

# Time-dependent neoclassical viscosity

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A time-dependent closure for the parallel viscous force is calculated in a bumpy cylinder magnetic field geometry using a Chapman-Enskog-like approach. The calculation is valid for all times and field modulations, and is expressed as a dynamic evolution in time. Two important applications are presented: modification of the frequency-dependent electrical conductivity due to the interaction between trapped and circulating particles, and the parallel flow evolution which can be extended to axisymmetric geometries.

## I. INTRODUCTION

Most present day plasma confinement devices operate in the low collisionality “banana” regime where the relevant closure moment is the parallel stress tensor. The parallel viscous force affects the dynamics of flows and is a drive in the momentum equation. In particular, the electron viscous drag in the parallel Ohm’s law introduces a modification of the electrical conductivity. Also, in the total momentum balance, the parallel flow is damped by this stress term. The equivalent effect in tokamak geometry produces poloidal flow damping.

The closure for the parallel viscous force has been calculated assuming a steady state [1]–[4]. Using the closures obtained in the steady state calculation, the damping rate of the poloidal flow in a toroidal plasma becomes comparable with the ion-ion collision frequency. This conclusion seems to violate the static assumption used to derive the closure [5].

Various authors have addressed the time dependent problem [6]–[10] both by a variational principle and using an expansion in Cordey eigenfunctions [11] of the pitch-angle Coulomb collision scattering operator. Flow damping rates are estimated [6]–[10] to be of the order of the ion collision frequency times some power of the inverse aspect ratio when a large aspect ratio limit is used.

In this work, the dynamic case is approached in a different way. The evolution of the parallel flow is found to be more than a simple exponential and thus cannot be characterized by a single damping rate. Instead of addressing each stage of the evolution separately as a frequency-dependent problem, the main objective here is to describe the *time evolution* of the parallel viscous force. This responds to the need for a complete picture of the transition to the steady state and the determination of the time scale on which an equilibrium assumption is formally valid. The relaxation of the poloidal flow occurs on a fairly fast time scale (about the ion-ion collision time) but is still relevant to some experiments, for instance just after a sawtooth crash [12] and in microturbulence [10].

The procedure presented here is based on a Chapman-Enskog-like approach [13] which is described in Section II. In Section III the drift kinetic equation (DKE) is stated and adapted to a simplified magnetic field geometry. A perturbation technique that provides a solution to the

lowest order DKE and obtains a closure is described in Section IV. Sections V and VI give two relevant applications of the result: the modification for the parallel electrical conductivity and the evolution of the parallel/poloidal flow, respectively. A summary and concluding remarks are included in Section VII. Some details of the Laplace-transform inversions are shown in Appendix A and heat flux effects are considered in Appendix B.

## II. CHAPMAN-ENSKOG-LIKE APPROACH

In a Chapman-Enskog-like approach, the distribution function of a system is assumed to be a dynamic, flow- and heat flow-shifted Maxwellian plus a small (compared with the equilibrium solution) kinetic deviation  $F$ . In most of this work, heat flux effects will be neglected for simplicity (they are discussed in Appendix B); thus, the distribution function for the system will be assumed to be [14]

$$f = f_M + F, \quad (1)$$

where  $f_M$  is a Maxwellian distribution function in the relative velocity  $\mathbf{v} = \mathbf{v}' - \mathbf{V}$ :

$$f_M(\mathbf{v}', \mathbf{x}, t) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left[ -\frac{m(\mathbf{v}' - \mathbf{V})^2}{2kT} \right]. \quad (2)$$

Here  $n(\mathbf{x}, t)$  is the number density,  $\mathbf{V}(\mathbf{x}, t)$  the flow velocity,  $T(\mathbf{x}, t)$  the temperature, and  $m$  the particle mass.

By the Chapman-Enskog hypothesis, the first three moments of the kinetic distortion vanish and the time dependence of the distribution function is given through the variations in the thermodynamic variables in the Maxwellian. Thus, for the present problem, the Chapman-Enskog Ansatz is subject to the constraint conditions

$$\int d^3v F = 0, \quad \int d^3v \mathbf{v} F = 0, \quad \int d^3v \frac{mv^2}{2} F = 0. \quad (3)$$

By including the first three moments of the distribution function in the Maxwellian part, the kinetic distortion does not add terms to the density, momentum and energy balance equations. Neoclassical effects will appear in this formulation through higher order moments

of  $F$ , in particular

$$\Pi = \int d^3v m \left( \mathbf{v}\mathbf{v} - \frac{1}{3}v^2\mathbf{I} \right) F, \quad (4)$$

which gives rise to the neoclassical effects that are of concern in this work. The relevant closure which is the focus of this work is the (flux-surface-averaged) parallel viscous force  $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle$ ; it will be obtained from the solution of the drift kinetic equation as will be shown in the next section.

### III. DRIFT KINETIC EQUATION

For a system of charged particles and independent variables  $\mathbf{v}$ ,  $\mathbf{x}$ , and  $t$ , the total time derivative on the left side of the plasma kinetic equation is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \left[ \frac{q}{m} \left( \mathbf{E} + \frac{1}{c} (\mathbf{V} + \mathbf{v}) \times \mathbf{B} \right) - \frac{\partial \mathbf{V}}{\partial t} - (\mathbf{V} + \mathbf{v}) \cdot \nabla \mathbf{V} \right] \cdot \frac{\partial f}{\partial \mathbf{v}}. \quad (5)$$

The distribution function given by Eqs. (1)–(3) can be introduced in Eq. (5) and using the density, momentum and energy balance equations the total time derivative of the distribution function can be written in terms of  $F$ . Using a model collision operator that separates the effects on  $f_M$  and  $F$ , Wang and Callen [13] recast the kinetic equation in a formal gyro-averaged drift kinetic equation for the kinetic distortion  $F$ . The full equation is not written here since only a simplified version of it will be considered. Including heat flux effects in the calculation is straightforward as shown in Appendix B. Thus, neglecting all heat flux terms and higher (than  $L_1^{1/2}$ ) order moments, Eq. (127) in Reference [13] reduces to

$$\frac{\partial F}{\partial t} + (v_{\parallel} \mathbf{b} + \mathbf{V}) \cdot \nabla F - C(F) = \left[ \left( \frac{v_{\perp}^2}{2} - v_{\parallel}^2 \right) G + \frac{v_{\parallel}}{p} \mathbf{b} \cdot \nabla \cdot \Pi_{\parallel} \right] f_M, \quad (6)$$

where the “flow drive” term is

$$G = \frac{m}{T} \left[ \mathbf{V} \cdot \nabla \ln B - \frac{1}{B} \mathbf{b} \cdot \nabla \times (\mathbf{B} \times \mathbf{V}) + \frac{2}{3} \nabla \cdot \mathbf{V} \right]. \quad (7)$$

The approximate collision operator we consider is

$$C(F) = \bar{\nu} \mathcal{L}(F), \quad (8)$$

in which  $\mathcal{L}$  is the Lorentz scattering operator and  $\bar{\nu} = \nu_{\perp}(v)/2$ . The full collision operator accounts for momentum conservation. Because the calculations are performed in the laboratory frame and from Eq. (3)  $\int d^3v \mathbf{v}F = 0$ , no momentum restoring term is needed. Additionally, since the kinetic distortion  $F$  represents the velocity-space angular effects of the magnetic field modulations, only perpendicular diffusion in velocity space

(pitch-angle scattering) will be considered in the collision operator.

A bumpy cylinder magnetic field model simplifies the calculation of the kinetic distortion  $F$  while retaining the effects of pitch angle scattering of particles in and out of trapped space. The functional form of such a field can be written as

$$B(\ell) = B_{\min} + \Delta B \sin^2 \frac{\pi \ell}{L}, \quad (9)$$

where  $\Delta B = B_{\max} - B_{\min} \equiv 2B_{\min}\epsilon$  and the axial length  $\ell$  follows the field line as it curves. Within the scale length of interest, which is the periodicity length of the cylinder  $L$ , only the magnetic field will be allowed to vary, retaining only the  $\nabla_{\parallel} B$  effects. (There are no radial drifts in an axisymmetric bumpy cylinder magnetic field).

