Role of bumpy fields on single particle orbit in near quasi-helically symmetric stellarators

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Abstract

The role of symmetry breaking on single particle orbits in near helically symmetric stellarators is investigated. In particular, the effect of a symmetry-breaking bumpy term is included in the analysis of trapped particle orbits. It is found that all trapped particle drift orbits are determined by surfaces on which $|B|_{min}$ is constant. Trapped particle orbits reside on these surfaces regardless of pitch angle and are determined solely by the initial position and the shape of the $|B|_{min}$ contour. Since $|B|_{min}$ contours do not depend on the direction of the banana center motion, superbanana orbits do not appear.

Collision-free trapped particles in a perfectly symmetric magnetic confinement system are confined. In contrast, collision-free particles trapped in the helical magnetic fields of a stellarator may be poorly confined because of symmetry breaking in the magnetic configuration. In this work, we investigate the effect of symmetry breaking on quasi-helically symmetric stellarators, and show that banana center motion of helically trapped particles can be described by a simple analytic expression.

Quasi-helically symmetric (QHS) stellarators such as the Helically Symmetric eXperiment(HSX) [1] are designed to have $B/B_{0,0} \simeq 1 - \epsilon_h \cos(m\theta - n\phi)$ where $m \neq 0, n \neq 0$ and $\epsilon_h = B_{m,n}/B_{0,0}$. Here B refers to the strength of magnetic field in Boozer coordinates [2]

$$B = \sum_{mn} B_{mn}(\psi) \cos(m\theta - n\phi)$$

where θ and ϕ are the poloidal and toroidal angles, respectively. The only perfect symmetry that is realizable in a toroidal magnetic confinement system is axisymmetry. Quasi-symmetry refers to the appropriate property that there is a symmetry direction. The expectation is that good confinement of collisonless trapped particles exists when quasi-symmetry is present. Auxiliary coils can be added to spoil the symmetry. In particular, the magnetic field strength of a configuration modified by a "bumpy" field is given by $B/B_0 \simeq 1 - \epsilon_h \cos(m\theta - n\phi) - \epsilon_m \cos n\phi$ where $\epsilon_m = B_{m,n}/B_{0,0}$. The symmetry-breaking bumpy field, $\epsilon_m \cos n\phi$, can cause the direct loss of trapped particles in the low collisionality regime. It is important to note that in both in both the standard quasi-helical case and in the case with bumpy fields, the prominent toroidal curvature term proportional to $\cos\theta$ is absent. In this respect, the configurations studied by QHS stellarators such as HSX are different than most other stellarator experiments.

In previous analytical studies [3, 4], the contours of constant $B_{min}(\psi, \theta)$ in Boozer coordinates have been considered in the description of deeply trapped particle orbits when the radial electric field is negligible. In this note, we show that not only deeply trapped particles but all trapped particles stay on contours of constant $B_{min}(\psi, \theta)$ when the toroidal curvature term is negligible; $B_{1,0}/B_{0,0} \cong 0$. Since particle drift orbits do not depend on the direction of the banana center motion, superbanana orbits do not appear in this limit. When the magnetic well along the contours of constant B_{min} is not deep enough to trap the particles, detrapping occurs. Detrapped particles move along the magnetic field line until they are trapped again in another magnetic well.

For a general case, the magnetic strength B in stellarators is written

$$\frac{B}{B_0} = 1 + \epsilon_t cos\theta + \epsilon_d cosm\theta + \sum_{n=-\infty}^{\infty} \epsilon^{(n)} cos(n\theta + \eta), \tag{1}$$

where ϵ_t, ϵ_d and $\epsilon^{(n)}$ are the amplitudes of the corresponding harmonics. In Eq.(1) $\eta \equiv m\theta - N\phi$, where m and N are the poloidal and toroidal mode numbers of the dominant helical component.

