Compressibility effect on magnetic-shear-localized ideal magnetohydrodynamic interchange instability

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Abstract

Eigenmode analysis of a magnetic-shear-localized ideal magnetohydrodynamic (MHD) interchange instability in the presence of plasma compressibility indicates the marginal stability criterion \(D_I = 1/4\) is not affected by the compressibility effects. Above the marginal stability criterion, plasma compressibility reduces the growth rate of ideal interchange instability. Including compressibility effects, robust shear-localized ideal MHD instabilities require \(D_I \gtrsim 3/4\), a factor of about 3 above the usual Suydam/Mercier instability criterion \((D_I > 1/4)\).

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I. INTRODUCTION

The Suydam criterion [1, 2] defines the stability condition for a localized ideal magnetohydrodynamics (MHD) interchange mode in an incompressible cylindrical plasma. It is given by

\[
D_I \leq 1/4, \quad \text{for stability.} (1)
\]

Here, the driving term for instability is

\[
D_I \equiv -\left(8\pi p_0'/B_z^2 r_s\right)(q/q')^2 \equiv \beta L_s^2/r_p R_c
\]

in which

\[
1/L_s \equiv (1/R_0 q)(r q'/q), \quad \beta/r_p \equiv -8\pi p_0'/B_z^2 \quad \text{and} \quad R_c \text{ is the average magnetic field curvature radius.}
\]

In a toroidal geometry, the stability regime is given by the Mercier criterion [1, 3], again with \( D_I \leq 1/4 \) for stability. For an axisymmetric tokamak in the limit of large aspect ratio, small \( \beta \) and circular cross-section, the form of \( D_I \) in the Mercier criterion is

\[
D_I = -\left(8\pi p_0'/B_0^2 r\right)(q/q')^2 (1 - q^2) \quad [1, 4].
\]

In these derivations, the mode is assumed to be localized arbitrarily close to the mode rational surface. Linear eigenmode analysis of the magnetic-shear-localized ideal MHD interchange instability indicates that the growth rate is exponentially small for \( D_I > 1/4 \) [5–7]. The criterion of “robust” growth appears to be almost a factor of two greater than the Suydam criterion. The usual non-ideal effects such as Finite Ion Larmor radius, electron diamagnetic flow and resistivity do not alter the higher criterion for robust growth [7].

Another possible effect that can alter the growth rate of the magnetic-shear-localized ideal interchange instability is the compressibility effect \( i.e., \) the effect of coupling to sound waves. From the pressure evolution equation, it is clear that the compressibility effects become important only when \( \omega^2 \gtrsim k_\parallel^2 C_s^2 \) where \( k_\parallel = k_y x / L_s \) is the sheared slab model. Thus, in the weak shear region \( i.e., \) large \( L_s \), this condition can be satisfied easily for any arbitrary \( \omega \). In this paper, we explore the effect of compressibility on the linear stability properties of the magnetic-shear-localized ideal MHD interchange instability in a sheared slab model. Our analytical analysis shows that the marginal stability condition \( i.e., \) \( D_I = 1/4 \) is unaffected by the compressional effects. For \( D_I > 1/4 \), compressibility has a stabilizing effect on the magnetic shear localized ideal interchange instabilities.

The paper is organized as follows. Section II describes the basic model used for studying the compressible interchange instability. In section III, the linear eigenvalue equation is solved analytically using a matched asymptotic analysis. Finally, conclusions are presented in section IV.
II. BASIC MODEL

We consider a sheared slab model where the magnetic field is locally represented by

$$\vec{B} = B_0 \left[ 1 + \frac{x}{L_B} \right] \hat{e}_z + \frac{z}{R_e} \hat{e}_x + \frac{x}{L_s} \hat{e}_y. \quad (2)$$

Here $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ are unit vectors along $x$, $y$ and $z$, $L_B = (d \ln B/dx)^{-1}$ is the scale-length of the perpendicular gradient of magnetic field strength, $L_s = B(dB_y/dx)^{-1}$ is the magnetic shear scale length and $\kappa = -1/R_e$ is the curvature of the magnetic field. Here, $R_c > 0$ represents good curvature and $R_c < 0$ represents bad curvature. In equilibrium, the radial force balance equation gives $d(P_0 + B^2/8\pi)/dx = B_0 B_z/4\pi R_c$, which implies $\beta/r_p = 1/R_c - 1/L_B$. On the other hand, in the direction parallel to the equilibrium magnetic field, pressure is constant, i.e., $\nabla \parallel P_0 = 0$.

