$

eliminates the first order distortion in Eq. (23). That is, if F_1 is a smooth distribution, its gradient vanishes upon integration over a closed orbit. The bounce-averaged first order equation provides a constraint to solve for the integration constant in the lowest order solution.

For trapped particles, if g_t is odd in ς, it vanishes when it is bounce averaged. If g_t is even, $g_t = 0$ at the tip of the orbit since to take account of density conservation $g_t(\varsigma) = -g_t(-\varsigma)$. Since g does not depend on l, $g_t = 0$ for all values of l.

From the bounce-averaged complete first order DKE (obtained in Appendix A), a partial differential equation for g_c for circulating particles is obtained

$$\begin{aligned}\left\{\left\langle\frac{B}{v_{\|}}\right\rangle\frac{d}{dt} - \frac{\nu_\perp}{2}\frac{2}{v^2}\frac{\partial}{\partial\lambda}\lambda\left\langle v_{\|}\right\rangle\frac{\partial}{\partial\lambda}\right\}g_c &= \frac{f_M}{p}\left\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle \\ &+ \frac{2}{v_t^2}f_M\left\langle B^2\right\rangle\left\{\frac{d}{dt}+\frac{\nu_\perp}{2}\right\}U \ ,\end{aligned} \tag{25}$$

where the Lorentz collision operator is expressed, with λ and v as independent variables, as follows

$$\mathcal{L} = \frac{2v_{\|}}{v^2}\frac{\partial}{\partial\lambda}\left(\frac{\lambda}{B}v_{\|}\frac{\partial}{\partial\lambda}\right) \ . \tag{26}$$

Equation (25) is an inhomogeneous, second order partial differential equation in time and pitch angle. In it energy (speed) is a parameter since only perpendicular diffusion effects are considered. In order to solve for the phase space variables dependence, a Laplace transform in time can be taken. Then, frequency will be treated as a parameter and the equation to be solved is

$$-i\omega \left\langle \frac{B}{v_{\|}} \right\rangle \widehat{g}_c - \frac{\nu_\perp}{2} \frac{2}{v^2} \frac{\partial}{\partial \lambda} \lambda \left\langle v_{\|} \right\rangle \frac{\partial \widehat{g}_c}{\partial \lambda} = f_M \hat{S}(v, \lambda, t) \ , \tag{27}$$

where hats denote Laplace-transformed quantities with tranform variable $-i\omega$ as defined below:

$$\hat{h}\left(\omega\right) = L\left\{h\left(t\right)\right\} \equiv \int_{0^-}^{\infty} dt \, e^{i\omega t} h\left(t\right) \ . \tag{28}$$

The drives and initial conditions are included in the source term

$$\hat{S}(v, \lambda, t) = \frac{m}{T} \left\langle B^2 \right\rangle \left(\frac{\nu_\perp}{2} - i\omega \right) \widehat{U} + \frac{1}{p} \langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi} \rangle - \left\langle \frac{B}{v_{||}} \right\rangle g_0(\lambda) - \left\langle B^2 \right\rangle \frac{2}{v_{th}^2} U_0(\psi) \ , \tag{29}$$

where $g_0 = g_c(t = 0)$, $U_0 = U(t = 0)$. Note that the λ dependence of the source is through the initial pitch-angle structure of the collisional correction g_c. That this equation should be treated as an initial value problem was first pointed out by Morris et al. [9]. Clearly, the collisional diffusion into trapped space of a distribution of untrapped particles will depend strongly (at least for early times) on how close the initial distribution is located to the boundary between the two types of particles. An initial distribution of particles is expected to damp more rapidly if it is peaked close to λ_c than if particles are introduced far away from the trapped-circulating boundary since then the portion of phase space they have to diffuse through is larger.

The speed dependence in $g_c\left(\lambda, v, t = 0\right)$ is arbitrary since the collision operator only operates on the pitch angle variable. It is convenient to choose

$g_0(v,\lambda) = \frac{m}{T}\langle B^2\rangle f_M g_0(\lambda)$ to match the rest of the terms in the source [9].

Noting that

$$\left\langle \frac{B}{v_{\|}} \right\rangle = -\frac{2\varsigma}{v}\frac{\partial}{\partial\lambda}\left\langle \sqrt{1-\lambda B}\right\rangle \ , \tag{30}$$

Eq. (27) can be expressed as

$$-\frac{2\varsigma}{v}\left\{ \frac{\nu_\perp}{2}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial}{\partial\lambda} - i\omega\frac{\partial}{\partial\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle\right\}\widehat{g}_c = f_M\hat{S}(v,\lambda,t)\ . \tag{31}$$

The speed and pitch-angle dependencies can now be separated using an eigenfunction expansion for the pitch angle operator in the left side of Eq. (31).

The function $\hat{g}_c$ is projected in Cordey eigenfunctions as done in references [7]-[9]:

$$\widehat{g}_c = \sum_{n=1}^{\infty} Y_n(v,i\omega)\Lambda_n(\lambda)\ . \tag{32}$$

The λ-dependent functions are solutions to the ordinary differential equation

$$\frac{d}{d\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{d\Lambda_n}{d\lambda} = \kappa_n\frac{d}{d\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle\Lambda_n\ , \tag{33}$$

where κ_n are the eigenvalues and the eigenfunctions Λ_n satisfy the orthogonality condition

$$\int_0^{\lambda_c}\Lambda_n\Lambda_m\frac{\partial\left\langle\sqrt{1-\lambda B}\right\rangle}{\partial\lambda}d\lambda = \delta_{nm}\int_0^{\lambda_c}\Lambda_n^2\frac{\partial\left\langle\sqrt{1-\lambda B}\right\rangle}{\partial\lambda}d\lambda\ . \tag{34}$$

The functions Λ_n for n odd vanish between $-l_c$ and l_c (i. e., for trapped particles) and are even in ς for circulating particles [15]. Thus, they are appropiate for the complete description of the solution inside the trapped region $g_t = 0$ and in the untrapped region for $g_c \neq 0$. At the boundary, the functions are continous.