Previous studies have calculated the bounce-averaged drift velocities, and in particular the drift motion of the banana center for the helically trapped particles from derivatives of the second adiabatic invariant $J \equiv \oint mv_\parallel dl$ where m is the particle mass, $v_\parallel$ the parallel velocity and dl the integral along the field lines [5]. To calculate the second adiabatic invariant J, we can simplify Eq.(1) to a single helicity. Without loss of generality, we keep only $n = 0$,± 1, and ± 2 terms. Using the relation $\cos(\pm n\theta + \eta) = \cos n\theta \cos\eta \mp \sin n\theta \sin\eta$, Eq.(1) can be simplified to

$$\frac{B}{B_0} = 1 + \epsilon_T + \epsilon_H \cos(\eta + \chi), \tag{2}$$

where $\epsilon_T = \epsilon_t \cos\theta + \epsilon_d \cos m\theta$ and $\epsilon_H = (C^2 + D^2)^{1/2}$, $C = \epsilon^{(0)} + (\epsilon^{(+1)} + \epsilon^{(-1)})\cos\theta + (\epsilon^{(+2)} + \epsilon^{(-2)})\cos 2\theta$, $D = (\epsilon^{(+1)} - \epsilon^{(-1)})\sin\theta + (\epsilon^{(+2)} - \epsilon^{(-2)})\sin 2\theta$, $\cos\chi = \frac{C}{(C^2+D^2)^{1/2}}$ and $\sin\chi = \frac{D}{(C^2+D^2)^{1/2}}$.

From Eq.(2), the longitudinal invariant of the helically trapped particles can be calculated in the absence of electric fields and given by

$$J \approx 16R/M(m\mu_m B_0 \epsilon_H)^{1/2}[E(\kappa) - (1-\kappa^2)K(\kappa)], \tag{3}$$

where

$$\kappa^2 \equiv \frac{W - \mu_m B_0(1 + \epsilon_T - \epsilon_H)}{2\mu_m B_0 \epsilon_H}, \tag{4}$$

W is the particle kinetic energy, μ_m is the magnetic moment, and $E(\kappa)$ and $K(\kappa)$ are elliptic integrals. Helically trapped particles satisfy $0 \le \kappa^2 \le 1$. To obtain Eq.(3), we have assumed that $N/\iota \gg m$ where ι is the rotational transform.

The bounce-averaged drift velocities for the helically trapped particles are

$$r\dot{\theta} = \frac{1}{qB_0}\frac{\partial J/\partial r}{\partial J/\partial W} = \frac{\mu_m}{q}\left[\frac{\partial \epsilon_H}{\partial r}\left(\frac{2E}{K} - 1\right) - \frac{\partial \epsilon_T}{\partial r}\right] \tag{5}$$

$$r\dot{r} = -\frac{1}{qB_0}\frac{\partial J/\partial \theta}{\partial J/\partial W} = -\frac{\mu_m}{q}\left[\frac{\partial \epsilon_H}{\partial \theta}\left(\frac{2E}{K} - 1\right) - \frac{\partial \epsilon_T}{\partial \theta}\right] \tag{6}$$

in cylindrical-like coordinates($\psi \cong B_0 r^2/2$) where q is the electric charge of the particle and W is the kinetic energy of the particle. Using Eq.(5) and Eq.(6), ϵ_H along the particle motion can be written as

$$\dot{\epsilon}_H(r,\theta) = \frac{dr}{dt}\frac{\partial\epsilon_H}{\partial r} + \frac{d\theta}{dt}\frac{\partial\epsilon_H}{\partial\theta} \quad (7)$$

$$= \frac{\mu_m}{qr}\left(\frac{\partial\epsilon_H}{\partial\theta}\frac{\partial\epsilon_T}{\partial r} - \frac{\partial\epsilon_H}{\partial r}\frac{\partial\epsilon_T}{\partial\theta}\right). \quad (8)$$

Note that if $\epsilon_T = 0$, then $\dot{\epsilon_H} = 0$. Therefore, when $\epsilon_T = 0$, the helically trapped particles stay on curves of constant ϵ_H regardless of the pitch angle. All helically trapped particles in a quasi-helically symmetric magnetic field system move on surfaces, $\dot{\epsilon_H} = 0$, independent of the particle pitch-angle(magnetic moment) as long as toroidal components of the field strength are absent. Since $\epsilon_H(r,\theta)$ provides the spatial variation of the magnetic field, the constant ϵ_H contours correspond to constant B_{min} contours.

We now consider the particular case relevant to HSX. In the following, we use the simple model for the field strength

$$\frac{B}{B_0} = 1 - \epsilon_h \cos(m\theta - n\phi) - \epsilon_m \cos(n\phi), \quad (9)$$

where the radial dependence of ϵ_h is taken to be $\epsilon_h = \epsilon_h^0 (r/a)^m$ for simplicity. We will distinguish between $\epsilon_m < 0$ and $\epsilon_m > 0$ in the following.