The linearized equations describing the magnetic-shear-localized ideal interchange mode for a compressible plasma in a low $\beta$ plasma (i.e., $\beta \equiv 8\pi P_0/B^2 < 1$) are:

$$\rho_m \frac{c^2}{B^2} \frac{d}{dt} \nabla_\perp \tilde{\phi} + \frac{2c}{B^2} \vec{B} \times \nabla \tilde{\rho} \cdot \tilde{\kappa} - (\vec{B} \cdot \nabla) \left( \frac{\tilde{J}_\parallel}{B} \right) = 0, \quad (3)$$

$$\vec{V}_\perp = \frac{c}{B^2} \vec{B} \times \nabla \tilde{\phi}, \quad \frac{1}{c} \frac{\partial \tilde{\psi}}{\partial t} = -\nabla || \tilde{\phi}, \quad (4)$$

$$\tilde{J}_\parallel = -\frac{c}{4\pi} \nabla_\perp \tilde{\psi}, \quad \frac{\partial \tilde{p}}{\partial t} = -\vec{V} \cdot \nabla \tilde{P}_0 - \Gamma_0 \tilde{P}_0 \nabla \cdot \vec{V}, \quad (5)$$

$$\rho_m \frac{\partial \tilde{V}_\parallel}{\partial t} = -\nabla || \tilde{\rho} + \frac{1}{B^2} \vec{B} \times \nabla \tilde{\psi} \cdot \nabla \tilde{P}_0, \quad (6)$$

where $\tilde{J}_\parallel \equiv (\vec{B} \cdot \tilde{J})/|\vec{B}|$ is the parallel component of the current perturbation. In a low $\beta$ plasma, we write $\vec{E}_\perp = -\nabla_\perp \tilde{\phi}$ and $\vec{E}_\parallel = -\nabla || \tilde{\phi} - (1/c) \partial \tilde{\psi}/\partial t$. The term with coefficient $\Gamma_0$ in the pressure equation represents the effect due to plasma compressibility. The term $\nabla \cdot \vec{V}_\perp \sim -\vec{V}_\perp \cdot (\nabla \ln B + \kappa)$ and it represents the perpendicular part of the plasma compressibility.

With perturbations of the form $f(\vec{x}, t) = f(x) \exp[i(k_y y - \omega t)]$, $\nabla || = ik || (x) = ik_y x/L_s$, the normalized eigenvalue equation becomes

$$\frac{d}{dX} \left[ (\omega^2 - X^2) \frac{d\phi}{dX} \right] - \left[ \frac{\omega^2 - X^2}{X^2} + D_l \left( 1 - 2\Gamma_0 \frac{r_p}{R_e} \frac{\omega^2}{\kappa^2} \beta X^2 \right) \right] = 0 \quad (7)$$

Here, we have defined a normalized distance $X \equiv k_y x$ and frequency $\tilde{\omega} \equiv \omega/(V_A/L_s)$, $D_l \equiv \beta L_s^2/(R_e r_p)$, $r_p = |d \ln P_0/dx|^{-1}$ is the equilibrium density scale length, and $\beta \equiv 8\pi P_0/B^2 \equiv \ldots$
\( C_s^2/V_A^2 \). The terms with coefficient \( \hat{\omega}^2 \) arise from the divergence of the polarization current, \( X^2 \) represents the magnetic field line bending terms brought about by magnetic shear and \( D_I \) is the combination of the pressure gradient and curvature effects which represents the source of free energy. The term with coefficient \( \Gamma_0 \) represents the sound wave coupling effect and is important when \( \hat{\omega}^2 \sim \beta X^2 \).

### III. EIGENVALUE ANALYSIS

We now solve the eigenvalue equation analytically using a matched asymptotic analysis method [13]. Here, the equation is divided into three regions: (1) Inner region where the inertial effects dominate; (2) Intermediate region where the compressible effects, \( i.e. \), sound effects, plays a dominant role and (3) Outer region, \( i.e. \), large \( X \) region. This matching procedure is valid for \( |\hat{\omega}| \ll 1 \) and \( \Gamma_0 \beta \ll 1 \). Since the equation is symmetric in \( X \), it is sufficient to solve this equation only for \( X \geq 0 \).