The operator in Eq. (31) is a combination of “time-dependence” and pitch-angle scattering. Thus, the eigenfunctions are similar in structure and reduce to the Legendre polynomials for a homogeneous magnetic field. Using these conditions, we can solve for the speed-dependent coefficients obtaining a expansion of the form (see Appendix B)

$$\widehat{g}_c = \frac{v\varsigma}{2} \sum_1^\infty \frac{\eta_n}{\left(\frac{\nu_\perp}{2}\kappa_n - i\omega\right)} \Lambda_n \; . \tag{35}$$

The coefficients η_n are calculated from

$$\eta_n = \frac{\int_0^{\lambda_c} \hat{S} \Lambda_n d\lambda}{\int_0^{\lambda_c} \Lambda_n^2 \frac{\partial}{\partial\lambda} \left\langle \sqrt{1-\lambda B} \right\rangle d\lambda} \; . \tag{36}$$

Note that even though these coefficients do not depend on λ, the initial structure of the distribution in pitch angle will enter as a drive for the viscosity through these integrals. Introducing Eq. (35) in the Laplace transform of Eq. (22), the lowest order solution for the departure from a Maxwellian can be expressed in frequency space as

$$\widehat{F_0} = -v_\| B \frac{2}{v_t^2} f_m \widehat{U} + \frac{v\sigma}{2} \sum_1^\infty \frac{\eta_n}{\left(\frac{\nu_\perp}{2}\kappa_n - i\omega\right)} \Lambda_n \; . \tag{37}$$

The viscous force, being a source in the differential equation for F, is part of the solution through the coefficient η_n. In the next section, the Chapman-Enskog constraint that the momentum moment of the small departure from the Maxwellian vanishes will be used to solve for $\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi} \rangle$.

3.1 Dynamic closure for $\left\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_\|} \right\rangle$

In this section an average collision frequency is introduced for simplicity in the notation. The speed-dependent $\nu_\perp$ can be restored easily by substituting $\bar{\nu} \to \int d^3v \, \frac{\nu_\perp}{2} f_M$.

The Chapman-Enskog Ansatz given in Eq. (3) can be used to express the parallel viscous force in terms of the parallel flow variable $\widehat{U}$ and the initial conditions. Taking the parallel momentum moment on both sides of Eq. (37) and setting $\int d^3v\, v_\| \widehat{F_0} = 0$ yields

$$\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_\|} \rangle = nm\bar{\nu} \left\langle B^2 \right\rangle \left(\frac{\widehat{f_t}}{\widehat{f_c}} \widehat{U} + \frac{1}{\bar{\nu}} U_0 + \frac{1}{\widehat{f_c}} \frac{1}{\bar{\nu}} \sum_1^\infty \frac{\chi_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} \right) , \qquad (38)$$

where $\gamma_n = \frac{3}{4} \frac{\left(\int_0^{\lambda_c} \Lambda_n d\lambda\right)^2}{\int_0^{\lambda_c} \Lambda_n^2 \frac{\partial}{\partial\lambda} \left\langle \sqrt{1-\lambda B} \right\rangle d\lambda}$, and $\widehat{f_c}$ and $\widehat{f_t}$ are defined by

$$\widehat{f_c}\,(\omega) = \sum \frac{\gamma_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} , \qquad (39)$$

$$\widehat{f_t}\,(\omega) = 1 - \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f_c} . \qquad (40)$$

Note that if initial conditions are ignored in Eq. (38), the expression obtained is similar to the known equilibrium result

$$\langle \mathbf{B}\cdot\nabla\cdot\Pi \rangle \sim nm \frac{f_t}{f_c} \left\langle B^2 \right\rangle U , \qquad (41)$$

where f_t and f_c correspond here to the fraction of trapped and circulating particles, respectively, in a static situation.

In this dynamical case, the fraction of trapped and circulating particles depend on frequency but, as will be shown in next section, reduce to the usual ones in the $\omega \ll \bar{\nu}$ limit. No interpretation of the fact that $\widehat{f_c} + \widehat{f_t}$ is a frequency dependent function is necessary since these quantities are defined only to obtain a simple expression for $\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_\|} \rangle$ similar in structure to the steady state solution.

Equation (38) is valid for any time scale and aspect ratio, since no approximations had been introduced so far. From it, a damping rate can be estimated numerically. For an explicit time-dependent expression, neither an analytical nor numerical Laplace inverse transform are trivial to perform since the expression involves infinite sums of terms that depend on integrals over the (numerically generated) eigenfunctions. Moreover, some of these infinite sums reside on the denominator and thus should be calculated to high accuracy if any poles of the response are to be found.

Analytically inverting the Laplace transform seems to be an impossible task. However, it will be shown that in a small ϵ expansion the time dependent closure can be obtained analytically. Before doing so, in the next section the long time asymptotic limit of Eq. (38) will be inspected and compared to the usual equilibrium result.

3.2 Static limit

In steady state or for long times ($\omega \ll \nu_\perp$), the initial conditions are completely damped away and will be neglected in calculating this limit for Eq. (38). Also, in order to compare with the closure $\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{||}} \rangle \sim nm\mu_{00} \langle B^2 \rangle U$, the speed dependence in the collision frequency is restored. That is, the small ω limit of the following expression will be considered:

$$\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{||}} \rangle = mn\widehat{U} \langle B^2 \rangle \int \frac{\nu_\perp}{v_{th}^2} \frac{v^2}{3} \frac{f_M}{n} \frac{\widehat{f}_t (v, \omega)}{\widehat{f}_c (v, \omega)} d^3 v \ , \tag{42}$$

for $\omega \sim 0$ where $\widehat{f}_t \simeq 1 - \widehat{f}_c$. The frequency dependence in the fractions of trapped and circlating particles is not present and thus the inverse Laplace tranform in Eq. (42) can be trivially taken.

Using the result

$$f_c \equiv \widehat{f}_c (\omega \to 0) = \sum \frac{\gamma_n}{\kappa_n} = \frac{3 \langle B^2 \rangle}{4} \int_0^{\lambda_c} \frac{\lambda' d\lambda'}{\langle \sqrt{1 - \lambda' B} \rangle} \ , \tag{43}$$

and taking the inverse Laplace transform of Eq. (42), the equilibrium result is obtained:

$$\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle = n_e m_e \mu_{00} \left\langle B^2 \right\rangle U \ . \tag{44}$$

Here, the viscosity coefficient μ_{00} is defined as

$$\mu_{00} = \frac{f_t}{f_c} \int \frac{\nu_\perp}{v_{th}^2} \frac{v^2}{3} \frac{f_M}{n} \, \mathrm{d}^3 v = \left[Z_i + \sqrt{2} - \ln\left(1 + \sqrt{2}\right)\right] \frac{f_t}{f_c} \nu_e \ . \tag{45}$$

The fraction of trapped and circulating particles can be estimated for $\sqrt{\epsilon} = \sqrt{\Delta B / 2 B_{min}} << 1$. The smallness of this parameter represents the large aspect ratio limit in toroidal geometries. In this model it reflects the limit of small variations of the magnetic field as one moves from the outboard to inboard side of the torus along the helical magnetic field lines. As one might expect, for small ΔB the fraction of trapped particles is small and almost all particles contribute to the flow. In Appendix B the following results are obtained:

$$f_c \approx 1 - 1.46\sqrt{\epsilon}, \quad f_t \approx 1.46\sqrt{\epsilon} \ . \tag{46}$$

