Compressibility effect on magnetic-shear-localized ideal magnetohydrodynamic interchange instability

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Abstract

Eigenmode analysis of a magnetic-shear-localized ideal magnetohydrodynamic (MHD) interchange instability in the presence of plasma compressibility indicates the marginal stability criterion ($D_I = 1/4$) is not affected by the compressibility effects. Above the marginal stability criterion, plasma compressibility reduces the growth rate of ideal interchange instability. Including compressibility effects, robust shear-localized ideal MHD instabilities require $D_I \gtrsim 3/4$, a factor of about 3 above the usual Suydam/Mercier instability criterion ($D_I > 1/4$).

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I. INTRODUCTION

The Suydam criterion [1, 2] defines the stability condition for a localized ideal magnetohydrodynamics (MHD) interchange mode in an incompressible cylindrical plasma. It is given by

$$D_I \leqslant 1/4, \quad \text{for stability.} \tag{1}$$

Here, the driving term for instability is $D_I \equiv -(8\pi p_0'/B_z^2 r_s)(q/q')^2 \equiv \beta L_s^2/r_p R_c$ in which $1/L_s \equiv (1/R_0 q)(rq'/q)$, $\beta/r_p \equiv -8\pi P_0'/B_z^2$ and R_c is the average magnetic field curvature radius. In a toroidal geometry, the stability regime is given by the Mercier criterion [1, 3], again with $D_I \leqslant 1/4$ for stability. For an axisymmetric tokamak in the limit of large aspect ratio, small β and circular cross-section, the form of D_I in the Mercier criterion is $D_I = -(8\pi p_0'/B_\phi^2 r)(q/q')^2(1-q^2)$ [1, 4]. In these derivations, the mode is assumed to be localized arbitrarily close to the mode rational surface. Linear eigenmode analysis of the magnetic-shear-localized ideal MHD interchange instability indicates that the growth rate is exponentially small for $D_I > 1/4$ [5–7]. The criterion of "robust" growth appears to be almost a factor of two greater than the Suydam criterion. The usual non-ideal effects such as Finite Ion Larmor radius, electron diamagnetic flow and resistivity do not alter the higher criterion for robust growth [7].

Another possible effect that can alter the growth rate of the magnetic-shear-localized ideal interchange instability is the compressibility effect *i.e.,* the effect of coupling to sound waves. From the pressure evolution equation, it is clear that the compressibility effects become important only when $\omega^2 \gtrsim k_\parallel^2 C_s^2$ where $k_\parallel = k_y x/L_s$ is the sheared slab model. Thus, in the weak shear region (*i.e.,* large L_s), this condition can be satisfied easily for any arbitrary ω. In this paper, we explore the effect of compressibility on the linear stability properties of the magnetic-shear-localized ideal MHD interchange instability in a sheared slab model. Our analytical analysis shows that the marginal stability condition (*i.e.,* $D_I = 1/4$) is unaffected by the compressional effects. For $D_I > 1/4$, compressibility has a stabilizing effect on the magnetic shear localized ideal interchange instabilities.

The paper is organized a follows. Section II describes the basic model used for studying the compressible interchange instability. In section III, the linear eigenvalue equation is solved analytically using a matched asymptotic analysis. Finally, conclusions are presented in section IV.

II. BASIC MODEL

We consider a sheared slab model where the magnetic field is locally represented by

$$\vec{B} = B_0\left[\left(1+\frac{x}{L_B}\right)\hat{e}_z + \frac{z}{R_c}\hat{e}_x + \frac{x}{L_s}\hat{e}_y\right]. \tag{2}$$

Here $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ are unit vectors along x, y and z, $L_B = (d\ln B/dx)^{-1}$ is the scale-length of the perpendicular gradient of magnetic field strength, $L_s = B(dB_y/dx)^{-1}$ is the magnetic shear scale length and $\kappa = -1/R_c$ is the curvature of the magnetic field. Here, $R_c > 0$ represents good curvature and $R_c < 0$ represents bad curvature. In equilibrium, the radial force balance equation gives $d(P_0 + B^2/8\pi)/dx = B_0B_z/4\pi R_c$, which implies $\beta/r_p = 1/R_c - 1/L_B$. On the other hand, in the direction parallel to the equilibrium magnetic field, pressure is constant, *i.e.,* $\nabla_\parallel P_0 = 0$.