In the inner region, \( i.e., \ |X^2| \sim |\hat{\omega}^2| \), for \( \Gamma_0 \beta \ll 1 \) with \( Z = X^2/\hat{\omega}^2 \) and \( \hat{\omega}^2 \ll 1 \), Eq. (7) reduces to

\[
Z(1 - Z) \frac{d^2 \phi}{dZ^2} + \left( \frac{1}{2} - \frac{3}{2} Z \right) \frac{d\phi}{dZ} - \frac{D_I}{4} \left( 1 - 2\Gamma_0 \frac{r_p}{R_e} \right) \phi = 0, \tag{8}
\]

which is a hypergeometric equation [14] whose general solution is

\[
\phi_I = A_I F\left( \frac{1}{4} + \frac{p}{2}, \frac{1}{4} - \frac{p}{2}; \frac{1}{2}; \frac{X^2}{\hat{\omega}^2} \right) + B_I \sqrt{X^2/\hat{\omega}^2} F\left( \frac{3}{4} + \frac{p}{2}, \frac{3}{4} - \frac{p}{2}; \frac{3}{2}; \frac{X^2}{\hat{\omega}^2} \right), \tag{9}
\]

with

\[
p = \sqrt{\frac{1}{4} - D_I \left( 1 - 2\Gamma_0 \frac{r_p}{R_e} \right)}. \]

The choice of the coefficients \( A_I, B_I \) depends on the parity condition at \( X = 0 \). For even modes, the boundary condition at \( X = 0 \) is \( d\phi/dX = 0 \), which demands that \( B_I = 0 \). For odd modes, the boundary condition at \( X = 0 \) is \( \phi = 0 \), which ensures that \( A_I = 0 \). Here, we consider only the even mode solution because it is the most unstable mode.

In the intermediate region, where sound effects become important, \( \Gamma_0 \beta X^2 \sim \hat{\omega}^2 \), Eq. (7) with \( Z = \Gamma_0 \beta X^2/2 \) becomes

\[
Z^2 \frac{d^2 \phi}{dZ^2} + \frac{3}{2} Z \frac{d\phi}{dZ} + \left( \frac{D_I}{4} - \frac{\Gamma_0}{2} \frac{r_p}{R_e} \frac{D_I}{1 - Z} \right) \phi = 0. \tag{10}
\]
With \( \phi = (-Z)^{-\frac{1}{4} + \frac{\beta}{2}} F(Z) \), the function \( F(Z) \) satisfies a hypergeometric equation and the general solution for \( \phi \) in the intermediate region is given by

\[
\phi_{II} = (-\frac{\Gamma_0}{2} \beta X^2)^{-\frac{1}{4} + \frac{\beta}{2}} \left[ A_{II} F(a, b; c; \frac{\Gamma_0}{2} \beta X^2) + B_{II} \left( \frac{\Gamma_0}{2} \beta X^2 \right)^{1-c} F(-b, -a; 2-c; \frac{\Gamma_0}{2} \beta X^2) \right],
\]

(11)

where \( a = (p + i\lambda)/2, b = (p - i\lambda)/2, c = 1 + p \) and \( \lambda = \pm \sqrt{D_I - 1/4} \).

Finally, in the outer region i.e., \(|X^2| \gg |\omega^2|\), Eq. (7) becomes

\[
\frac{d}{dX} \left( X^2 \frac{d\phi}{dX} \right) + (D_I - X^2) \phi = 0.
\]

(12)

In the limit \( D_I \gg X^2 \), this equation is the same as that analyzed by Suydam [1, 2] from which he obtained the stability criterion \( D_I \leq 1/4 \). With \( Z = 2X \) and \( \phi = \hat{\phi}/Z \), Eq. (12) becomes a Whittaker equation [12, 14]. The Whittaker function solution for a growing mode, which decays as \( X \to \infty \), is given by

\[
\phi_o = A_{II} \frac{\exp(-X)}{\sqrt{2X}} (2X)^{i\lambda} U \left( \frac{1}{2} + i\lambda, 1 + 2i\lambda, 2X \right),
\]

(13)

where \( U \) is Kummer’s confluent hypergeometric function.