To obtain a correction for the electrical conductivity caused by the parallel viscous force $\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle$, the bounce-averaged parallel component of Eq. (7) for electrons in the static limit is considered. Using $\mathbf{B} \cdot \mathbf{F}_{0e} = n_e e B J_{\|} / \sigma_{\|}$, one obtains an equilibrium Ohm's law

$$\left\langle J_{\|} B \right\rangle = \frac{n_e e^2}{m_e \nu_e} \left\langle E_{\|} B \right\rangle + \frac{e}{m_e \nu_e} \left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle \ . \tag{47}$$

The closure in Eq. (44) with the value for μ_{00} ($\simeq 1.533 \frac{f_t}{f_c} \nu_e$ for hydrogenic ions) in the small $\sqrt{\epsilon}$ approximation can be introduced in Eq. (47). The usual correction to the static electrical conductivity is obtained:

$$\sigma = \frac{n_e e^2 / m_e \nu_e}{1 + \mu_{00}/\nu_e} = \frac{n_e e^2}{m_e \nu_e} \frac{1}{1 + 2.24\sqrt{\Delta B / B_{\min}}} \ . \tag{48}$$

At this point, it is important to note that the fraction of trapped particles scales as $\sqrt{\epsilon} = \sqrt{\Delta B / 2B_{\min}} \ll 1$. This result will be used soon for an equivalent expansion in the dynamic problem.

3.3 Dynamic calculation for small field modulations

To obtain the corresponding correction to the electrical conductivity in the time-dependent case, the complete bounce-averaged and Laplace-tranformed parallel electron momentum equation has to be considered. Introducing the frequency-dependent closure yields a ω-dependent electrical conductivity:

$$\hat{\sigma}(\omega) = \frac{n_e e^2 / m_e \nu_e}{\left\{1 + \frac{1}{\nu_e} \int \frac{\nu_\perp}{v_{th}^2} \frac{v^2}{3} \frac{f_M}{n} \frac{\hat{f}_t(v,\omega)}{\hat{f}_c(v,\omega)} d^3 v - \frac{i\omega}{\nu_e}\right\}} \ . \tag{49}$$

The $\frac{i\omega}{\nu_e}$ term in the denominator arises from the inertia term in the momentum equation. Here, as well as in the static case, the ion parallel flow has been neglected for time scales $t \sim 1/\nu_e$.

The dynamic conductivity in Eq. (49) is valid for any frequency (time) and magnetic field modulation. Once again, the analytical process cannot be carried out further. However, a numerical computation could give the time evolution of σ to some appropiate accuracy .

For small field variations, one can make an expansion in the small parameter $\sqrt{\epsilon}$ as was done for the static limit. Thus, a comparison of the long time limit of the result with the static calculations will also be possible. In order to make the expansion, we invoke the $f_t \sim \sqrt{\epsilon}$ result pointed out earlier. For the dynamic case a similar behavior in the function f_t, which is now frequency dependent, can be assumed. Based on this argument, we propose an expansion in powers of $\widehat{f}_t$ and will expect the lowest order terms to dominate.

Here the initial distribution $\hat{g}_0$ will be neglected for simplicity but will be taken into account in the next section when it will become crucial in the analysis of the time evolution of the parallel flow. The relevant factor to be inverted is $\widehat{f}_t/\widehat{f}_c$ for which we propose an expansion as follows

$$\frac{\widehat{f}_t}{\widehat{f}_c} = \left(1 - \frac{2i\omega}{\nu_\perp}\right)\frac{\widehat{f}_t}{1-\widehat{f}_t} \simeq \left(1 - \frac{2i\omega}{\nu_\perp}\right)\widehat{f}_t + \left(1 - \frac{2i\omega}{\nu_\perp}\right)\hat{f}_t^2 + \ldots . \tag{50}$$

Introducing Eq. (50) in the closure given by Eq. (38), we obtain a much simpler expression for the parallel viscous force and the Laplace transform can be inverted term by term. In particular, to lowest order in f_t we have

$$\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle = mn\left\langle B^2\right\rangle \int \frac{\nu_\perp}{v_{th}^2}\frac{v^2}{3}\frac{f_M}{n}L^{-1}\left\{\left(1 - \frac{2i\omega}{\nu_\perp}\right)\widehat{f}_t\hat{U}\right\} d^3v \, . \tag{51}$$

After introducing the average collision frequency and some manipulation (see Appendix C), Eq. (51) can be written as

$$\begin{aligned}\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle \;\; = \;\; & \left\langle B^2\right\rangle mn\bar{\nu}\left\{U(t)\left[1-\sum\frac{\gamma_n}{\kappa_n}\right]\right. \\ & \left. +\frac{1}{\bar{\nu}}\frac{\partial U(t)}{\partial t}\left[1-\sum\gamma_n\right] + \sum\frac{\gamma_n}{\kappa_n}\left(\kappa_n-1\right)^2\int_0^t \frac{dU}{d\tau}e^{-\bar{\nu}\kappa_n(t-\tau)/2}d\tau\right\} \, .\end{aligned} \tag{52}$$

In Eq. (52), the explicit behavior in time of $\langle \mathbf{B}\cdot\nabla\cdot\Pi\rangle$ is exhibited. The first term is proportional to the parallel flow and will be the dominant in the long times asymptotic limit. The second term contains the time variations of the flow and is important only for times of the order of $1/\bar{\nu}$. The last term arises from the convolution of the time-dependent trapped particle fraction and the intrinsic time dependence of the flow of circulating particles. To higher order in this expansion, this term will develops a series in powers of $\bar{\nu}t$.

Another path leading to the same results from considering, from the static calculation,

$$\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{\|}} \rangle \sim f_t \sim \sqrt{\epsilon}\,. \tag{53}$$

Based on this, one can go back to Eq. (31) and propose $g = g^0 + \sqrt{\epsilon} g^1 + \epsilon g^2 + \ldots$. Then, to lowest order, the term in the source $\hat{S}$ due to the parallel viscous force can be neglected. The resulting equation becomes

$$-i\omega \left\langle \tfrac{B}{v_{||}} \right\rangle \widehat{g} - C_R\left(\widehat{g}\right) = f_M \tfrac{m}{T} \left\langle B^2 \right\rangle \left(\bar{\nu} - i\omega\right) \widehat{U} \quad , \tag{54}$$

which can be solved to obtain a lowest order $\widehat{g}^0$. With the kinetic correction obtained in this way, the lowest order distortion $\hat{F}_0$ is completely determined to this order. The viscous force is not present in this solution but can be obtained from

$$\begin{aligned} \langle \mathbf{B} \cdot \nabla \cdot \Pi \rangle &= \left\langle mB \int d^3v\, mv_{\|}^2 \left[(\mathbf{b} \cdot \nabla F_1) \right] \right\rangle \\ &= \left\langle mB \int d^3v\, mv_{\|} \left\{ C_R(F_0) - \frac{\partial F_0}{\partial t} \right\} \right\rangle\,. \end{aligned} \tag{55}$$

This solution can be introduced as a lowest order approximation in the DKE and used to solve for a next order result. This procedure leads to the same result in Eq. (52). It is described here because it provides an easy method to numerically obtain the closure to any order.