In this simple configuration

$$\nabla \times (\mathbf{B} \times \mathbf{V}) = 0. \quad (10)$$

Hence the second term in the drive  $G$  is absent. For the last term in Eq. (7), a particle continuity equation is used. For bounce time scales we have

$$0 \simeq \frac{\partial n}{\partial t} = -n \nabla \cdot \mathbf{V} - \mathbf{V} \cdot \nabla n, \quad (11)$$

which leads to an incompressible flow to lowest order for density constant along the magnetic field. As a consequence, by noting that

$$\nabla \cdot \mathbf{V} = (\mathbf{B} \cdot \nabla) \left( \frac{V_{\parallel}}{B} \right) = 0, \quad (12)$$

the incompressibility constraint can then be satisfied by defining a parallel flow variable  $V_{\parallel}(\ell, t)/B(\ell) = U(t)$  on a given magnetic flux surface.

Introducing these simplifications in Eq. (7), and using the relation  $v_{\parallel} \mathbf{b} \cdot \nabla (v_{\parallel} B) = (v_{\parallel}^2 - v_{\perp}^2/2) \mathbf{B} \cdot \nabla \ln B$ , the flow drive term reduces to

$$\left( v_{\parallel}^2 - \frac{v_{\perp}^2}{2} \right) G = \frac{m}{T} v_{\parallel} \mathbf{b} \cdot \nabla (v_{\parallel} B) U. \quad (13)$$

The DKE can then be written as

$$\frac{\partial F}{\partial t} + v_{\parallel} \mathbf{b} \cdot \nabla \left( F + \frac{m}{T} v_{\parallel} B U f_M \right) - C(F) = \frac{v_{\parallel}}{p} f_M \mathbf{b} \cdot \nabla \cdot \Pi_{\parallel}. \quad (14)$$

Thus, the dynamic evolution of the kinetic distortion is affected by spatial variations of both the kinetic distortion and the free streaming flow of the circulating particles, and collisions that can drive particles through a perpendicular diffusion process (in velocity space) into or out of the trapped region of velocity space. The parallel viscous force is a source for the evolution of the unknown distribution  $F$  and hence will be present in the solution. However, the Chapman-Enskog constraints given

in Eq. (3) will allow an expression to be obtained for  $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle$  that has no explicit dependence on  $F$ .

Equation (14) is solved by a standard perturbation technique in the low collisionality banana regime where trapped particles can complete their orbits before being scattered by collisions. Then, the parameter  $\nu_* = \nu/\epsilon^{3/2}\omega_b$  is small and will be used as an expansion parameter for  $F$ . In defining  $\nu_*$ ,  $\nu$  is the collision frequency and  $\omega_b$  is the bounce frequency. The kinetic distortion is then expanded as follows

$$F = F_0 + \nu_* F_1 + \dots \quad (15)$$

Assuming the time derivative is of order  $\nu_*$ , the lowest order DKE is

$$v_{\parallel} \mathbf{b} \cdot \nabla \left( F + \frac{m}{T} v_{\parallel} B U f_M \right) = 0. \quad (16)$$

Since for the model magnetic field considered  $\mathbf{b} \cdot \nabla = \partial/\partial \ell$ , the term in square brackets cannot depend on  $\ell$ . Thus, the lowest order distortion is given by

$$F_0 = -\frac{m}{T} v_{\parallel} B U f_M + g(v, \lambda, \varsigma, t). \quad (17)$$

Here,  $g$  is an integration ‘‘constant,’’ which is a function of all the variables of the system other than the length  $\ell$  over which the integration was performed. The first term in Eq. (17) represents the free streaming part of  $F$  while the second term is a collisional correction. The variable  $\varsigma$  depends on the direction in which the particles circulate and is defined as  $\varsigma = v_{\parallel}/|v_{\parallel}|$ .

To next order in  $\nu_*$  the DKE is

$$\frac{\partial F_0}{\partial t} + v_{\parallel} \mathbf{b} \cdot \nabla F_1 - C(F_0) = v_{\parallel} \frac{1}{p} \mathbf{b} \cdot \nabla \cdot \Pi_{\parallel} f_M. \quad (18)$$

Since  $v_{\parallel} \mathbf{b} \cdot \nabla F_1 = v_{\parallel} \partial F_1 / \partial \ell$ , application of the integral

$$\oint \frac{d\ell}{v_{\parallel}} = \begin{cases} \int_0^L \frac{d\ell}{v_{\parallel}}, & \text{untrapped particles,} \\ \sum_{\varsigma} \int_{-\ell_c}^{\ell_c} \frac{d\ell}{|v_{\parallel}|}, & \text{trapped particles,} \end{cases} \quad (19)$$

annihilates the first order distortion in Eq. (18). Here  $\pm \ell_c$  are the turning points of the closed, trapped particle orbits where  $v_{\parallel} \rightarrow 0$ . Thus, it is not necessary to solve for  $F_1$  since the bounce-averaged first order equation provides a constraint that can be solved for the integration constant in the lowest order solution.

For trapped particles, to take account of density conservation at the tip of the bounce orbits we must have  $g_t(\varsigma) = -g_t(-\varsigma)$ . Then, for  $g_t$  even, we have  $g_t = 0$  at  $\ell_c$ . Since  $g$  does not depend on  $\ell$ ,  $g_t = 0$  for all values of  $\ell$ .

For circulating particles, the bounce-averaged complete first order DKE yields a differential equation for  $g_c$ . When a Laplace transform is taken, both energy (speed) and frequency can be treated as parameters. The resulting equation, which can be solved for the pitch angle dependence in  $g_c$ , is

$$-2 \left\{ i\omega \left\langle \frac{B}{v_{\parallel}} \right\rangle + \frac{\bar{v}}{v^2} \frac{\partial}{\partial \lambda} \lambda \langle v_{\parallel} \rangle \frac{\partial}{\partial \lambda} \right\} \hat{g}_c = f_M \hat{S}. \quad (20)$$

Hats denote Laplace-transformed quantities with transform variable  $-i\omega$  and the flux-surface-average is

$$\langle A \rangle \equiv \oint \frac{d\ell}{B} A / \oint \frac{d\ell}{B}. \quad (21)$$

In Eq. (20)  $\lambda = 2\mu/v^2$  is the pitch angle variable and the drives and initial conditions are included in the source term:

$$\begin{aligned} \hat{S}(v, \lambda, \omega) &\equiv \frac{m}{T} \langle B^2 \rangle (\bar{v} - i\omega) \hat{U} - \langle B^2 \rangle \frac{m}{T} U_0 \\ &+ \frac{1}{p} \langle \mathbf{B} \cdot \widehat{\nabla} \cdot \Pi_{\parallel} \rangle - \frac{m}{T} \left\langle \frac{B}{v_{\parallel}} \right\rangle g_0(\lambda), \end{aligned} \quad (22)$$

where  $g_0 \equiv g_c(t=0)$ ,  $U_0 \equiv U(t=0)$ . The speed dependence in  $g_c(\lambda, v, t=0)$  is arbitrary since the collision operator only operates on the pitch angle variable. Thus, it is conveniently chosen as

$$g_0(v, \lambda) \sim \frac{m}{T} f_M(v) g_0(\lambda),$$

to match the rest of the terms in the source [9]. Note that the  $\lambda$  dependence of the source is only through the initial pitch-angle structure of the collisional correction  $g_c$ .

Morris et al. [9] were the first to recognize that this should be treated as an initial value problem. The collisional diffusion into trapped space of a distribution of untrapped particles will depend strongly (at least for early times) on how close the initial distribution is located relative to the boundary between the two types of particles. An initial distribution of particles is expected to damp more rapidly if it is peaked close to  $\lambda_c$  than if particles are introduced far away from the trapped-circulating boundary, since then the portion of phase space they have to diffuse through is larger.