Next, we match the inner (\( \phi_I \)) and intermediate (\( \phi_{II} \)) solutions in their overlap region. For the outer limit of the inner region solution (i.e., when \(|X| \to \infty \)), we obtain

\[
\phi_I^o \sim \frac{A_I \sqrt{\pi} \Gamma(p)}{[\Gamma(\frac{1}{4} + \frac{p}{2})]^2} \left( -\frac{X^2}{\omega^2} \right)^{-\frac{1}{4} + \frac{\beta}{2}} \left[ 1 + \frac{\Gamma(-p)}{\Gamma(p)} \left( \frac{\Gamma(\frac{1}{4} + \frac{p}{2})}{\Gamma(\frac{1}{4} - \frac{p}{2})} \right)^2 \left( -\frac{X^2}{\omega^2} \right)^{-p} \right].
\]

For the inner limit of the intermediate region solution (i.e., when \(|X| \to 0\)), we obtain

\[
\phi_{II}^i \sim \left( -\frac{\Gamma_0 \beta X^2}{2 \omega^2} \right)^{-\frac{1}{4} + \frac{\beta}{2}} A_{II} \left[ 1 + \frac{B_{II}}{A_{II}} \left( \frac{\Gamma_0 \beta X^2}{2 \omega^2} \right)^{-p} \right].
\]

On comparing \( \phi_I^o \) with \( \phi_{II}^i \), we obtain

\[
A_{II} = \frac{A_I \sqrt{\pi} \Gamma(p)}{[\Gamma(\frac{1}{4} + \frac{p}{2})]^2} \left( \frac{2}{\Gamma_0 \beta} \right)^{-\frac{1}{4} + \frac{\beta}{2}}
\]

(14)

and

\[
\frac{B_{II}}{A_{II}} = \frac{\Gamma(-p)}{\Gamma(p)} \left( \frac{\Gamma(\frac{1}{4} + \frac{p}{2})}{\Gamma(\frac{1}{4} - \frac{p}{2})} \right)^2 \left( \frac{2}{\Gamma_0 \beta} \right)^{-p}.
\]

(15)

We next match the intermediate region solution to the outer solution in their overlapping region. The outer limit of the intermediate region solution with coefficient \( A_{II} \) and \( B_{II} \) [given by Eq.(14) and (15)] is

\[
\phi_{II}^o \sim \frac{A_I \sqrt{\pi} \Gamma(p) \Gamma(i\lambda)}{[\Gamma(\frac{1}{4} + \frac{p}{2})]^2} \left( -\frac{X^2}{2 \omega^2} \right)^{-\frac{1}{4} + \frac{i\beta}{2}} C_1^\infty \left[ 1 + \frac{\Gamma(-i\lambda) C_1^\infty}{\Gamma(i\lambda) C_1^\infty} \left( -\frac{\omega^2}{X^2} \right)^{i\lambda} \right]
\]

\[
\phi_{II}^o \sim \frac{A_I \sqrt{\pi} \Gamma(p) \Gamma(i\lambda)}{[\Gamma(\frac{1}{4} + \frac{p}{2})]^2} \left( -\frac{X^2}{2 \omega^2} \right)^{-\frac{1}{4} + \frac{i\beta}{2}} C_1^\infty \left[ 1 + \frac{\Gamma(-i\lambda) C_1^\infty}{\Gamma(i\lambda) C_1^\infty} \left( -\frac{\omega^2}{X^2} \right)^{i\lambda} \right]
\]