4 Flow Evolution

With the closure for the parallel viscous force obtained in the previous section, the evolution of the parallel flow can be calculated as an initial value problem. Two schemes are considered: 1) the parallel flow damping within the bumpy cylinder model, and 2) an extension to an axysimmetric geometry and thus to the time dynamics of the poloidal flow in a magnetically-confined toroidal plasma.

4.1 Parallel flow evolution

In this section the evolution of the parallel flow will be examined. The simplicity of the model permits a full calculation including initial condition effects. Consider the total flux-surface-averaged momentum equation for the plasma

$$mn \left\langle B^2 \right\rangle \frac{\partial U(t)}{\partial t} = - \left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle \ . \tag{56}$$

Since the time dependence of the left side is of interest, we start by taking again a Laplace transform and work from the full frequency-dependent closure in Eq. (38). The last term, which includes an initial distribution in pitch angle, is considered for this calculation. For this term, the factor $1/\hat{f}_c$ has to be expanded. For small $\sqrt{\epsilon}$ we propose

$$\frac{1}{\hat{f}_c} = \left(1 - \frac{i\omega}{\bar{\nu}}\right) \frac{1}{1 - \hat{f}_t} \simeq \left(1 - \frac{i\omega}{\bar{\nu}}\right) \left(1 - \hat{f}_t\right) \ . \tag{57}$$

Introducing the approximations (50) and (57) in Eq. (38) yields

$$\begin{aligned} \left\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{\|}} \right\rangle &= nm\bar{\nu} \left\langle B^2 \right\rangle \left[\left(1 - \frac{i\omega}{\bar{\nu}}\right) \hat{f}_t \widehat{U} + \frac{1}{\bar{\nu}} U_0 \right. \\ &+ \left. \left(1 - \frac{i\omega}{\bar{\nu}}\right) \left(1 - \hat{f}_t\right) \frac{1}{\bar{\nu}} \sum_1^\infty \frac{\chi_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} \right] \ . \end{aligned} \tag{58}$$

After introducing the closure given in Eq. (58) in the Laplace transform of Eq. (56) and rearranging terms, the evolution equation for the parallel flow may be written as

$$\left[(\bar{\nu} - i\omega)\,\widehat{f}_t - i\omega\right]\widehat{U} = \left(1 - \frac{i\omega}{\bar{\nu}}\right)\left(1 - \widehat{f}_t\right)\sum_{1}^{\infty}\frac{\chi_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} \; . \tag{59}$$

Solving for $\hat{U}$ at this point would lead again to infinite sums in the denominator. Instead, the inverse Laplace transform can be taken on both sides and after some manipulation (see Appendix C) one obtains an inhomogeneous integral equation for $U(t)$:

$$U(t) = h(t) + \bar{\nu}\int_0^t K(t;\tau)\,U(\tau) \; . \tag{60}$$

The integration kernel is given by

$$K(t;\tau) = \frac{1}{\left(2 - \sum\gamma_m\right)}\left\{\sum_m \frac{\gamma_m}{\kappa_m} - 1 - \sum_m \frac{\gamma_m}{\kappa_m}\left(\kappa_m - 1\right)^2 e^{-\kappa_m\bar{\nu}(t-\tau)}\right\} \; . \tag{61}$$

The initial pitch-angle structure is buried in the inhomogeneous “initial” term

$$\begin{aligned} h(t) \;=\; & \frac{1}{\left(2 - \sum\gamma_m\right)}\left\{\sum_{m,n}\frac{\kappa_m - 1}{\kappa_n\kappa_m}\left[\chi_m\gamma_n\kappa_n + \chi_n\gamma_m\right]e^{-\kappa_m\bar{\nu}t}\right. \qquad (62)\\ & \left. - \sum_{m,n}\frac{\bar{\nu}\chi_m\gamma_n}{\kappa_m}\left(\kappa_m - 1\right)\left(\kappa_n - 1\right)e^{-\kappa_m\bar{\nu}t}\int_0^t e^{-\bar{\nu}\tau(\kappa_m - \kappa_n)}d\tau + \sum_{m,n}\frac{\chi_n\gamma_m}{\kappa_n\kappa_m}\right\} \; . \end{aligned}$$

This equation gives the time evolution of the parallel flow and has “memory” of the localization of the initial distribution relative to the boundary with trapped particle space.

4.2 Poloidal flow evolution

In this section, the dynamic evolution of the "parallel" flow in an axisymmetric configuration is considered. An initial distribution in pitch angle variable is not taken into account for simplicity. That is, the initial perturbation introduced in the system is only composed of untrapped particles that will contribute to the flow.

In a toroidal configuration, the magnetic field can be written as

$$\mathbf{B} = B_T\hat{\zeta} + B_P\hat{\theta} \ , \tag{63}$$

where B_T and B_P are the components of the magnetic field in the toroidal (ζ) direction and poloidal (θ) directions respectively. To apply the model developed in the previous sections to this geometry some modifications have to be introduced.

In such a configuration, the field modulations along a field line are not unidimensional. This is, the relevant flow variable to be considered is the poloidal flow defined by [16]

$$U_\theta \equiv \frac{\mathbf{V}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\theta} = \frac{V_\parallel}{B} + \frac{\mathrm{I}}{B^2}\left(\frac{d\phi}{d\psi} + \frac{1}{nq}\frac{dp}{d\psi}\right) \ , \tag{64}$$

where the first term is due to the parallel flow velocity and the second is due to the perpendicular flows in the plasma. The perpendicular flow to lowest order in gyroradius is a combination of the $\mathbf{E}\times\mathbf{B}$ and diamagnetic flows:

$$\mathbf{V}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}\times\nabla p}{nqB^2} \ . \tag{65}$$

In Eq. (64) $\mathbf{E} = -\nabla\phi$ and ψ is the magnetic flux function.

From the parallel (to $\mathbf{B}$) momentum balance, an evolution equation for U_θ

will include a contribution ($\propto q^2$) from the toroidal flow [5]

$$nm\langle B_P^2\rangle\left(1+2q^2\right)\frac{\partial U_\theta}{\partial t}=-\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle+4\pi\langle B_P^2\rangle\langle\frac{\partial E_r}{\partial t}\rangle\ , \tag{66}$$

where the safety factor is defined as $q=\epsilon B_T/B_p$.