The banana center drift motion can be solved by means of the longitudinal adiabatic invariant $J = \oint m v_\parallel dl$ along the banana orbit [6]. Eq. (9) can be reduced to the form [5]:

$$\frac{B}{B_0} = 1 - \epsilon_H \cos(n\phi - \chi), \quad (10)$$

where $\epsilon_H = \left[\epsilon_h^2 + \epsilon_m^2 + 2\epsilon_h\epsilon_m\cos(m\theta)\right]^{1/2}$ and $\chi = \arctan\left(\frac{\epsilon_h \sin(m\theta)}{\epsilon_h\cos(m\theta)+\epsilon_m}\right)$.

Since there is no toroidal component term (ϵ_T), all trapped particles in the magnetic well stay on constant ϵ_H surfaces so long as they are trapped. For the model corresponding to Eq.(9), ϵ_H can be written as

$$\epsilon_H = \frac{\epsilon_h^0}{a^m}\left((r^m \cos m\theta + \frac{a^m\epsilon_m}{\epsilon_h^0})^2 + r^{2m}\sin^2 m\theta\right)^{1/2}. \quad (11)$$

When eq.(11) for m=1 is expressed in cartesian coordinates, it is the equation of circle whose center is displaced from the magnetic axis by $a\epsilon_m/\epsilon_h^0$

$$\frac{a\epsilon_H}{\epsilon_h^0} = \left(x + \frac{a\epsilon_m}{\epsilon_h^0}\right)^2 + y^2. \quad (12)$$

The particle's initial position solely determines what contour the particle resides on. If a contour intersects the boundary, trapped particles on that contour are lost.

The guiding center motion of trapped particles are obtained by solving numerically the particle drift equations to verify the analytic model. The equations are solved by a 4th-order Runge-Kutta method with an error tolerance=1e-6. The collisionless particle orbit is investigated using the guiding center drift equations [2].

$$\dot{\psi} = \frac{\delta}{\gamma}\left(\frac{\partial B}{\partial \phi} I - \frac{\partial B}{\partial \theta} g\right), \tag{13}$$

$$\dot{\theta} = \left(\delta\frac{\partial B}{\partial \psi} + e\frac{\partial \Phi}{\partial \psi}\right)\frac{g}{\gamma} - \frac{e^2 B^2}{m}\rho_c\left(\frac{\rho_c g' - \iota}{\gamma}\right), \tag{14}$$

$$\dot{\phi} = -\frac{I}{\gamma}\left(\delta\frac{\partial B}{\partial \psi} + e\frac{\partial \Phi}{\partial \psi}\right) + \frac{e^2 B^2}{m}\rho_c\left(\frac{\rho_c I' + 1}{\gamma}\right), \tag{15}$$

$$\dot{\rho}_c = \frac{\delta}{\gamma}\left(\frac{\partial B}{\partial \theta}(\rho_c \gamma' - \iota) - \frac{\partial B}{\partial \varphi}(\rho_c I' + 1)\right), \tag{16}$$

where $2\pi\psi$ is the toroidal flux, $2\pi I(\psi)$ is the toroidal current within a flux surface, $2\pi g(\psi)$ is the poloidal current outside a flux surface, $\Phi(\psi)$ is the radial electrostatic potential, $\rho_c = mv_\parallel/eB$, and θ and ϕ are the poloidal and toroidal angles in the Boozer coordinates, respectively. The following quantities have also been defined: $\delta = e^2\rho_c^2 B/m + \mu_m$, $\gamma = e[g(\rho_c I' + 1) - I(\rho_c g' - \iota)]$.

Figure 1 shows the solution of these drift orbit equation for $\epsilon_m > 0$. It demonstrates that trapped particles' orbits do not depend on the pitch angle, only on initial conditions. Figure 2 shows the solution for $\epsilon_m < 0$. The center of the circle is displaced in the opposite direction due to the sign change of ϵ_m.

Figure 3 shows a transition for trapped particles. The banana centers of the particles trapped at the point c of Figure 3(b) move along the contour B_{min} and detrapping occurs at the point a where the magnetic well is not deep enough. The guiding centers of the detrapped particles follow the magnetic field line until the particles are trapped again. Figure 3(c) shows that the magnetic well (point a) is not deep enough, yet neighboring well (point b) is deep enough. When the particles are trapped again, their banana centers move along the contour of constant B_{min}.