Speed and pitch-angle variables can now be separated using an eigenfunction expansion for the pitch-angle operator on the left side of Eq. (20). The function  $\hat{g}_c$  is projected in Cordey eigenfunctions as was done in references [7]–[9]:

$$\hat{g}_c = \sum_{n=1}^{\infty} Y_n(v, i\omega) \Lambda_n(\lambda). \quad (23)$$

The  $\lambda$ -dependent functions  $\Lambda_n$  are eigenfunctions of the ordinary differential equation

$$\frac{d}{d\lambda} \lambda \left\langle \sqrt{1 - \lambda B} \right\rangle \frac{d\Lambda_n}{d\lambda} = \kappa_n \frac{d}{d\lambda} \left\langle \sqrt{1 - \lambda B} \right\rangle \Lambda_n, \quad (24)$$

where  $\kappa_n$  are the eigenvalues and the orthogonality condition is given by

$$\int \Lambda_n \Lambda_m \frac{\partial \langle \sqrt{1 - \lambda B} \rangle}{\partial \lambda} d\lambda = \delta_{nm} \int \Lambda_n^2 \frac{\partial \langle \sqrt{1 - \lambda B} \rangle}{\partial \lambda} d\lambda, \quad (25)$$

in which the integration is performed over circulating space  $0 < \lambda < \lambda_c$ .

Using equations (24) and (25), one can solve for the speed- and frequency-dependent coefficients. The lowest order solution for the departure from a Maxwellian can then be expressed in frequency space as

$$\widehat{F}_0 = -v_{\parallel} B \frac{m}{T} f_M \widehat{U} + \frac{v_S}{2} f_M \sum_{n=1}^{\infty} \frac{\eta_n}{(\bar{v} \kappa_n - i\omega)} \Lambda_n, \quad (26)$$

where the coefficients  $\eta_n$  are calculated from

$$\eta_n = \frac{\int_0^{\lambda_c} \widehat{S} \Lambda_n d\lambda}{\int_0^{\lambda_c} \Lambda_n^2 \frac{\partial}{\partial \lambda} \langle \sqrt{1 - \lambda B} \rangle d\lambda}. \quad (27)$$

Note that even though these coefficients do not depend on  $\lambda$ , the initial structure of the distribution in pitch angle will enter as a drive for the viscosity through these integrals. The viscous force, being a source in the differential equation for  $F_0$  [Eq. (18)], is part of the solution through the coefficients  $\eta_n$ . In the next section, the Chapman-Enskog constraint that the momentum moment of the small departure from the Maxwellian vanishes will be used to solve for  $\langle \widehat{\mathbf{B}} \cdot \widehat{\nabla} \cdot \Pi_{\parallel} \rangle$ .

#### IV. DYNAMIC CLOSURE FOR $\langle \widehat{\mathbf{B}} \cdot \widehat{\nabla} \cdot \Pi_{\parallel} \rangle$

The Chapman-Enskog Ansatz given in Eq. (3) can be used to express the parallel viscous force in terms of the parallel flow variable  $\widehat{U}$  and the initial conditions. Taking the parallel momentum moment on both sides of Eq. (26) and setting  $\int d^3 v v_{\parallel} \widehat{F}_0 = 0$  yields

$$\langle \widehat{\mathbf{B}} \cdot \widehat{\nabla} \cdot \Pi_{\parallel} \rangle = nm \langle B^2 \rangle \left[ \widehat{U} v(\omega) + U_0 + \Upsilon(\omega) \right]. \quad (28)$$

The first term shows the usual proportionality between the viscous force and the parallel flow. The coefficient  $v(\omega)$  is defined as

$$v(\omega) = \frac{m}{T} \int d^3 v \bar{v} \frac{v^2}{3} \frac{f_M}{n} \frac{\widehat{f}_t(v, \omega)}{\widehat{f}_c(v, \omega)}, \quad (29)$$

and  $\widehat{f}_c$  and  $\widehat{f}_t$  are [8]

$$\widehat{f}_c(\omega, v) = \sum \frac{\gamma_n}{\kappa_n - i\omega/\bar{v}}, \quad (30)$$

$$\widehat{f}_t(\omega, v) = 1 - \left( 1 - \frac{i\omega}{\bar{v}} \right) \widehat{f}_c, \quad (31)$$

where

$$\gamma_n = \frac{3}{4} \frac{\left( \int_0^{\lambda_c} \Lambda_n d\lambda \right)^2}{\int_0^{\lambda_c} \Lambda_n^2 \left[ (\partial/\partial \lambda) \langle \sqrt{1 - \lambda B} \rangle \right] d\lambda}. \quad (32)$$

In Eq. (28) the initial pitch angle distribution is contained in the last term which is defined as

$$\Upsilon(\omega) = \frac{m}{T} \int d^3 v \frac{v^2}{3} \frac{f_M}{n} \frac{1}{\widehat{f}_c} \sum_1^{\infty} \frac{\chi_n}{\kappa_n - i\omega/\bar{v}}, \quad (33)$$

and the coefficient  $\chi_n$  is

$$\chi_n = \frac{\gamma_n}{\int_0^{\lambda_c} \Lambda_n d\lambda} \int_0^{\lambda_c} \left\langle \frac{B}{v_{\parallel}} \right\rangle g_0(\lambda) \Lambda_n d\lambda. \quad (34)$$

Note that for long times, when initial conditions are completely damped and can thus be neglected, one can use the result [8]

$$f_c \equiv \widehat{f}_c(\omega = 0) = \sum \frac{\gamma_n}{\kappa_n} = \frac{3 \langle B^2 \rangle}{4} \int_0^{\lambda_c} \frac{\lambda d\lambda}{\langle \sqrt{1 - \lambda B} \rangle}, \quad (35)$$

and the steady state limit of Eq. (28) yields the standard result

$$\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle = nm \mu \langle B^2 \rangle U. \quad (36)$$

Here, the viscosity coefficient  $\mu = v(0)$  is defined (for electrons) as [15]

$$\mu_e = \frac{m_e}{T_e} \int d^3 v \bar{v} \frac{v^2}{3} \frac{f_M}{n_e} = \left[ Z + \sqrt{2} - \ln(1 + \sqrt{2}) \right] \frac{f_t}{f_c} \nu_e, \quad (37)$$

and  $f_t \equiv 1 - f_c$ .

Equation (28) is valid for any time scale and  $\epsilon$  (or aspect ratio), since no approximations have been introduced so far. From it, a damping rate can be estimated numerically. For an explicit time-dependent expression, neither an analytical nor numerical Laplace inverse transform are trivial to perform since the expression involves infinite sums of terms that depend on integrals over the (numerically generated) eigenfunctions. Moreover, some of these infinite sums reside in the denominator and thus should be calculated to high accuracy if all relevant poles of the response are to be found.