The linearized equations describing the magnetic-shear-localized ideal interchange mode for a compressible plasma in a low β plasma (*i.e.,* $\beta \equiv 8\pi P_0/B^2 \ll 1$) are:

$$\rho_m\frac{c^2}{B^2}\frac{d}{dt}\nabla_\perp^2\tilde{\phi} + \frac{2c}{B^2}\vec{B}\times\vec{\nabla}\tilde{p}\cdot\vec{\kappa} - \left(\vec{B}\cdot\vec{\nabla}\right)\left(\frac{\tilde{J}_\parallel}{B}\right) = 0, \tag{3}$$

$$\tilde{\vec{V}}_\perp = \frac{c}{B^2}\vec{B}\times\vec{\nabla}\tilde{\phi}, \quad \frac{1}{c}\frac{\partial\tilde{\psi}}{\partial t} = -\nabla_\parallel\tilde{\phi}, \tag{4}$$

$$\tilde{J}_\parallel = -\frac{c}{4\pi}\nabla_\perp^2\tilde{\psi}, \quad \frac{\partial\tilde{p}}{\partial t} = -\tilde{V}\cdot\vec{\nabla}P_0 - \Gamma_0P_0\vec{\nabla}\cdot\tilde{V}, \tag{5}$$

$$\rho_m\frac{\partial\tilde{V}_\parallel}{\partial t} = -\nabla_\parallel\tilde{p} + \frac{1}{B^2}\vec{B}\times\vec{\nabla}\tilde{\psi}\cdot\vec{\nabla}P_0, \tag{6}$$

where $\tilde{J}_\parallel \equiv (\vec{B}\cdot\tilde{\vec{J}})/|B|$ is the parallel component of the current perturbation. In a low β plasma, we write $\tilde{\vec{E}}_\perp = -\vec{\nabla}_\perp\tilde{\phi}$ and $\tilde{E}_\parallel = -\nabla_\parallel\tilde{\phi} - (1/c)\partial\tilde{\psi}/\partial t$. The term with coefficient Γ_0 in the pressure equation represents the effect due to plasma compressibility. The term $\vec{\nabla}\cdot\tilde{\vec{V}}_\perp \sim -\tilde{V}_\perp\cdot(\vec{\nabla}\ln B + \vec{\kappa})$ and it represents the perpendicular part of the plasma compressibility.

With perturbations of the form $f(\vec{x},t) = f(x)\exp[i(k_yy - \omega t)]$, $\nabla_\parallel = ik_\parallel(x) = ik_yx/L_s$, the normalized eigenvalue equation becomes

$$\frac{d}{dX}\left[\left(\hat{\omega}^2 - X^2\right)\frac{d\phi}{dX}\right] - \left[\hat{\omega}^2 - X^2 + D_I\left\{1 - 2\Gamma_0\frac{r_p}{R_c}\frac{\hat{\omega}^2}{(\hat{\omega}^2 - \frac{\Gamma_0}{2}\beta X^2)}\right\}\right] = 0 \tag{7}$$

Here, we have defined a normalized distance $X \equiv k_yx$ and frequency $\hat{\omega} \equiv \omega/(V_A/L_s)$, $D_I \equiv \beta L_s^2/(R_cr_p)$, $r_p = |d\ln P_0/dx|^{-1}$ is the equilibrium density scale length, and $\beta \equiv 8\pi P_0/B^2 \equiv$

C_s^2/V_A^2. The terms with coefficient $\hat{\omega}^2$ arise from the divergence of the polarization current, X^2 represents the magnetic field line bending terms brought about by magnetic shear and D_I is the combination of the pressure gradient and curvature effects which represents the source of free energy. The term with coefficient Γ_0 represents the sound wave coupling effect and is important when $\hat{\omega}^2 \sim \beta X^2$.

III. EIGENVALUE ANALYSIS

We now solve the eigenvalue equation analytically using a matched asymptotic analysis method [13]. Here, the equation is divided into three regions: (1) Inner region where the inertial effects dominate; (2) Intermediate region where the compressible effects, *i.e.,* sound effects, plays a dominant role and (3) Outer region, *i.e.,* large X region. This matching procedure is valid for $|\hat{\omega}^2| \ll 1$ and $\Gamma_0\beta \ll 1$. Since the equation is symmetric in X, it is sufficient to solve this equation only for $X \geqslant 0$.