The closure in Eq. (38) may now be introduced. In this problem parallel stress damps the poloidal component of the parallel flow since toroidal momentum is only damped by the (higher order) perpendicular stress. Then, one can write

$$\langle\widehat{\mathbf{B}\cdot\nabla\cdot\Pi_{\|}}\rangle=nm\bar{\nu}\left\langle B^2\right\rangle\left(\frac{\widehat{f}_t}{\widehat{f}_c}\widehat{U_\theta}+\frac{1}{\bar{\nu}}U_{\theta 0}\right)\ , \tag{67}$$

which, when introduced in the Laplace transform of Eq. (66) in a large aspect ratio approximation, yields

$$\begin{aligned}\left\{(\bar{\nu}-i\omega)\,\hat{f}_t-i\omega\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\left(1+2q^2\right)\right\}\widehat{U_\theta} &= \frac{4\pi}{nm}\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\langle\phi'\left(0\right)-i\omega\widehat{\phi'}\rangle \\ &- \left\{1+\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\left(1+2q^2\right)\right\}U_\theta\left(0\right)\ .\end{aligned} \tag{68}$$

In this geometry, the flux surface average is defined as

$$\langle A\rangle=\frac{\oint\frac{d\theta}{\mathbf{B}\cdot\nabla\theta}A}{\oint\frac{d\theta}{\mathbf{B}\cdot\nabla\theta}}\ . \tag{69}$$

Equation (69) is inverted to lowest order in $\hat{f}_t$ in Appendix C and the dynamic evolution of the poloidal evolution is expressed again as an integral equation as

$$U_\theta\left(t\right)=U_0+\bar{\nu}\int_0^t K\left(t;\tau\right)U_\theta\left(t\right)d\tau\ . \tag{70}$$

Here the integration kernel is given by

$$K\left(t;\tau\right) \sim \frac{\left\langle B^2\right\rangle \left\{1-\sum \frac{\gamma_n}{\kappa_n}\left[1+\left(\kappa_n-1\right)^2 e^{-\kappa_n \bar{\nu}(t-\tau)}\right]\right\}}{\left\langle B^2\right\rangle \left(1-\sum \gamma_n\right)+\left(1+2q^2\right)} \ . \tag{71}$$

Equation (70) gives the time evolution of the poloidal component of the flow in a torus induced by U_0.

5 Summary

A simple inhomogeneous magnetic field model has been used to calculate the kinetic closure for the parallel viscous force $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \rangle$ in the banana collisionality regime. This model retains the effect of trapped particles and can be extended to more complicated three-dimensional geometries.

In carrying out the calculation a formal Laplace transform is introduced which retains the initial value character of the problem. The complexity of the expressions obtained for the closure and the conductivity call for an expansion for small magnetic field modulations. In such a case, the inverse Laplace tranform can be calculated to any order both analytically and numerically. The key result [Eq. (42)] is an *explicit-time-dependent* closure for the parallel viscous force which may be useful in numerical codes as well as for the theoretical modeling of fast phenomena ($t \lesssim 1/\nu$).

The evolution of the parallel/poloidal flow driven by the expressions obtained are described by integral equations which are still to be solved. The results obtained here have various applications that are yet to be exploited. For example, they can be used to explore the dynamics of the transition of the perpendicular dielectric from regular Alfvenic to enhanced neoclassical [17] and for the effects of flow dynamics on neoclassical tearing modes [18].

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Appendix A

In this appendix the calculations that led to the key results in Section 3 are shown. Consider the lowest order distortion $F_0 = -\frac{2}{v_t^2} v_\| BU f_m + g$. Applying the bounce-average operator defined in Eq. (24) to the next order DKE given in Eq. (23) yields

$$\frac{\partial}{\partial t} \oint \frac{dl}{v_\|} \left(-\frac{2}{v_t^2} v_\| BU f_M + g \right) - \frac{\nu_\perp}{2} \oint \frac{dl}{v_\|} \mathcal{L}\left(F_0\right) = \frac{2}{\nu_\perp} \oint dl \frac{1}{p} \mathbf{b} \cdot \nabla \cdot \mathbf{\Pi}_\| f_M \ . \quad (72)$$

Since $v_\| = v\varsigma\sqrt{1-\lambda B}$:

$$\pounds \left(-\frac{2}{v_t^2} v_\| BU f_m + g \right) = \frac{2}{v_t^2} v_\| BU f_M + \frac{2 v_\|}{v^2} \frac{\partial}{\partial \lambda} \left(\frac{\lambda}{B} v_\| \frac{\partial g}{\partial \lambda} \right) \ . \quad (73)$$

Then, the left side of (72) can be written as

$$-\frac{2}{v_t^2} f_M \frac{\partial U}{\partial t} \oint \frac{dl}{B} B^2 + \frac{\partial g}{\partial t} \oint \frac{dl}{B} \frac{B}{v_\|} - \frac{\nu_\perp}{v_t^2} U f_M \oint \frac{dl}{B} B^2 - \frac{\nu_\perp}{v^2} \frac{\partial}{\partial \lambda} \left(\lambda \frac{\partial g}{\partial \lambda} \oint \frac{dl}{B} v_\| \right) \ . \quad (74)$$

Defining $\langle A \rangle = \oint \frac{dl}{B} A / \oint \frac{dl}{B}$, the bounce-averaged first order equation is

$$\begin{aligned} \left\{ \left\langle \frac{B}{v_\|} \right\rangle \frac{\partial}{\partial t} - \frac{\nu_\perp}{2} \frac{2}{v^2} \frac{\partial}{\partial \lambda} \lambda \left\langle v_\| \right\rangle \frac{\partial}{\partial \lambda} \right\} g \quad &= \quad \frac{2}{v_t^2} f_M \left\langle B^2 \right\rangle \left\{ \frac{\partial}{\partial t} + \frac{\nu_\perp}{2} \right\} U \\ &+ \quad \frac{2}{\nu_\perp} \frac{f_M}{p} \left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_\| \right\rangle \ . \end{aligned} \quad (75)$$

In Section 3, Eq. (75) was written as

$$\left\{ \frac{\nu_\perp}{2} \frac{\partial}{\partial \lambda} \lambda \left\langle \sqrt{1-\lambda B} \right\rangle \frac{\partial}{\partial \lambda} - i\omega \frac{\partial}{\partial \lambda} \left\langle \sqrt{1-\lambda B} \right\rangle \right\} \widehat{g} = \frac{v\varsigma}{2} f_M \hat{S}(\lambda, t) \ . \quad (76)$$

Introducing $g = \sum_1^\infty Y_n(v, i\omega) \Lambda_n(\lambda)$ in Eq. (76) yields

$$Y_n \left(\frac{\nu_\perp}{2} \kappa_n - i\omega \right) \frac{\partial}{\partial \lambda} \left\langle \sqrt{1-\lambda B} \right\rangle \Lambda_n = \frac{v\varsigma}{2} f_M \hat{S}\left(\lambda\right) \ . \quad (77)$$