An analytical inversion of the Laplace transform in general is not obtainable. However, for small field variations ( $\Delta B \ll B_{min}$ ) an analytical solution can be obtained through a power series expansion in the small parameter  $\sqrt{\epsilon}$ . In order to make such an expansion, we invoke the  $f_t \sim \sqrt{\epsilon} \ll 1$  result obtained by various authors [4, 8] in both dynamic and static situations. The relevant factors to be inverted are  $\widehat{f}_t/\widehat{f}_c$  and  $1/\widehat{f}_c$ , for which we propose the following expansions:

$$\frac{\widehat{f}_t}{\widehat{f}_c} \simeq \left( 1 - \frac{i\omega}{\bar{v}} \right) \widehat{f}_t + \left( 1 - \frac{i\omega}{\bar{v}} \right) \widehat{f}_t^2 + \dots, \quad (38)$$

$$\frac{1}{\widehat{f}_c} \simeq \left( 1 - \frac{i\omega}{\bar{v}} \right) \left( 1 - \widehat{f}_t \right) + \dots \quad (39)$$

Introducing equations (38) and (39) in the closure given by Eq. (28), a much simpler expression for the parallel

viscous force is obtained and the Laplace transform can be inverted term by term. In particular, to lowest order in  $f_t$  we have

$$\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle \simeq nm \langle B^2 \rangle \times \frac{m}{T} \int d^3v \frac{v^2}{3} \frac{f_M}{n} \bar{\nu} L^{-1} \left\{ \left( 1 - \frac{i\omega}{\bar{\nu}} \right) \hat{f}_t \hat{U} \right\}, \quad (40)$$

in which  $L^{-1}$  is the inverse Laplace transform and initial conditions have been neglected for simplicity but can be easily introduced using Eq. (80). After calculating the inverse Laplace transform of the term in curly brackets in Eq. (40) (see Appendix A), the time dependent closure for the parallel viscous force in this small  $\epsilon$  limit is

$$\begin{aligned} \langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle &= \langle B^2 \rangle mn \int d^3v \frac{v^2}{3} \frac{f_M}{n} \bar{\nu} \times \\ &\frac{m}{T} \left\{ U(t) (1 - f_c) + \frac{1}{\bar{\nu}} \frac{\partial U(t)}{\partial t} \left( 1 - \sum \gamma_n \right) \right. \\ &\left. + \sum \frac{\gamma_n}{\kappa_n} (\kappa_n - 1)^2 \int_0^t \frac{dU}{d\tau} e^{-\bar{\nu} \kappa_n (t-\tau)} d\tau \right\}. \quad (41) \end{aligned}$$

This equation exhibits the explicit behavior in time of  $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle$ . The first term is proportional to the parallel flow and will be dominant in the long time asymptotic limit. The second term contains the time variation of the flow and is important only for times of the order of  $1/\bar{\nu}$ . The last term contains the time-history through the convolution integral of the time-dependent trapped particle fraction and the intrinsic time dependence of the flow of circulating particles. To higher order in this expansion, this last term develops a series in powers of  $\bar{\nu}t$  inside the (convolution) time integral. For estimates of the coefficients in Eq. (41) see Eqs. (51)–(54) below.

## V. ELECTRICAL CONDUCTIVITY

To obtain the modification of the electrical conductivity caused by the parallel viscous force  $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle$ , the bounce-averaged parallel component of the momentum equation for electrons is considered:

$$-i\omega \langle J_{\parallel} B \rangle = \frac{n_e e^2}{m_e} \langle E_{\parallel} B \rangle + \frac{e}{m_e} \langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel e} \rangle - \nu_e \langle J_{\parallel} B \rangle. \quad (42)$$

Introducing the frequency-dependent closure in Eq. (28) yields an  $\omega$ -dependent electrical conductivity:

$$\hat{\sigma}(\omega) = \frac{n_e e^2}{m_e \nu_e} \frac{1}{\alpha_e(\omega)}, \quad (43)$$

where

$$\alpha_e(\omega) = 1 + \frac{1}{\nu_e} [v(\omega) - i\omega], \quad (44)$$

and initial conditions have been neglected.

The dynamic conductivity in Eq. (43) is valid for any frequency (time) and magnetic field modulation. Once again, the analytical process cannot be carried out further. However, a numerical computation could give the frequency dependence and upon taking the inverse Laplace transform, the time evolution of  $\sigma$  to some appropriate accuracy.

To obtain the static limit, we have

$$\alpha_e(0) = 1 + \mu_e / \nu_e. \quad (45)$$

The fraction of trapped and circulating particles can be estimated for  $\sqrt{\epsilon} \ll 1$  as  $f_t \simeq 1.46\sqrt{\epsilon}$  [4]. Considering the value for  $\mu_e$  ( $\simeq 1.53 \nu_e f_t / f_c$  for hydrogenic ions) in the small  $\sqrt{\epsilon}$  approximation, the usual correction to the static electrical conductivity is obtained:

$$\sigma = \frac{n_e e^2 / m_e \nu_e}{1 + \mu_e / \nu_e} = \frac{n_e e^2}{m_e \nu_e} \frac{1}{1 + 2.24 \sqrt{\Delta B / 2 B_{\min}}}. \quad (46)$$

For  $\omega \neq 0$  we consider the low and high frequency limits of

$$\alpha_e(\omega) \simeq 1 + \frac{\hat{f}_t(\omega, \nu_e)}{\hat{f}_c(\omega, \nu_e)} - \frac{i\omega}{\nu_e}, \quad (47)$$

where we are considering  $\bar{\nu} = \nu_e$  for simplicity. Using Eqs. (30)–(31) and the expansion in (38), for  $\omega \ll \nu_e$  we obtain

$$\begin{aligned} \alpha_e(\omega) &\simeq 1 + f_t - \frac{\omega^2}{\nu_e^2} \left( 2 \sum \frac{\gamma_n}{\kappa_n^2} - f_c \right) \\ &- \frac{i\omega}{\nu_e} \left( 2f_t + \sum \frac{\gamma_n}{\kappa_n^2} \right). \quad (48) \end{aligned}$$

Thus, for small  $\omega$  the real part of the conductivity will decay from the static limit as  $-(\omega/\nu_e)^2$  while the imaginary (“reactive”) part grows as  $\omega/\nu_e$ . The sums in Eq. (48) can be estimated using the small  $\epsilon$  approximation for the eigenfunctions in terms of Legendre functions given in Reference [11]. That is, since in the  $\epsilon = 0$  case the eigenfunctions are Legendre polynomials, one can consider for small  $\epsilon$  [11]

$$\Lambda_n \sim P_{\nu_n}, \quad \kappa_n \sim \nu_n (\nu_n + 1), \quad (49)$$

where the index  $\nu_n$  is an integer plus a small correction proportional to  $\sqrt{\epsilon}$ :

$$\nu_n = n + \sqrt{2\epsilon} \frac{4}{\pi} \frac{\Gamma^2(1+n/2)}{\Gamma^2(1/2+n/2)}. \quad (50)$$

In this approximation, the relevant sums can be obtained and are shown below:

$$\sum \gamma_n \simeq 1 - 2.93\epsilon^{3/2} + \mathcal{O}(\epsilon^2), \quad (51)$$

$$\sum \gamma_n \kappa_n \simeq 1 + 1.48\sqrt{\epsilon} + \mathcal{O}(\epsilon), \quad (52)$$

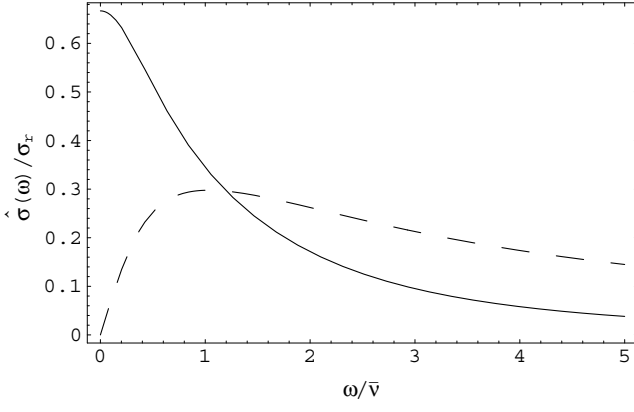


Fig. 1. The real (solid) and imaginary (dashed) parts of  $\hat{\sigma}/\sigma_r$  are plotted against  $\omega/\bar{\nu}$  in the low frequency limit for  $\epsilon = 0.1$ .