In the inner region, *i.e.,* $|X^2| \sim |\hat{\omega}^2|$, for $\Gamma_0\beta \ll 1$ with $Z = X^2/\hat{\omega}^2$ and $\hat{\omega}^2 \ll 1$, Eq. (7) reduces to

$$Z(1-Z)\frac{d^2\phi}{dZ^2} + \Big(\frac{1}{2} - \frac{3}{2}Z\Big)\frac{d\phi}{dZ} - \frac{D_I}{4}\Big(1 - 2\Gamma_0\frac{r_p}{R_c}\Big)\phi = 0, \tag{8}$$

which is a hypergeometric equation [14] whose general solution is

$$\phi_I = A_I\, F\Big(\frac{1}{4} + \frac{p}{2}, \frac{1}{4} - \frac{p}{2}; \frac{1}{2}; \frac{X^2}{\hat{\omega}^2}\Big) + B_I\sqrt{\frac{X^2}{\hat{\omega}^2}}\, F\Big(\frac{3}{4} + \frac{p}{2}, \frac{3}{4} - \frac{p}{2}; \frac{3}{2}; \frac{X^2}{\hat{\omega}^2}\Big), \tag{9}$$

with

$$p = \sqrt{\frac{1}{4} - D_I\Big(1 - 2\Gamma_0\frac{r_p}{R_c}\Big)}.$$

The choice of the coefficients A_I, B_I depends on the parity condition at $X = 0$. For even modes, the boundary condition at $X = 0$ is $d\phi/dX = 0$, which demands that $B_I = 0$. For odd modes, the boundary condition at $X = 0$ is $\phi = 0$, which ensures that $A_I = 0$. Here, we consider only the even mode solution because it is the most unstable mode.

In the intermediate region, where sound effects become important, $\Gamma_0\beta X^2 \sim \hat{\omega}^2$, Eq. (7) with $Z = \Gamma_0\beta X^2/2$ becomes

$$Z^2\frac{d^2\phi}{dZ^2} + \frac{3}{2}Z\frac{d\phi}{dZ} + \Big[\frac{D_I}{4} - \frac{\Gamma_0}{2}\frac{r_p}{R_c}\frac{D_I}{(1-Z)}\Big]\phi = 0. \tag{10}$$

With $\phi = (-Z)^{-\frac{1}{4}+\frac{p}{2}}F(Z)$, the function $F(Z)$ satisfies a hypergeometric equation and the general solution for ϕ in the intermediate region is given by

$$\phi_{II} = (-\frac{\Gamma_0}{2}\beta X^2)^{-\frac{1}{4}+\frac{p}{2}}\left[A_{II}F\left(a,b;c;\frac{\Gamma_0}{2}\beta X^2\right)+B_{II}\left(\frac{\Gamma_0}{2}\beta X^2\right)^{1-c}F\left(-b,-a;2-c;\frac{\Gamma_0}{2}\beta X^2\right)\right], \tag{11}$$

where $a = (p + i\lambda)/2$, $b = (p - i\lambda)/2$, $c = 1 + p$ and $\lambda = \pm\sqrt{D_I - 1/4}$.

Finally, in the outer region i.e., $|X^2| \gg |\hat{\omega}^2|$, Eq. (7) becomes

$$\frac{d}{dX}\left(X^2\frac{d\phi}{dX}\right) + \left(D_I - X^2\right)\phi = 0. \tag{12}$$

In the limit $D_I \gg X^2$, this equation is the same as that analyzed by Suydam [1, 2] from which he obtained the stability criterion $D_I \leqslant 1/4$. With $Z = 2X$ and $\phi = \hat{\phi}/Z$, Eq. (12) becomes a Whittaker equation [12, 14]. The Whittaker function solution for a growing mode, which decays as $X \to \infty$, is given by

$$\phi_o = A_{III}\frac{\exp(-X)}{\sqrt{2X}}(2X)^{i\lambda}U\left(\frac{1}{2} + i\lambda, 1 + 2i\lambda, 2X\right), \tag{13}$$

where U is Kummer's confluent hypergeometric function.