To solve for the energy-dependent function $Y_n(v, i\omega)$, Eq. (77) is multiplied on both sides by Λ_m and integrated as follows

$$\sum_1^\infty Y_n \left(\frac{\nu_\perp}{2}\kappa_n - i\omega\right) \int_0^{\lambda_c} \Lambda_m \Lambda_n \frac{\partial}{\partial\lambda} \left\langle \sqrt{1-\lambda B} \right\rangle d\lambda = \frac{v\varsigma}{2} f_M \int_0^{\lambda_c} \hat{S}(\lambda)\, \Lambda_m d\lambda \ . \tag{78}$$

Using the orthogonality property of Λ_n, one can solve for the coefficients Y_n:

$$Y_n = \frac{v}{2\varsigma} \frac{1}{\left(\frac{\nu_\perp}{2}\kappa_n - i\omega\right)} \frac{\int_0^{\lambda_c} \hat{S}(\lambda)\, \Lambda_n d\lambda}{\int_0^{\lambda_c} \Lambda_n^2 \frac{\partial}{\partial\lambda} \left\langle \sqrt{1-\lambda B} \right\rangle d\lambda} \ . \tag{79}$$

Approximating $\int \frac{\nu_\perp}{2} \sim \bar{\nu}$, the lowest order $\widehat{F_0}$ is expressed as

$$\widehat{F_0} = -v_\| B \frac{2}{v_t^2} f_M \widehat{U} + \frac{v\varsigma}{2} \sum_1^\infty \frac{\eta_n}{(\bar{\nu}\kappa_n - i\omega)} \Lambda_n \ . \tag{80}$$

To solve for the driving term in $\hat{S}$, the $v_\|$ moment is taken and set to zero:

$$\int \mathrm{d}^3 v\, v_\| \widehat{F_0} = 0, \tag{81}$$

where the integration is over all velocity space and is expressed in spherical velocity-space coordinates as follows:

$$\int d^3 v = \sum_\sigma \pi \int_0^\infty v^3 dv \int_0^{1/B} \frac{d\lambda B}{|v_\||} \ . \tag{82}$$

For the first term in Eq. (80),

$$\sum_\sigma \pi \int_0^{1/B} \frac{d\lambda B}{|v_\||} v_\| \left(-v_\| B \frac{2}{v_t^2} f_M \widehat{U} \right) = -\frac{8\pi}{3 v_t^2} \widehat{U} B v f_M \ , \tag{83}$$

and the second term,

$$\sum_\sigma \pi \int_0^{1/B} \frac{d\lambda B}{|v_\parallel|} v_\parallel \frac{v\sigma}{2} \left(\sum_1^\infty \frac{\eta_n}{\bar{\nu}\kappa_n - i\omega} \Lambda_n \right) = \pi v B \sum_1^\infty \frac{1}{\bar{\nu}\kappa_n - i\omega} \left(\int_0^{1/B} d\lambda \eta_n \Lambda_n \right) . \tag{84}$$

These results are introduced in Eq. (81) and the speed integration calculated on both sides to yield

$$\begin{aligned} \widehat{U} &= \left\{ \left[(\bar{\nu} - i\omega) \widehat{U} - U_0 + \frac{1}{nm} \frac{\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi} \rangle}{\langle B^2 \rangle} \right] \sum_1^\infty \frac{\gamma_n}{\bar{\nu}\kappa_n - i\omega} \right. \\ &- \left. \sum_1^\infty \frac{\chi_n}{\bar{\nu}\kappa_n - i\omega} \right\} \end{aligned} \tag{85}$$

After rearranging terms, it is convenient to define

$$\hat{f}_c (v, \omega) = \sum \frac{\gamma_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} , \tag{86}$$

$$\hat{f}_t (v, \omega) = 1 - \left(1 - \frac{i\omega}{\bar{\nu}} \right) \hat{f}_c . \tag{87}$$

Then the parallel viscous force may be written as

$$\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_\parallel} \rangle = nm\bar{\nu} \langle B^2 \rangle \left(\frac{\hat{f}_t}{\hat{f}_c} \widehat{U} + \frac{1}{\bar{\nu}} U_0 + \frac{1}{\hat{f}_c} \frac{1}{\bar{\nu}} \sum_1^\infty \frac{\chi_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} \right) . \tag{88}$$

Appendix B

Here, the numerical factors for the fractions of trapped and untrapped particles in the small ϵ approximation are calculated. To recover the static result, the long-time (small ω) limit is considered. For the fraction of untrapped particles we have:

$$\widehat{f}_c\left(\omega \to 0\right) = \sum \frac{\gamma_n}{\kappa_n} = \frac{3\left\langle B^2\right\rangle}{4} \int_0^{\lambda_c} \frac{\lambda' d\lambda'}{\left\langle\sqrt{1-\lambda' B}\right\rangle} \; . \tag{89}$$

Introducing the functional form of the magnetic field given in Eq. (14) the circulating flow fraction f_c is

$$\begin{aligned} f_c &\equiv \frac{3\left\langle B^2\right\rangle}{4} \int_0^{1/B_{\max}} \frac{\lambda' d\lambda'}{\left\langle\sqrt{1-\lambda'\left(B_{\min}+\Delta B\sin^2\frac{\pi l}{L}\right)}\right\rangle} \\ &= \frac{3\left\langle B^2\right\rangle}{4} \int_0^{1/B_{\max}} \frac{\lambda' d\lambda'}{\left\langle\sqrt{\left(1-\lambda' B_{\min}\right)\left(1-\frac{\lambda'\Delta B}{1-\lambda' B_{\min}}\sin^2\frac{\pi l}{L}\right)}\right\rangle} \\ &= \frac{3\left\langle B^2\right\rangle}{2B_{\min}^2}\sqrt{\frac{\Delta B}{B_{\min}}} \int_0^{1} \frac{k^3 dk}{\left(k^2+\Delta B/B_{\min}\right)^{5/2}\left\langle\sqrt{1-k^2\sin^2\frac{\pi l}{L}}\right\rangle} \; , \end{aligned} \tag{90}$$

where the variable k is given by

$$k^2 = \frac{\lambda'\Delta B}{1-\lambda' B_{\min}} \; . \tag{91}$$

The integral in Eq. (90) can be done in the limit $\Delta B/B_{\min} \ll 1$ by expanding in this small parameter

$$\begin{aligned} f_c &\simeq 1 - \frac{3}{2}\sqrt{\Delta B/B_{\min}}\left[1 - \int_0^1 \frac{dk}{k^2}\left(\frac{1}{\left\langle\sqrt{1-k^2\sin^2\frac{\pi l}{L}}\right\rangle} - 1\right)\right] \\ &\simeq 1 - 1.46\sqrt{\Delta B/B_{\min}} \; , \end{aligned} \tag{92}$$

and

$$f_t \simeq \frac{3}{2}\sqrt{\Delta B/B_{\min}}\left[1-\int_0^1 \frac{dk}{k^2}\left(\frac{1}{\left\langle\sqrt{1-k^2\sin^2\frac{\pi l}{L}}\right\rangle}-1\right)\right] \tag{93}$$

$$\simeq 1.46\sqrt{\Delta B/B_{\min}}\ .$$

Appendix C

In this appendix the inverse Laplace transform of the frequency-dependent closure and the parallel and poloidal flows momentum equations are developed in some detail. For the parallel stress, to lowest order in $\widehat{f}_t$,

$$\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle = mn \left\langle B^2 \right\rangle \int \frac{2\bar{\nu}}{v_{th}^2} \frac{v^2}{3} \frac{f_M}{n} L^{-1} \left\{ \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f}_t \widehat{U} \right\} d^3v \ . \tag{94}$$