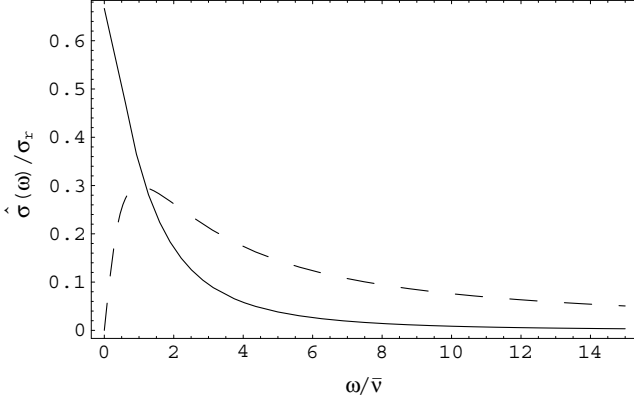


Fig. 2. The asymptotic decay of the real (solid) and imaginary (dashed) parts of  $\hat{\sigma}/\sigma_r$  is plotted against  $\omega/\bar{\nu}$  in the high frequency limit for  $\epsilon = 0.1$ .

$$\sum \frac{\gamma_n}{\kappa_n} \simeq 1 - 1.48\sqrt{\epsilon} + \mathcal{O}(\epsilon), \quad (53)$$

$$\sum \frac{\gamma_n}{\kappa_n^2} \simeq 1 - 2 \times 1.48\sqrt{\epsilon} + \mathcal{O}(\epsilon). \quad (54)$$

Figure 1 shows the frequency-dependence of the electrical conductivity using this expansion.

For high frequencies where  $\omega > \nu_e$ , the  $\omega$ -dependent factor in the conductivity can be expressed as

$$\alpha_e(\omega) \simeq 2 + \sum \gamma_n (\kappa_n - 2) - \frac{\nu_e^2}{\omega^2} \sum \gamma_n \kappa_n - \frac{i\omega}{\nu_e} \left[ 2 - \sum \gamma_n + \frac{\nu_e^2}{\omega^2} \sum \gamma_n (1 - 2\kappa_n) \right]. \quad (55)$$

In this limit, the real part of  $\hat{\sigma}(\omega)$  decreases as  $(\nu_e/\omega)^2$ . The imaginary part also decreases, but at a slower rate ( $\sim \nu_e/\omega$ ), and both asymptote to zero as shown in Fig. 2. Since in Eqs. (48)–(55) a small (or large)  $\omega/\nu_e$  is considered after a small  $\sqrt{\epsilon}$  assumption, what is considered a small (or large) frequency is to be compared with the magnitude of the field modulation. That is, the high frequency asymptotic behavior is expected to be seen only for  $\omega/\nu_e \gg 1/\sqrt{\epsilon}$ .

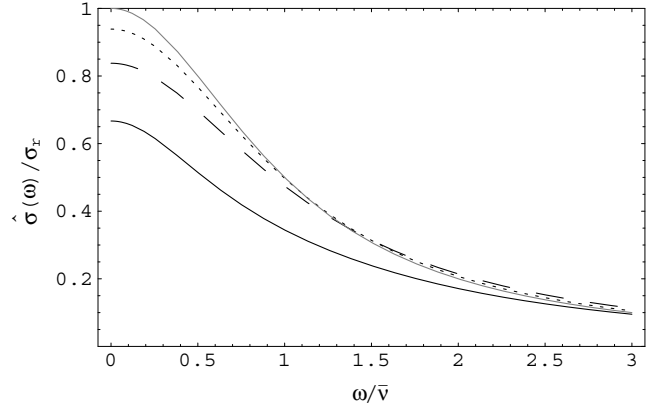


Fig. 3. The real part of the electrical conductivity is shown for  $\epsilon = 0$  (gray),  $\epsilon = 10^{-3}$  (dotted),  $\epsilon = 10^{-2}$  (dashed) and  $\epsilon = 0.1$  (solid) in the low  $\omega$  limit.

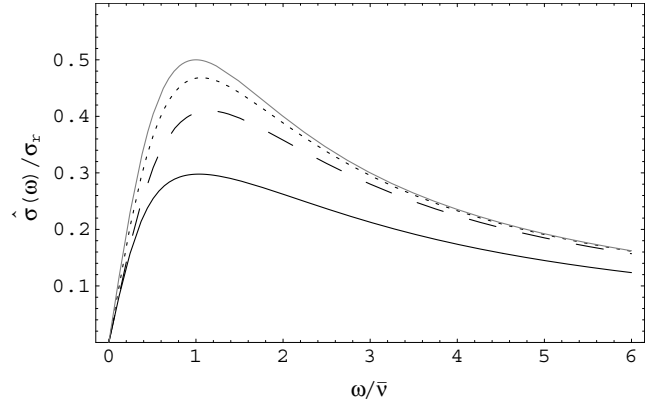


Fig. 4. The imaginary part of the electrical conductivity is shown for  $\epsilon = 0$  (gray),  $\epsilon = 10^{-3}$  (dotted),  $\epsilon = 10^{-2}$  (dashed) and  $\epsilon = 0.1$  (solid) in the low  $\omega$  limit.

Note that in the static, low frequency limit, the effects of trapped particles are present for  $\omega = 0$ . On the other hand, as  $\omega \rightarrow \infty$  (initial times) there are no significant trapped particles effects. Figures 3 and 4 show the effects of  $\epsilon$  on the real and imaginary parts of  $\hat{\sigma}(\omega)$ . It can be shown, using the estimates in Eqs. (51)–(54), that

$$\text{Re}[\hat{\sigma}(\omega)] \sim \begin{cases} 1 + \mathcal{O}(\sqrt{\epsilon}) + \mathcal{O}(\sqrt{\epsilon})(\omega/\nu_e)^2, & \omega \ll \nu_e, \\ [1 + \mathcal{O}(\sqrt{\epsilon})](\nu_e/\omega)^2, & \omega \gg \nu_e, \end{cases} \quad (56)$$

$$\text{Im}[\hat{\sigma}(\omega)] \sim \begin{cases} [1 + \mathcal{O}(\sqrt{\epsilon})]\omega/\nu_e, & \omega \ll \nu_e, \\ [1 + \mathcal{O}(\epsilon^{3/2})]\nu_e/\omega, & \omega \gg \nu_e. \end{cases} \quad (57)$$

These relations are qualitatively consistent with the behavior shown in Fig. 3-4.

## VI. FLOW EVOLUTION

With the closure for the parallel viscous force obtained previously in Eq. (28), the evolution of the parallel flow can be calculated as an initial value problem.

Two schemes are considered: the parallel flow damping within the bumpy cylinder model, and an extension to an axisymmetric geometry and thus to the time dynamics of the poloidal flow in a magnetically-confined toroidal plasma.

The simplicity of the one-dimensional, bumpy cylinder magnetic field model permits a full calculation of the parallel flow dynamics including effects of initial conditions. Consider the total flux-surface-averaged momentum equation for the plasma:

$$mn \langle B^2 \rangle \frac{\partial U(t)}{\partial t} = - \langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle. \quad (58)$$

Since the time dependence of the left side is of interest, we start by again taking a Laplace transform and work from the full frequency-dependent closure in Eq. (28) as follows

$$[v(\omega) - i\omega] \widehat{U}(t) = -\Upsilon(\omega). \quad (59)$$

Considering the small  $\sqrt{\epsilon}$  approximation in Eqs. (38) and (39) for  $v$  and  $\Upsilon$ , we have

$$v \simeq \frac{m}{T} \int d^3v \bar{v} \frac{v^2}{3} \left(1 - \frac{i\omega}{\bar{v}}\right) \widehat{f}_t \frac{f_M}{n}, \quad (60)$$

$$\Upsilon \simeq \frac{m}{T} \int d^3v \frac{v^2}{3} \frac{f_M}{n} \left(1 - \frac{i\omega}{\bar{v}}\right) (1 - \widehat{f}_t) \sum \frac{\chi_n}{\kappa_n - i\omega/\bar{v}}. \quad (61)$$

Solving for  $\widehat{U}$  in Eq. (59) at this point would lead again to infinite sums in the denominator. Instead, the inverse Laplace transform can be taken on both sides and after some manipulation (see Appendix A) one obtains an inhomogeneous integral equation for  $U(t)$ :

$$U(t) = h(t) + \int_0^t K(t; \tau) U(\tau) d\tau. \quad (62)$$

This equation gives the time evolution of the parallel flow and has “memory” of the localization of the initial distribution relative to the boundary with trapped particle space.