Next, we match the inner (ϕ_I) and intermediate (ϕ_{II}) solutions in their overlap region. For the outer limit of the inner region solution (*i.e.,* when $|X| \to \infty$), we obtain

$$\phi_I^o \sim \frac{A_I\sqrt{\pi}\Gamma(p)}{[\Gamma(\frac{1}{4}+\frac{p}{2})]^2}\left(-\frac{X^2}{\hat{\omega}^2}\right)^{-\frac{1}{4}+\frac{p}{2}}\left[1 + \frac{\Gamma(-p)}{\Gamma(p)}\left\{\frac{\Gamma(\frac{1}{4}+\frac{p}{2})}{\Gamma(\frac{1}{4}-\frac{p}{2})}\right\}^2\left(-\frac{X^2}{\hat{\omega}^2}\right)^{-p}\right].$$

For the inner limit of the intermediate region solution (*i.e.,* when $|X| \to 0$), we obtain

$$\phi_{II}^i \sim \left(-\frac{\Gamma_0\beta X^2}{2\hat{\omega}^2}\right)^{-\frac{1}{4}+\frac{p}{2}}A_{II}\left[1 + \frac{B_{II}}{A_{II}}\left(\frac{\Gamma_0\beta X^2}{2\hat{\omega}^2}\right)^{-p}\right].$$

On comparing ϕ_I^o with ϕ_{II}^i, we obtain

$$A_{II} = \frac{A_I\sqrt{\pi}\Gamma(p)}{[\Gamma(\frac{1}{4}+\frac{p}{2})]^2}\left(\frac{2}{\Gamma_0\beta}\right)^{-\frac{1}{4}+\frac{p}{2}} \tag{14}$$

and

$$\frac{B_{II}}{A_{II}} = \frac{\Gamma(-p)}{\Gamma(p)}\left\{\frac{\Gamma(\frac{1}{4}+\frac{p}{2})}{\Gamma(\frac{1}{4}-\frac{p}{2})}\right\}^2\left(\frac{2}{\Gamma_0\beta}\right)^{-p}. \tag{15}$$

We next match the intermediate region solution to the outer solution in their overlapping region. The outer limit of the intermediate region solution with coefficient A_{II} and B_{II} [given by Eq.(14) and (15)] is

$$\phi_{II}^o \sim \frac{A_I\sqrt{\pi}\Gamma(p)\Gamma(i\lambda)}{[\Gamma(\frac{1}{4}+\frac{p}{2})]^2}\left(-\frac{X^2}{2\hat{\omega}^2}\right)^{-\frac{1}{4}+i\frac{\lambda}{2}}C_1^\infty\left[1 + \frac{\Gamma(-i\lambda)}{\Gamma(i\lambda)}\frac{C_2^\infty}{C_1^\infty}\left(-\frac{\hat{\omega}^2}{X^2}\right)^{i\lambda}\right]$$

where

$$C_1^{\infty} = \frac{\Gamma(1+p)}{\Gamma(a)\Gamma(1+a)}\left(\frac{\Gamma_0\beta}{2}\right)^{-b} + \frac{\Gamma(-p)\Gamma(1-p)}{\Gamma(p)\Gamma(-b)\Gamma(1-b)}\left\{\frac{\Gamma(\frac{1}{4}+\frac{p}{2})}{\Gamma(\frac{1}{4}-\frac{p}{2})}\right\}^2\left(\frac{\Gamma_0\beta}{2}\right)^{a} \tag{16}$$

and

$$C_2^{\infty} = \frac{\Gamma(1+p)}{\Gamma(b)\Gamma(1+b)}\left(\frac{\Gamma_0\beta}{2}\right)^{-a} + \frac{\Gamma(-p)\Gamma(1-p)}{\Gamma(p)\Gamma(-a)\Gamma(1-a)}\left\{\frac{\Gamma(\frac{1}{4}+\frac{p}{2})}{\Gamma(\frac{1}{4}-\frac{p}{2})}\right\}^2\left(\frac{\Gamma_0\beta}{2}\right)^{b}. \tag{17}$$