In summary, trapped particles in the helical magnetic wells in stellarators with multiple helicity stay on contours of constant B_{min} regardless of the pitch angle when the toroidal curvature components are negligible. For cases relevant to HSX, the contours of constant

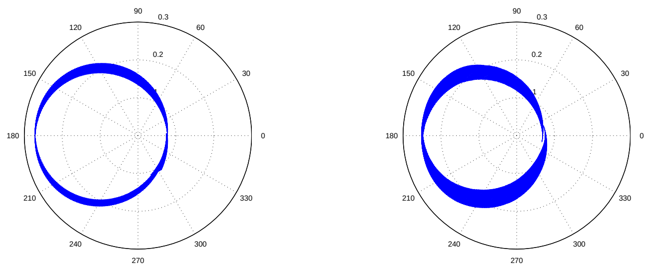

FIG. 1: Drift motion of the banana center with $\epsilon_m > 0$, a=0.15, $\epsilon_m = 0.1$, $\theta = 0$, $\phi = 0$ and $r/a = 0.5$. The particle trajectory is described by a circle whose center is displaced from the magnetic axis. The banana center motion of the trapped particles does not depend on the pitch angle, and is described by the contour given in Eq.(11).

B_{min} are circles whose centers are displaced outward or inward horizontally in the poloidal plasma cross-section depending on the sign of ϵ_m.

The confinement condition for helically trapped particles can be easily understood. The trapped particles that start on contours of B_{min} that do not intersect the boundary are confined. When a contour of constant B_{min} intersects the outer boundary of plasmas, trapped particles are lost unless detrapping happens before particles cross the outer boundary. Since the displacement of the circles is proportional to the magnitude of the mirror bumps, the confinement region gets smaller as the mirror bumps gets larger.

Since the banana centers of trapped particles stay on contours of constant B_{min} regardless of the pitch angle, the banana centers follow the same path even when the poloidal motion changes direction. Therefore, superbanana orbits are not present in these configurations.

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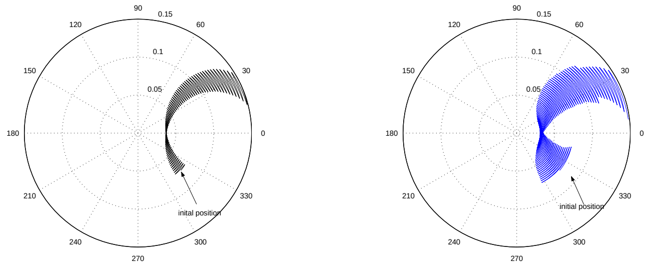
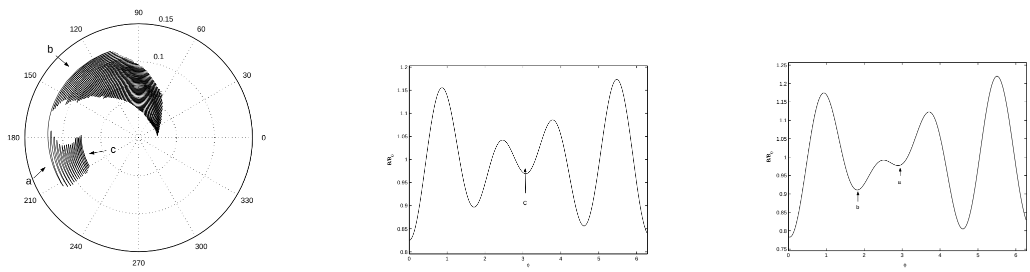
(a) pitch angle= 85°

(b) pitch angle= 70°



(a)pitch angle= 85° (b)pitch angle= 75°

FIG. 2: Drift motion of the banana center with $\epsilon_m < 0$ a=0.15, $\epsilon_m = -0.1$, $\theta = 0$, $\phi = 0$, $r/a = 0.5$ and pitch angle= 85°. The center of the circle is displaced in the opposite direction relative to Fig.1 due to the sign of ϵ_m.



(a)particles orbit in the plasma cross section

(b)magnetic field line along which the particles are just trapped

(c)magnetic field line which passes the point a and the point

FIG. 3: A transition particle orbit that is launched at a=0.15, $\epsilon_m = 0.1$, $\theta = 3.3825$, $\phi = 3.075$ and $r/a = 0.5$, pitch angle=80°. The particle detraps and transitions between two trapped particle trajectories described by Eq.(11).

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