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where
\[ C_1^\infty = \frac{\Gamma(1 + p)}{\Gamma(a)\Gamma(1 + a)} \left( \frac{\Gamma_0 \beta}{2} \right)^{-b} + \frac{\Gamma(-p)\Gamma(1 - p)}{\Gamma(p)\Gamma(-b)\Gamma(1 - b)} \left\{ \frac{\Gamma \left( \frac{1}{4} + \frac{b}{2} \right)}{\Gamma \left( \frac{1}{4} - \frac{b}{2} \right)} \right\}^2 \left( \frac{\Gamma_0 \beta}{2} \right)^a \tag{16} \]
and
\[ C_2^\infty = \frac{\Gamma(1 + p)}{\Gamma(b)\Gamma(1 + b)} \left( \frac{\Gamma_0 \beta}{2} \right)^{-a} + \frac{\Gamma(-p)\Gamma(1 - p)}{\Gamma(p)\Gamma(-a)\Gamma(1 - a)} \left\{ \frac{\Gamma \left( \frac{1}{4} + \frac{a}{2} \right)}{\Gamma \left( \frac{1}{4} - \frac{a}{2} \right)} \right\}^2 \left( \frac{\Gamma_0 \beta}{2} \right)^b. \tag{17} \]
Finally, the inner limit of the outer solution \( \phi_{III} \) is
\[ \phi_{III}^i \sim A_{III} \sqrt{\pi} \frac{1}{2X} \left( \frac{1}{2\lambda \Gamma(i\lambda) \sinh \lambda \pi} \right) \left[ 1 + \frac{\Gamma(i\lambda)}{\Gamma(-i\lambda)} \left( \frac{2}{X} \right)^{2i\lambda} \right] \left( \frac{X}{2} \right)^{i\lambda}. \]
On comparing the outer limit of the intermediate region solution with the inner limit of outer solution, we obtain the eigenvalue equation
\[ \left( -\frac{\hat{\omega}^2}{4} \right)^{i\lambda/2} = \exp(-i\pi) \frac{\Gamma(1 + i\lambda)}{\Gamma(1 - i\lambda)} \sqrt{\frac{C_1^\infty}{C_2^\infty}}, \tag{18} \]
where \( C_1^\infty \) and \( C_2^\infty \) are given by Eqs.(16) and (17) respectively.

A. \( p^2 \geq 0 \) Case.

This corresponds to the case when Suydam’s criterion is weakly violated. For \( D_I \leq (1/4)(1 - \Gamma_0 \tau_p/2R_c)^{-1} \), the value of the parameter \( p = \pm \sqrt{1/4 - D_I(1 - 2\Gamma_0 \tau_p/R_c)} \) is real. The coefficient \( C_1^\infty \) is the complex conjugate of \( C_2^\infty \). Thus, the normalized growth rate \( -\hat{\omega}^2 \equiv \hat{\gamma}^2 \) of the mode is given by
\[ \hat{\gamma} = 2 \exp \left( -\frac{(\pi - 2\Theta_1 - \Theta_2)}{\lambda} \right), \tag{19} \]
where \( \Theta_1 \) is the phase of the Gamma function \( \Gamma(1 + i\lambda) \) and \( \Theta_2 \) is the phase of the coefficient \( C_1^\infty \), both depending on \( \lambda = \sqrt{D_I - 1/4} \). As \( | \lambda | \to 0 \), coefficients \( C_1^\infty \) and \( C_2^\infty \) become real which results into \( \Theta_2 \to 0 \). Also the phase angle \( \Theta_1 \to 0 \). Thus, the above expression clearly indicates that marginal stability can be approached from the unstable region when \( D_I \to 1/4 \) (\( | \lambda | \to 0 \)). The normalized growth for \( D_I \geq 1/4 \) is \( \hat{\gamma} \approx \exp(-\pi/ | \lambda |) \to 0 \). Thus, the marginal stability condition in the compressible case is same as Suydam criterion in an incompressible plasma.
B. \( p^2 < 0 \) Case.

This case corresponds to a strong violation of Suydam’s criterion. Here, the parameter \( p = i\lambda_0 \) is purely imaginary and \( \lambda_0 = \pm \sqrt{D_I(1 - 2\Gamma_0 r_p/R_c) - (1/4)} \). In this limit, the coefficient \( C_2^\infty \) is given as

\[
C_2^\infty = \frac{\Gamma(-i\lambda_0)}{\Gamma(i\lambda_0)} \left\{ \frac{\Gamma\left(\frac{1}{4} + i\frac{\lambda_0}{2}\right)}{\Gamma\left(\frac{1}{4} - i\frac{\lambda_0}{2}\right)} \right\}^2 (C_1^\infty)^*.
\]