The analysis for the bumpy cylinder can also be employed for a toroidal magnetic field. For the dynamic evolution of the “parallel” flow in an axisymmetric configuration the initial distribution in the pitch angle variable is not taken into account for simplicity. In this configuration, the magnetic field can be written as

$$\mathbf{B} = B_T \hat{\zeta} + B_P \hat{\theta}, \quad (63)$$

where  $B_T$  and  $B_P$  are the components of the magnetic field in the toroidal ( $\zeta$ ) and poloidal ( $\theta$ ) directions respectively.

To apply the model developed in the previous sections to this geometry some modifications have to be introduced. In such a configuration, the field modulations along a field line are not unidimensional. That is, the

relevant flow variable to be considered is the poloidal flow defined by [15]

$$U_{\theta}(\psi, t) \equiv \frac{\mathbf{V} \cdot \nabla \theta}{\mathbf{B} \cdot \nabla \theta} = \frac{V_{\parallel}}{B} + \frac{1}{B^2} \left( \frac{d\phi}{d\psi} + \frac{1}{nq} \frac{dp}{d\psi} \right), \quad (64)$$

where the first term is due to the parallel flow velocity and the second is due to the perpendicular flows in the plasma, which to lowest order in gyroradius are a combination of the  $\mathbf{E} \times \mathbf{B}$  and diamagnetic flows:

$$\mathbf{V}_{\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mathbf{B} \times \nabla p}{nqB^2}. \quad (65)$$

In Eq. (64) we have used  $\mathbf{E} = -\nabla\phi$  and  $\psi$  is the poloidal magnetic flux function,  $B_P \hat{\theta} = \nabla\zeta \times \nabla\psi$ .

From the parallel (to  $\mathbf{B}$ ) momentum balance, an evolution equation for  $U_{\theta}$  will include a contribution ( $\propto q^2$ ) from the toroidal flow [5]:

$$nm(1 + 2q^2) \frac{\partial U_{\theta}}{\partial t} = - \frac{\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle}{\langle B_P^2 \rangle} + 4\pi \langle \frac{\partial \phi'}{\partial t} \rangle, \quad (66)$$

where the safety factor is defined as  $q = \epsilon B_T / B_P$  in which now  $\epsilon \equiv r/R_0 \ll 1$  is the inverse aspect ratio.

The closure in Eq. (28) may now be introduced. In this problem parallel stress damps the poloidal component of the parallel flow since toroidal momentum is only damped by the (higher order) perpendicular stress. With the flux surface average in this case being defined as

$$\langle A \rangle \equiv \oint \frac{d\theta A(\theta)}{\mathbf{B} \cdot \nabla \theta} / \oint \frac{d\theta}{\mathbf{B} \cdot \nabla \theta}, \quad (67)$$

one can write

$$\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle = nm \langle B^2 \rangle \left[ \widehat{U}_{\theta} v(\omega) + U_{\theta 0} \right], \quad (68)$$

Once again, Eqs. (60) and (61) can be used to obtain an approximate closure to lowest order in  $\widehat{f}_t$ . Introducing this closure in Eq. (66) yields

$$\left[ \frac{\langle B^2 \rangle}{\langle B_P^2 \rangle} v(\omega) - i\omega(1 + 2q^2) \right] \widehat{U}_{\theta} = \left[ (1 + 2q^2) - \frac{\langle B^2 \rangle}{\langle B_P^2 \rangle} \right] U_{\theta 0} + \frac{4\pi}{nm} \langle \frac{\partial \phi'}{\partial t} \rangle, \quad (69)$$

and taking the inverse Laplace transform on both sides yields the integral equation

$$U_{\theta}(t) = h_{\theta}(t) + \int_0^t K_{\theta}(t; \tau) U_{\theta}(\tau) d\tau, \quad (70)$$

where  $h_{\theta}(t)$  and  $K_{\theta}(t; \tau)$  are defined in Appendix A. Using the approximation described in Sec. V, the asymptotic limits of the poloidal flow evolution for short and long times can be inspected. For  $t \ll \bar{v}$ , Eq. (70) can be roughly approximated by

$$U_{\theta}(t) \simeq t K_{\theta}(t; t) U_{\theta}(t) \quad (71)$$

where we have assumed the initial condition is already damped. Then the characteristic damping time is initially given by  $\tau_p \simeq 1/K_\theta(t; t)$  which yields (see Appendix A)

$$1/\tau_p \simeq \frac{0.51}{\epsilon} \frac{m}{T} \int d^3v \bar{v} \frac{v^2}{3} \frac{f_M}{n}. \quad (72)$$

The numerical result in Eq. (72) is similar to the estimate for the damping rate in Ref. [8] for this limit. This result is also obtained in Ref. [10] when taking a time average including the transient behavior due to the initial perturbation.

In the present  $\sqrt{\epsilon} \ll 1$  expansion, in addition to the damping rate  $\nu_p$  there is a small oscillatory ( $\omega_r$ ) response [8]. It can be recovered by writing  $i\omega = \nu_p + i\omega_r$  and expanding  $v(\omega)$  about the damping rate assuming  $\nu_p \gg \omega_r$ . By taking account of the Landau-type pole [8] in the denominator of  $f_c$  and equating the imaginary parts one obtains an imaginary component of the frequency  $\omega_r$ . This effect contributes slight oscillatory reponses in  $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle$  and in the kernels of the time-history integrals in Eqs. (70). This feature of the flow evolution is neglected in this paper because we are concentrating on the dominant, damping effect of the parallel viscosity.

For long times, in recovering the static limit, one can rewrite the flow evolution as

$$U_\theta(t) \simeq U_\theta(t) \int_0^t K_\theta(t; \tau) d\tau, \quad (73)$$

and thus, the characteristic damping rate is obtained from

$$\int_0^t K_\theta(t; \tau) d\tau = 1. \quad (74)$$

In the small  $\epsilon$  approximation the damping rate for the static calculation is obtained (see Appendix A):

$$\nu_p \simeq \frac{0.31}{\sqrt{\epsilon}} \frac{m}{T} \int d^3v \bar{v} \frac{v^2}{3} \frac{f_M}{n}. \quad (75)$$

This result clearly violates the static assumption, as pointed out by other authors (see for example [5]–[9]).

## VII. SUMMARY

A simple inhomogeneous magnetic field model has been used to calculate the time-dependent kinetic closure for the parallel viscous force  $\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle$  in the banana collisionality regime. This model retains the effect of trapped particles and can be extended to more complicated geometries.

In carrying out the calculation, a formal Laplace transform is introduced which retains the initial value character of the problem. Further analytic progress can be made by employing an expansion for small field modulations. In such a case, the inverse Laplace transform

can be calculated to any order both analytically and numerically. The key result, Eq. (41), is an *explicit time-dependent* closure for the parallel viscous force which may be useful in numerical codes as well as for the theoretical modeling of fast phenomena ( $t \lesssim 1/\nu$ ).

The evolution of the driven parallel/poloidal flow in the expressions obtained are described by integral equations which are still to be evaluated or solved. The results obtained here have various applications that are yet to be exploited. For example, they can be used to explore the dynamics of the transition of the perpendicular dielectric from regular Alfvénic to enhanced neoclassical regimes [16] and for the effects of flow dynamics on neoclassical tearing modes [17].

Extension to these various results to include heat flux effects are given in Appendix B. These extensions are required to obtain net electrical conductivity and poloidal flow damping effects that are not just correct to within about a factor of 2, but instead are within about the  $1/\ln\Lambda \sim 7\%$  intrinsic accuracy of the Coulomb collision operator.