To invert the term in curly brackets consider $\widehat{f}_t = 1 - \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f}_c$. The calculation is performed below term by term. Starting with the fraction of trapped and circulating particles:

$$f_c = L^{-1} \left\{ \sum \frac{\gamma_n}{\kappa_n - \frac{i\omega}{\bar{\nu}}} \right\} = \sum \nu \gamma_n e^{-\kappa_n \bar{\nu} t} \ , \tag{95}$$

$$f_t = L^{-1} \left\{ 1 - \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f}_c \right\} = -\sum \kappa_n \nu^2 \gamma_n e^{-\kappa_n \bar{\nu} t} + \delta(t) \sum \bar{\nu} \gamma_n \ . \tag{96}$$

Defining the integral $I_n(t) = \int_0^t e^{\kappa_n \bar{\nu} \tau} \frac{\partial U(\tau)}{\partial \tau} d\tau$, the convolution with the parallel flow variable is

$$L^{-1} \left\{ \widehat{f}_t \widehat{U} \right\} = \left[1 - \sum \frac{\gamma_n}{\kappa_n} \right] U(t) - \sum \frac{\gamma_n}{\kappa_n} (\kappa_n - 1) e^{-\kappa_n \bar{\nu} t} \left[U_0 + I_n(t) \right] \ . \tag{97}$$

Finally, the triple convolution can be easily calculated using the previous results as follows

$$\begin{aligned} L^{-1} \left\{ -\frac{i\omega}{\bar{\nu}} \widehat{f}_t \widehat{U} \right\} &= \frac{1}{\bar{\nu}} \frac{\partial}{\partial t} L^{-1} \left\{ \widehat{f}_t \widehat{U} \right\} + \frac{1}{\bar{\nu}} L^{-1} \left\{ L^{-1} \left\{ \widehat{f}_t \widehat{U} \right\} \Big|_{t=0} \right\} \\ &= \left[1 - \sum \gamma_n \right] \frac{1}{\bar{\nu}} \frac{\partial U(t)}{\partial t} + \sum \gamma_n (\kappa_n - 1) e^{-\kappa_n \bar{\nu} t} U_0 \\ &+ \sum \gamma_n (\kappa_n - 1) e^{-\kappa_n \bar{\nu} t} I_n(t) + \frac{1}{\bar{\nu}} U_0 \delta(t) \left[1 - \sum \gamma_n \right] \ . \end{aligned} \tag{98}$$

Then,

$$L^{-1}\left\{\left(1-\frac{i\omega}{\bar{\nu}}\right)\widehat{f}_t\widehat{U}\right\}+\frac{1}{\bar{\nu}}L^{-1}\left\{U_0\right\} = \frac{1}{\bar{\nu}}U_0\delta\left(t\right)\left[2-\sum\gamma_n\right] + \sum\frac{\gamma_n}{\kappa_n}\left(\kappa_n-1\right)^2 e^{-\kappa_n\bar{\nu}t}\left[U_0+I_n\left(t\right)\right] + \left[1-\sum\frac{\gamma_n}{\kappa_n}\right]U\left(t\right)+\left[1-\sum\gamma_n\right]\frac{1}{\bar{\nu}}\frac{\partial U\left(t\right)}{\partial t}\ , \tag{99}$$

and to lowest order in f_t the time-dependent closure is given by

$$\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle = \left\langle B^2\right\rangle mn\left\{\left[1-\sum\frac{\gamma_n}{\kappa_n}\right]\bar{\nu}U\left(t\right)+\left[1-\sum\gamma_n\right]\frac{\partial U\left(t\right)}{\partial t} + \sum\bar{\nu}\frac{\gamma_n}{\kappa_n}\left(\kappa_n-1\right)^2 e^{-\kappa_n\bar{\nu}t}\left[U_0+I_n\left(t\right)\right]+U_0\delta\left(t\right)\left[2-\sum\gamma_n\right]\right\}\ , \tag{100}$$

which is Eq. (52) for $t>0$.

For the time evolution of the parallel flow variable, consider Eq. (59):

$$\left[\left(\bar{\nu}-i\omega\right)\hat{f}_t-i\omega\right]\widehat{U}=\left(1-\frac{i\omega}{\bar{\nu}}\right)\left(1-\hat{f}_t\right)\sum_1^{\infty}\frac{\chi_n}{\kappa_n-\frac{i\omega}{\bar{\nu}}}\ . \tag{101}$$

One could solve for $\widehat{U}$ and then invert the transform numerically. For an analytical result, the infinite sums in the denominators complicate the integrals. Instead, one can take the inverse Laplace transform on both sides and solve for $\widehat{U}$ afterwards. Using the results in Eqs. (95)-(99), the inverse Laplace transform of the right side is

$$RHS = \left(2-\sum\gamma_n\right)\left(\frac{\partial}{\partial t}U\left(t\right)+U\left(0\right)\delta\left(t\right)\right) + \left(\sum\gamma_n\left(\kappa_n-2\right)+1\right)\bar{\nu}U\left(t\right)-\sum\bar{\nu}^2\gamma_n\left(\kappa_n-1\right)^2 e^{-\kappa_n\bar{\nu}t}\int_0^t e^{\kappa_n\bar{\nu}\tau}U\left(\tau\right)d\tau\ , \tag{102}$$

and of the left side

$$LHS = \delta(t)\sum_{m,n}\gamma_n\chi_m - \sum_{m,n}\nu(\chi_m\gamma_n + \chi_n\gamma_m)(\kappa_m - 1)e^{-\kappa_m\bar{\nu}t} + \sum_{m,n}\bar{\nu}^2\chi_m\gamma_n(\kappa_m - 1)(\kappa_n - 1)e^{-\kappa_m\bar{\nu}t}\int_0^t e^{\bar{\nu}\tau(\kappa_m-\kappa_n)}d\tau \ . \tag{103}$$