Thus, the dispersion relation can be written as

\[
\left( -\frac{\hat{\omega}^2}{4} \right)^{i\lambda/2} = \exp(-i\pi) \frac{\Gamma(1 + i\lambda) \Gamma\left(\frac{1}{4} - i\frac{\lambda_0}{2}\right)}{\Gamma(1 - i\lambda) \Gamma\left(\frac{1}{4} + i\frac{\lambda_0}{2}\right)} \left\{ \frac{\Gamma(i\lambda_0)}{\Gamma(-i\lambda_0)} \frac{C_1^\infty}{(C_1^\infty)^*} \right\}^{1/2}.
\]

This gives a purely growing mode with normalized growth rate \( \hat{\gamma}^2 \equiv -\hat{\omega}^2 \)

\[
\hat{\gamma} = 2 \exp \left[ -\frac{1}{|\lambda|} \left( \pi - 2\Theta_1 - \Theta_2 - 2\Theta_3 - \Theta_4 \right) \right],
\]

where \( \Theta_1 \) is the phase of the Gamma function \( \Gamma(1 + i\lambda) \), \( \Theta_2 \) is the phase of the coefficient \( C_1^\infty \), \( \Theta_3 \) is the phase of the Gamma function \( \Gamma(1/4 - i\lambda_0/2) \) and \( \Theta_4 \) is the phase of the Gamma function \( \Gamma(i\lambda_0) \). All are functions of \( \lambda \) and \( \lambda_0 \).

The growth rate in the case of an incompressible ideal interchange instability can be easily derived from above equation. With \( \Gamma_0\beta \to 0 \), the parameter \( \lambda_0 \to \lambda \). Hence, the coefficient \( C_1^\infty \to 1/\Gamma(i\lambda) \) and the dispersion relation is given as

\[
\left( -\frac{\hat{\omega}_{inc}^2}{4} \right)^{i\lambda/2} = \exp(-i\pi) \frac{\Gamma(1 + i\lambda) \Gamma\left(\frac{1}{4} - i\frac{\lambda}{2}\right)}{\Gamma(1 - i\lambda) \Gamma\left(\frac{1}{4} + i\frac{\lambda}{2}\right)} ,
\]

which yields the normalized growth rate \( \hat{\gamma}_{inc} = -\hat{\omega}_{inc}^2 \)

\[
\hat{\gamma}_{inc} = 2 \exp \left[ -\frac{(\pi - 2\Theta_1 - 2\Theta_5)}{|\lambda|} \right]
\]

where \( \Theta_5 \) is the phase of the Gamma function \( \Gamma(1/4 - i\lambda/2) \).

Finally, we solve the full dispersion relation Eq.(18) numerically for different value of \( D_I \) assuming \( \Gamma_0\beta_0 = 0.01 \). The normalized growth rate of an ideal interchange instability for both the incompressible and compressible cases are shown in Figure 1. The figure clearly indicates that compressibility has a stabilizing effect on the interchange instability. It also indicates that near \( D_I = 1/4 \) the growth rate is similar to the one obtained in the
incompressible case. Also taking the (arbitrary) criterion for robust ideal MHD growth to be $\hat{\gamma} > 0.05$, we see that $D_I \gtrsim 3/4$ is required. Thus, including compressibility effects the criterion for robust growth is about a factor of 3 larger than the Suydam criterion $D_I > 1/4$.

IV. CONCLUSION

In this paper, we have investigated the effect of plasma compressibility on the linear eigenmode and growth rate for a magnetic-shear-localized ideal interchange instability in a simple shear slab geometry. A linear eigenvalue equation was derived and solved using a matched asymptotic analysis. The equation is solved in three subregions: (1) inner region where inertial effects are dominant, (2) intermediate region, where the sound effects are dominant and (3) large $|X|$ region where the dynamics is governed by the pressure drive and field line bending terms. A marginal stability condition at $D_I = 1/4$ is found consistent with the conventional Suydam analysis. The effect of compressibility does not alter the marginal stability condition. However, when the instability boundary is crossed, compressibility reduces the amplitude of ideal MHD interchange instability growth rate. Including compressibility effects, robust ideal MHD interchange instabilities ($\hat{\gamma} > 0.05$) in a sheared magnetic field require $D_I \gtrsim 3/4$, a factor of about 3 above the usual Suydam/Mercier criterion.
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