## Acknowledgments

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## APPENDIX A

In this appendix the inverse Laplace transform of the frequency-dependent closure and the parallel and poloidal flow evolution equations are developed in some detail. For the parallel stress, the term in curly brackets in Eq. (40) can be inverted term by term:

$$L^{-1} \left\{ \hat{f}_t \right\} = \sum (\kappa_n - 1) \bar{v} \gamma_n e^{-\kappa_n \bar{v} t} + \left( 1 - \sum \gamma_n \right) \delta(t). \quad (76)$$

Defining the integral  $I_n(t) = \int_0^t d\tau e^{-\kappa_n \bar{v}(t-\tau)} U(\tau)$ , the convolution with the parallel flow variable is

$$L^{-1} \left\{ \hat{f}_t \hat{U} \right\} = \left( 1 - \sum \gamma_n \right) U(t) + \bar{v} \sum \gamma_n (\kappa_n - 1) I_n(t), \quad (77)$$

and the triple convolution can be calculated as follows:

$$L^{-1} \left\{ -\frac{i\omega}{\bar{v}} \hat{f}_t \hat{U} \right\} = \frac{1}{\bar{v}} \left( 1 - \sum \gamma_n \right) \left( \frac{\partial U(t)}{\partial t} + U_0 \delta(t) \right) + \sum \gamma_n (\kappa_n - 1) (U(t) - \kappa_n \bar{v} I_n(t)). \quad (78)$$



Putting these results together we have

$$L^{-1} \left\{ \frac{\widehat{f}_t}{\widehat{f}_c} \widehat{U} \right\} \simeq \frac{1}{\bar{v}} \left( 1 - \sum \gamma_n \right) \left( \frac{\partial U(t)}{\partial t} + U_0 \delta(t) \right) + \left[ 1 + \sum \gamma_n (\kappa_n - 2) \right] U(t) - \sum \gamma_n (\kappa_n - 1)^2 \bar{v} I_n(t), \quad (79)$$

which, after some manipulation leads to the dynamic closure in Eq. (41) for  $t > 0$ .

To obtain the time evolution of the parallel flow variable, the inverse Laplace transform is calculated on both sides of Eq. (59). For the right side, using Eq. (39) and the previous results one can obtain

$$L^{-1} \left\{ \frac{1}{\widehat{f}_c} \sum \frac{\chi_n}{\kappa_n - i\omega/\bar{v}} \right\} \simeq \sum_{m,n} \{ \delta(t) \gamma_n \chi_m - \bar{v} (\kappa_m - 1) e^{-\kappa_m \bar{v} t} [(\chi_m \gamma_n + \chi_n \gamma_m) - \bar{v} \chi_m \gamma_n (\kappa_n - 1) \int_0^t e^{\bar{v} \tau (\kappa_m - \kappa_n)} d\tau] \}. \quad (80)$$

To eliminate the delta functions, a time integral is taken on both sides and after a simple calculation one can solve for  $U(t)$ . Equation (62) is obtained for which we define the inhomogeneous term and the integration kernel as follows:

$$h(t) = \frac{1}{(\sum \gamma_m - 2)} \int d^3 v \frac{m v^2}{T} \frac{f_M}{3} \sum_{m,n} \left\{ \frac{\chi_n \gamma_m}{\kappa_m \kappa_n} - (\kappa_m - 1) \frac{\chi_m \gamma_n}{\kappa_m} e^{-\kappa_m \bar{v} t} \left[ \left( 1 + \frac{\chi_n \gamma_m}{\kappa_n \chi_m \gamma_n} \right) - \bar{v} (\kappa_n - 1) \int_0^t e^{\bar{v} \tau (\kappa_m - \kappa_n)} d\tau \right] \right\}, \quad (81)$$

$$K(t; \tau) = \frac{1}{(\sum \gamma_m - 2)} \int d^3 v \bar{v} \frac{m v^2}{T} \frac{f_M}{3} \left\{ 1 + \sum \frac{\gamma_m}{\kappa_m} \left[ (\kappa_m - 1)^2 e^{-\kappa_m \bar{v} (t-\tau)} - 1 \right] \right\}. \quad (82)$$

For the toroidal geometry, the inverse Laplace transform of each term in Eq. (69) is calculated as before. A time integral has to be calculated in this case also and one can obtain Eq. (70) where the kernel and inhomogeneous term are defined by

$$h_\theta(t) = a \left[ \left( 1 + 2q^2 - \frac{\langle B^2 \rangle}{\langle B_P^2 \rangle} \right) U_{\theta 0} - \frac{4\pi}{nm} \langle \phi'(t) - \phi'_0 \rangle \right], \quad (83)$$

$$K_\theta(t; \tau) = -a \frac{\langle B^2 \rangle}{\langle B_P^2 \rangle} \frac{m}{T} \int d^3 v \bar{v} \frac{v^2}{3} \frac{f_M}{n} \left\{ 1 - \sum \frac{\gamma_n}{\kappa_n} \left[ 1 + (\kappa_n - 1)^2 e^{-\kappa_n \bar{v} (t-\tau)} \right] \right\}, \quad (84)$$

and the factor  $a$  for both expressions is

$$a = \left[ 1 + 2q^2 + \frac{\langle B^2 \rangle}{\langle B_P^2 \rangle} \left( 1 - \sum \gamma_n \right) \right]^{-1} \simeq \frac{\langle B^2 \rangle}{\langle B_P^2 \rangle} \left( 1 - \sum \gamma_n \right), \quad (85)$$

where the second line is valid for small  $\epsilon$ . Using Eq. (85) and the values in Eqs. (51)–(54), the kernel of integration  $K_\theta(t; t) \sim 1/\tau_p$  for the  $t \ll \bar{v}$  case in Section VI is

$$K_\theta(t; t) \simeq \left( \frac{0.51}{\epsilon} - 2.69 \right) \frac{m}{T} \int d^3 v \bar{v} \frac{v^2}{3} \frac{f_M}{n}. \quad (86)$$

In the long time case, for the static limit at the end of Sect. VI we have, for  $\int_0^t K_\theta(t; \tau) d\tau = 1$ ,

$$t - \sum \frac{\gamma_n}{\kappa_n} \left[ t + \int_0^t (\kappa_n - 1)^2 e^{-\kappa_n \bar{v} (t-\tau)} d\tau \right] = 1 - \sum \gamma_n, \quad (87)$$

or, solving for  $t = \tau_p$ ,

$$\tau_p \simeq \left( \frac{\sqrt{\epsilon}}{0.31} \right) / \frac{m}{T} \int d^3 v \bar{v} \frac{v^2}{3} \frac{f_M}{n}, \quad (88)$$

which yields the damping rate in Eq. (75)

## APPENDIX B

To include heat flux effects, the distribution function to be considered in the Chapman-Enskog procedure is an equilibrium flow and heat-flux shifted Maxwellian plus a small kinetic distortion:

$$f = f_M \left[ 1 + \frac{m}{T} \mathbf{v} \cdot \left( \frac{2}{5nT} \mathbf{q} \right) L_1^{3/2} \right] + F, \quad (89)$$

where  $f_M$  is defined in Eq. (2). In this case, the recast DKE includes two extra terms and, upon defining a parallel heat flux variable  $q_{\parallel} = \mathbf{q} \cdot \mathbf{B}/B^2$  to satisfy heat flux incompressibility, can be written as [15]:

$$\frac{dF}{dt} + v_{\parallel} \mathbf{b} \cdot \nabla \left[ F + \frac{m}{T} v_{\parallel} B \left( U - \frac{2}{5p} q_{\parallel} L_1^{3/2} \right) f_M \right] = C(F) + \frac{v_{\parallel}}{p} f_M \left( \mathbf{b} \cdot \nabla \cdot \Pi_{\parallel} + \frac{2}{5} L_1^{3/2} \mathbf{b} \cdot \nabla \cdot \Theta_{\parallel} \right). \quad (90)$$