The double sums show up from the triple convolution involved in calculating the inverse transform for the last term. To eliminate the delta functions, a time integral can be taken on both sides as follows

$$\int_0^t RHS(s)\,ds = \sum_{m,n}\frac{\chi_n\gamma_m}{\kappa_m\kappa_n} + \sum_{m,n}\frac{\kappa_m - 1}{\kappa_m\kappa_n}e^{-\kappa_m\bar{\nu}t}\left[\chi_m\gamma_n\kappa_n + \chi_n\gamma_m\right] - \sum_{m,n}\bar{\nu}\chi_m\gamma_n(\kappa_m - 1)(\kappa_n - 1)\frac{1}{\kappa_m}e^{-\kappa_m\bar{\nu}t}\int_0^t e^{\bar{\nu}\tau(\kappa_m-\kappa_n)}d\tau \ , \tag{104}$$

$$\int_0^t LHS(s)\,ds = \left(1 - \sum\frac{\gamma_n}{\kappa_n}\right)\bar{\nu}\int_0^t U(t)\,dt + \left(2 - \sum\gamma_n\right)U(t) + \sum\bar{\nu}\frac{\gamma_n}{\kappa_n}(\kappa_n - 1)^2 e^{-\kappa_n\bar{\nu}t}\int_0^t e^{\kappa_n\bar{\nu}\tau}U(\tau)\,d\tau \ . \tag{105}$$

Equating both sides and solving for $U(t)$, one gets

$$U(t) = h(t) + \bar{\nu}\int_0^t K(t;\tau)\,U(\tau)\,d\tau \ , \tag{106}$$

where

$$h(t) = \frac{1}{\left(2 - \sum\gamma_m\right)}\left\{\sum_{m,n}\frac{\kappa_m - 1}{\kappa_m\kappa_n}e^{-\kappa_m\bar{\nu}t}\left[\chi_m\gamma_n\kappa_n + \chi_n\gamma_m\right] - \sum_{m,n}\frac{\chi_n\gamma_m}{\kappa_m\kappa_n} + \sum_{m,n}\bar{\nu}\chi_m\gamma_n(\kappa_m - 1)(\kappa_n - 1)\frac{1}{\kappa_m}e^{-\kappa_m\bar{\nu}t}\int_0^t e^{\bar{\nu}\tau(\kappa_m-\kappa_n)}d\tau\right\} \ , \tag{107}$$

and

$$K(t;\tau) = \frac{1}{\left(2 - \sum \gamma_m\right)} \left\{ \sum \frac{\gamma_m}{\kappa_m} - 1 - \sum \frac{\gamma_m}{\kappa_m} \left(\kappa_m - 1\right)^2 e^{-\kappa_m \bar{\nu}(t-\tau)} \right\} . \quad (108)$$

For the toroidal geometry in Section 4, initial conditions were ignored and thus one has

$$\begin{aligned} \left\{ (\bar{\nu} - i\omega)\, \hat{f}_t - i\omega \frac{\langle B_p^2 \rangle}{\langle B^2 \rangle} \left(1 + 2q^2\right) \right\} \widehat{U_\theta} &= \frac{4\pi}{nm} \frac{\langle B_p^2 \rangle}{\langle B^2 \rangle} \langle \phi'(0) - i\omega \widehat{\phi'} \rangle \qquad (109) \\ &- \left\{ 1 + \frac{\langle B_p^2 \rangle}{\langle B^2 \rangle} \left(1 + 2q^2\right) \right\} U_\theta(0) \ . \end{aligned}$$

Defining $I_n(t) = \int_0^t e^{\kappa_n \bar{\nu} \tau} \frac{\partial U_\theta(\tau)}{\partial \tau} d\tau$, the inverse Laplace transform of each term is calculated as before and the time integral is taken to eliminate singularities. So, proceding as before one obtains for the left side

$$\begin{aligned} \int_0^t LHS\, dt &= \langle B_p^2 \rangle \left\{ \left[\left(1 + 2q^2\right) + \frac{\langle B^2 \rangle}{\langle B_p^2 \rangle} \left(1 - \sum \gamma_n\right) \right] U_\theta(t) \right. \qquad (110) \\ &+ \frac{\langle B^2 \rangle}{\langle B_p^2 \rangle} \left[1 - \sum \frac{\gamma_n}{\kappa_n} \right] \bar{\nu} \int_0^t U_\theta(\tau)\, d\tau \\ &\left. + \frac{\langle B^2 \rangle}{\langle B_p^2 \rangle} \bar{\nu} \left[\sum \frac{\gamma_n}{\kappa_n} \left(\kappa_n - 1\right)^2 I_n(t)\, e^{-\kappa_n \bar{\nu} t} \right] \right\} , \end{aligned}$$

and for the right side

$$\int_0^t RHS\, dt = \frac{4\pi}{nm} \langle B_p^2 \rangle \langle \widehat{\phi'} - \phi'(0) \rangle \ . \qquad (111)$$

Finally, equating both sides and rearranging terms, the following evolution equation for the poloidal flow is obtained:

$$\left[\left(1+2q^2\right)+\frac{\langle B^2\rangle}{\langle B_p^2\rangle}\left(1-\sum\gamma_n\right)\right]U_\theta\left(t\right) = \frac{4\pi}{nm}\langle\widehat{\phi}'-\phi'\left(0\right)\rangle \tag{112}$$

$$- \int_0^t \frac{\langle B^2\rangle}{\langle B_p^2\rangle}\left\{1-\sum\frac{\gamma_n}{\kappa_n}\left[1+\left(\kappa_n-1\right)^2 e^{-\kappa_n\bar{\nu}\left(t-\tau\right)}\right]\right\}\bar{\nu}U_\theta\left(\tau\right)d\tau\ , \tag{113}$$

which can be expressed as an integral equation for the poloidal flow:

$$U_\theta\left(t\right)=h\left(t\right)+\bar{\nu}\int_0^t K\left(t;\tau\right)U_\theta\left(t\right)d\tau\ . \tag{114}$$

Here the kernel and inhomogeneous term are defined by

$$h\left(t\right)=\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\frac{4\pi}{nm}\frac{\langle\widehat{\phi}'-\phi'\left(0\right)\rangle}{\left(1-\sum\gamma_n\right)+\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\left(1+2q^2\right)}\ , \tag{115}$$

$$K\left(t;\tau\right)=\frac{-\left\{1-\sum\frac{\gamma_n}{\kappa_n}\left[1+\left(\kappa_n-1\right)^2 e^{-\kappa_n\bar{\nu}\left(t-\tau\right)}\right]\right\}}{\left(1-\sum\gamma_n\right)+\frac{\langle B_p^2\rangle}{\langle B^2\rangle}\left(1+2q^2\right)}\ . \tag{116}$$

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