Finally, the inner limit of the outer solution ϕ_{III} is

$$\phi_{III}^{i} \sim A_{III}\sqrt{\frac{\pi}{2X}}\left(\frac{1}{2\lambda\Gamma(i\lambda)\sinh\lambda\pi}\right)\left[1+\frac{\Gamma(i\lambda)}{\Gamma(-i\lambda)}\left(\frac{2}{X}\right)^{2i\lambda}\right]\left(\frac{X}{2}\right)^{i\lambda}.$$

On comparing the outer limit of the intermediate region solution with the inner limit of outer solution, we obtain the eigenvalue equation

$$\left(-\frac{\hat{\omega}^2}{4}\right)^{i\lambda/2} = \exp(-i\pi)\frac{\Gamma(1+i\lambda)}{\Gamma(1-i\lambda)}\sqrt{\frac{C_1^{\infty}}{C_2^{\infty}}} \quad , \tag{18}$$

where C_1^{∞} and C_2^{∞} are given by Eqs.(16) and (17) respectively.

A. $p^2 \geqslant 0$ Case.

This corresponds to the case when Suydam's criterion is weakly violated. For $D_I \leq (1/4)(1-\Gamma_0 r_p/2R_c)^{-1}$, the value of the parameter $p = \pm\sqrt{1/4 - D_I(1-2\Gamma_0 r_p/R_c)}$ is real. The coefficient C_1^{∞} is the complex conjugate of C_2^{∞}. Thus, the normalized growth rate $(-\hat{\omega}^2 \equiv \hat{\gamma}^2)$ of the mode is given by

$$\hat{\gamma} = 2\exp\left(-\frac{(\pi - 2\Theta_1 - \Theta_2)}{\mid\lambda\mid}\right), \tag{19}$$

where Θ_1 is the phase of the Gamma function $\Gamma(1+i\lambda)$ and Θ_2 is the phase of the coefficient C_1^{∞}, both depending on $\lambda = \sqrt{D_I - 1/4}$. As $\mid\lambda\mid\rightarrow 0$, coefficients C_1^{∞} and C_2^{∞} become real which results into $\Theta_2 \rightarrow 0$. Also the phase angle $\Theta_1 \rightarrow 0$. Thus, the above expression clearly indicates that marginal stability can be approached from the unstable region when $D_I \rightarrow 1/4 \quad (\mid\lambda\mid\rightarrow 0)$. The normalized growth for $D_I \geqslant 1/4$ is $\hat{\gamma} \approx \exp(-\pi/\mid\lambda\mid) \rightarrow 0$. Thus, the marginal stability condition in the compressible case is same as Suydam criterion in an incompressible plasma.

B. $p^2 < 0$ Case.

This case corresponds to a strong violation of Suydam's criterion. Here, the parameter $p = i\lambda_0$ is purely imaginary and $\lambda_0 = \pm\sqrt{D_I(1-2\Gamma_0 r_p/R_c) - (1/4)}$. In this limit, the coefficient C_2^∞ is given as

$$C_2^\infty = \frac{\Gamma(-i\lambda_0)}{\Gamma(i\lambda_0)} \left\{ \frac{\Gamma(\frac{1}{4} + i\frac{\lambda_0}{2})}{\Gamma(\frac{1}{4} - i\frac{\lambda_0}{2})} \right\}^2 (C_1^\infty)^*.$$

Thus, the dispersion relation can be written as

$$\left(-\frac{\hat{\omega}^2}{4} \right)^{i\lambda/2} = \exp(-i\pi) \frac{\Gamma(1+i\lambda)}{\Gamma(1-i\lambda)} \frac{\Gamma(\frac{1}{4} - i\frac{\lambda_0}{2})}{\Gamma(\frac{1}{4} + i\frac{\lambda_0}{2})} \left\{ \frac{\Gamma(i\lambda_0)}{\Gamma(-i\lambda_0)} \frac{C_1^\infty}{(C_1^\infty)^*} \right\}^{1/2}. \quad (20)$$

This gives a purely growing mode with normalized growth rate ($\hat{\gamma}^2 \equiv -\hat{\omega}^2$)

$$\hat{\gamma} = 2 \exp\left[-\frac{1}{|\lambda|} \left(\pi - 2\Theta_1 - \Theta_2 - 2\Theta_3 - \Theta_4 \right) \right], \quad (21)$$

where Θ_1 is the phase of the Gamma function $\Gamma(1+i\lambda)$, Θ_2 is the phase of the coefficient C_1^∞, Θ_3 is the phase of the Gamma function $\Gamma(1/4 - i\lambda_0/2)$ and Θ_4 is the phase of the Gamma function $\Gamma(i\lambda_0)$. All are functions of λ and λ_0.