In Eqs. (89) and (90),  $L_1^{3/2} = 5/2 - x$  is the Laguerre polynomial in the usual variable  $x = v^2/v_t^2$  and the heat stress tensor is given by

$$\Theta = \int d^3 v m \left( \mathbf{v} \mathbf{v} - \frac{1}{3} v^2 \mathbf{I} \right) L_1^{3/2} f. \quad (91)$$

Solving Eq. (90) to lowest order as done previously in Section II the lowest order distortion is obtained:

$$F_0 = -\frac{m}{T} v_{\parallel} B \left( U - \frac{2}{5p} q_{\parallel} L_1^{3/2} \right) f_M + g(v, \lambda, \varsigma, t). \quad (92)$$

Calculating the annihilator (19) in the next order DKE and taking a Laplace transform as before, one obtains a differential equation for  $\hat{g}$  for circulating particles similar to Eq. (20). The source term in this case includes heat flux and heat stress drives:

$$\begin{aligned} \widehat{S}(v, \lambda, \omega) \equiv & \frac{m}{T} \left\{ \langle B^2 \rangle \left[ (\bar{\nu} - i\omega) \left( \widehat{U} - \frac{2}{5p} \widehat{q} L_1^{3/2} \right) - \right. \right. \\ & \left. \left. \left( U_0 - \frac{2}{5p} \widehat{q}_0 L_1^{3/2} \right) \right] - \left\langle \frac{B}{v_{\parallel}} \right\rangle g_0(\lambda) + \right. \\ & \left. \frac{1}{mn} \left( \langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle + \frac{2}{5} \langle \mathbf{B} \cdot \nabla \cdot \Theta_{\parallel} \rangle L_1^{3/2} \right) \right\}. \end{aligned} \quad (93)$$

Thus, the lowest order solution is

$$\widehat{F}_0 = -v_{\parallel} B \frac{m}{T} f_M \left( \widehat{U} - \frac{2}{5p} \widehat{q}_{\parallel} L_1^{3/2} \right) + \frac{v\zeta}{2} \sum_1^{\infty} \frac{\eta_n \Lambda_n}{(\bar{\nu} \kappa_n - i\omega)}, \quad (94)$$

where  $\eta_n$  is defined in Eq. (27). Since two unknowns are included in  $\widehat{F}_0$ , two Chapman-Enskog constraints are required. Thus, it is convenient to consider  $[L_0^{3/2} \equiv 1]$

$$\int d^3 v v_{\parallel} L_0^{3/2} \widehat{F}_0 = 0, \quad (95)$$

$$\int d^3 v v_{\parallel} L_1^{3/2} \widehat{F}_0 = 0. \quad (96)$$

Using equation (95) and the Laguerre polynomials orthogonality condition one obtains

$$\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle = nm \langle B^2 \rangle \left( \widehat{U} v_{00} - \frac{2}{5p} \widehat{q} v_{01} + U_0 + \Upsilon_0 \right), \quad (97)$$

where  $v_{ij}(\omega)$  are the ‘‘viscosity coefficients’’ [for  $v_{ij}(0) = \mu_{ij}$  numerical values see Reference [15]]. Both  $v_{ij}(\omega)$  and  $\Upsilon_i(\omega)$  are obvious generalizations of the coefficients defined in Eqs. (29) and (33):

$$v_{ij}(\omega) = \frac{m}{T} \int d^3 v \bar{\nu} \frac{v^2}{3} \frac{f_M}{n} L_i^{3/2} L_j^{3/2} \frac{\hat{f}_t(v, \omega)}{\hat{f}_c(v, \omega)}, \quad (98)$$

$$\Upsilon_i(\omega) = \frac{m}{T} \int d^3 v \frac{v^2}{3} \frac{f_M}{n} L_i^{3/2} \frac{1}{\hat{f}_c} \sum_1^{\infty} \frac{\chi_n}{\kappa_n - i\omega/\bar{\nu}}. \quad (99)$$

Equation (96), which is a combination of the density and the energy constraints in Eq. (3), yields

$$\langle \mathbf{B} \cdot \nabla \cdot \Theta_{\parallel} \rangle = nm \langle B^2 \rangle \left( \widehat{U} v_{01} - \frac{2}{5p} \widehat{q} v_{11} - \frac{1}{p} \widehat{q}_0 + \Upsilon_1 \right). \quad (100)$$

Thus, the frequency dependent closure, including heat flux effects, can be expressed as a matrix equation as follows:

$$\begin{aligned} \begin{bmatrix} \langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \rangle \\ \langle \mathbf{B} \cdot \nabla \cdot \Theta_{\parallel} \rangle \end{bmatrix} &= nm \langle B^2 \rangle \begin{bmatrix} v_{00} & v_{01} \\ v_{01} & v_{11} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ -\frac{2}{5p} \widehat{q}_{\parallel} \end{bmatrix} \\ &+ \begin{bmatrix} U_0 + \Upsilon_0(\omega) \\ -\frac{1}{p} \widehat{q}_0 + \Upsilon_1(\omega) \end{bmatrix}. \end{aligned} \quad (101)$$

In order to obtain the heat flux correction to expression (43), we use the moment approach. This is, taking the momentum and heat flux moments (i.e.  $m \int d^3 v v_{\parallel} L_i^{3/2}$  for  $i = 1, 2$ ) of the kinetic equation yields a matrix equation for the evolution of  $V_{\parallel}$  and  $q_{\parallel}$ . The parallel component of these balance equations for electrons can be written in matrix form as

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} V_{\parallel} B \\ -q_{\parallel} B/p \end{bmatrix} &= -\frac{e}{m_e} \begin{bmatrix} E_{\parallel} B \\ 0 \end{bmatrix} \\ -\nu_e [\mathbb{L}] \begin{bmatrix} V_{\parallel} B \\ -2q_{\parallel} B/5p \end{bmatrix} &+ \frac{1}{n_e m_e} \begin{bmatrix} \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel e} \\ \mathbf{B} \cdot \nabla \cdot \Theta_{\parallel e} \end{bmatrix} \end{aligned} \quad (102)$$

where  $\mathbb{L}$  denotes the  $2 \times 2$  matrix containing the usual [4] electron friction coefficients  $L_{ij}^e$ . When a bounce average is calculated and the closure (101) is introduced, the Laplace transform of Eq. (102) can be written as

$$\begin{bmatrix} \langle E_{\parallel} B \rangle \\ 0 \end{bmatrix} \simeq \frac{m_e \nu_e}{n_e e^2} [\mathbb{F}] \begin{bmatrix} \langle J_{\parallel} B \rangle \\ \frac{2e}{5T_e} \langle q_{\parallel e} B \rangle \end{bmatrix}, \quad (103)$$

where we defined

$$[\mathbb{F}] = [\mathbb{L}] + \begin{bmatrix} (v_{00} - i\omega)/\nu_e & v_{01}/\nu_e \\ v_{01}/\nu_e & (v_{11} - 5i\omega/2)/\nu_e \end{bmatrix}. \quad (104)$$

Thus, the electrical conductivity in this more precise approximation can be calculated as

$$\hat{\sigma}(\omega) \simeq \left( \frac{\mathbb{F}_{22}}{\mathbb{F}_{11} \mathbb{F}_{22} - \mathbb{F}_{10}^2} \right) \sigma_r. \quad (105)$$

For the parallel flow damping correction, we generalize the quantities in Eqs. (81) and (82) by introducing  $L_i^{3/2}$  [for  $h(t) \rightarrow h_i(t)$ ] and  $L_i^{3/2} L_j^{3/2}$  [for  $K(t; \tau) \rightarrow K_{ij}(t; \tau)$ ] in the velocity space integrals. Doing so, and proceeding as in the last part of Appendix A one can write the evolution for the flow including heat flux effects as an integral (matrix) equation as follows

$$\begin{bmatrix} U(t) \\ -\frac{1}{p} q(t) \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} + \int_0^t \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix} \begin{bmatrix} U(\tau) \\ -\frac{2}{5p} q(\tau) \end{bmatrix} d\tau. \quad (106)$$

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