The growth rate in the case of an incompressible ideal interchange instability can be easily derived from above equation. With $\Gamma_0\beta \to 0$, the parameter $\lambda_0 \to \lambda$. Hence, the coefficient $C_1^\infty \to 1/\Gamma(i\lambda)$ and the dispersion relation is given as

$$\left(-\frac{\hat{\omega}_{inc}^2}{4} \right)^{i\lambda/2} = exp(-i\pi) \frac{\Gamma(1+i\lambda)}{\Gamma(1-i\lambda)} \frac{\Gamma(\frac{1}{4} - i\frac{\lambda}{2})}{\Gamma(\frac{1}{4} + i\frac{\lambda}{2})} \quad , \quad (22)$$

which yields the normalized growth rate ($\hat{\gamma}_{inc} = -\hat{\omega}_{inc}^2$)

$$\hat{\gamma}_{inc} = 2 \exp\left[-\frac{(\pi - 2\Theta_1 - 2\Theta_5)}{|\lambda|} \right] \quad (23)$$

where Θ_5 is the phase of the Gamma function $\Gamma(1/4 - i\lambda/2)$).

Finally, we solve the full dispersion relation Eq.(18) numerically for different value of D_I assuming $\Gamma_0\beta_0 = 0.01$. The normalized growth rate of an ideal interchange instability for both the incompressible and compressible cases are shown in Figure 1. The figure clearly indicates that compressibility has a stabilizing effect on the interchange instability. It also indicates that near $D_I = 1/4$ the growth rate is similar to the one obtained in the

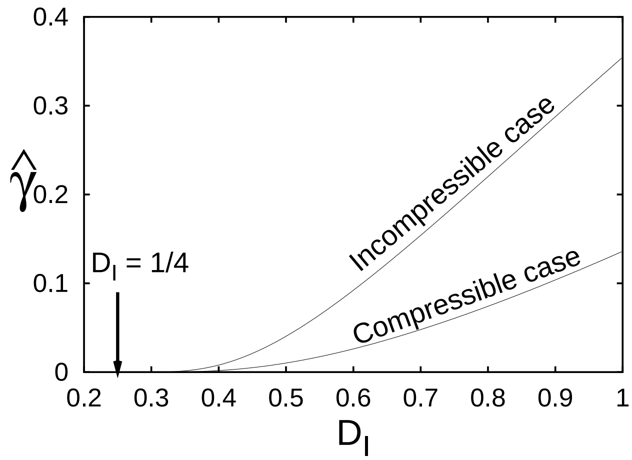

FIG. 1: Normalized Growth rate $\hat{\gamma}$ versus D_I shows the stabilizing effect of compressibility.

incompressible case. Also taking the (arbitrary) criterion for robust ideal MHD growth to be $\hat{\gamma} > 0.05$, we see that $D_I \gtrsim 3/4$ is required. Thus, including compressibility effects the criterion for robust growth is about a factor of 3 larger than the Suydam criterion $D_I > 1/4$.

IV. CONCLUSION

In this paper, we have investigated the effect of plasma compressibility on the linear eigenmode and growth rate for a magnetic-shear-localized ideal interchange instability in a simple shear slab geometry. A linear eigenvalue equation was derived and solved using a matched asymptotic analysis. The equation is solved in three subregions: (1) inner region where inertial effects are dominant, (2) intermediate region, where the sound effects are dominant and (3) large $|X|$ region where the dynamics is governed by the pressure drive and field line bending terms. A marginal stability condition at $D_I = 1/4$ is found consistent with the conventional Suydam analysis. The effect of compressibility does not alter the marginal stability condition . However, when the instability boundary is crossed, compressibility reduces the amplitude of ideal MHD interchange instability growth rate. Including compressibility effects, robust ideal MHD interchange instabilities ($\hat{\gamma} > 0.05$) in a sheared magnetic field require $D_I \gtrsim 3/4$, a factor of about 3 above the usual Suydam/Mercier criterion.

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