Time and space dependent parallel viscous force

by

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Fluid moment equations describing the macroscopic behavior of magnetized plasmas require kinetic-based closures for the viscous forces. In particular, the parallel component of the force balance is crucial for the analysis of the macroscopic dynamics of toroidal plasmas in the low collisionality regime since the relevant forces and dynamics are in the direction of the magnetic field. The damping force in this equation is the parallel viscous force, $\mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|}$, for which a time and space dependent closure is required.

In this work, the needed closure is obtained using a Chapman-Enskog-like procedure in a drift-kinetic theory. A bumpy cylinder magnetic field model is used; it causes us to have to distinguish between the effects of circulating particles which carry parallel flow and trapped particles which do not.

A frequency-dependent closure is obtained for which the inverse Laplace transform does not have a simple analytical expression. Introducing this result in the parallel Ohm's law, we obtain corrections to the frequency-dependent electrical conductivity. Also, a parallel flow evolution equation is obtained from the total parallel force balance; it can be inverted to yield an integral equation in time. The poloidal flow damping in a toroidal geometry is calculated by extending the closure obtained with the simpler bumpy cylinder model. Expressions for the explicit time dependence of $\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle$ and of the flow evolution are calculated in a small field strength modulation ($\epsilon \equiv \Delta B/B \ll 1$) approximation.

The variation of $\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}$ within a flux surface is here shown to be large compared to its average. To calculate this varying component, the first order drift-kinetic equation is integrated along field lines. We consider small magnetic field modulations and a steady state situation to calculate the spatially-dependent stress tensor $\mathbf{\Pi}_{\|}$ which leads to the closure for the parallel viscous force. Unlike the collisional regime, the anisotropy $p_{\|} - p_{\perp}$, and hence $\Pi_{\|}$, is not simply sinusoidal but instead expressed in terms of incomplete elliptic functions. The magnitude of the closure scales as $\epsilon^{-1/2}$. Advances in combining the full spatial and temporal variations into a single, comprehensive closure relation are also presented.

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And finally, since this page is entitled "acknowledgments" I want to tap myself in the head. It was hard, but against all odds... I made it!

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Contents

I. INTRODUCTION

In the pursuit of controlled fusion deuterium and tritium nuclei must collide with kinetic energies of about 10 keV or greater. Such energetic ions are usually in the plasma state. Confinement of such hot plasmas is being sought in toroidal magnetic systems. The macroscopic dynamics of such magnetized toroidal plasmas is often described by fluid moment equations. However, kinetic effects need to be included in the closure moments for such plasmas because of the long collision lengths compared to the physical size of the toroidal magnetic configurations.

Most plasma confinement devices operate in the low collisionality banana or neoclassical regime where the collision length is much longer than the circumference of the torus. Then, the most important closure moment is the parallel stress tensor. This closure has usually been derived and used in its flux-surface-averaged form and in steady state situations. In this work, we explore both the time-dependent averaged closure and the variation of the parallel viscous force within a flux surface.

The closure for the flux-surface-averaged parallel viscous force, $\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle$, has been calculated in steady state situations by various authors [1]–[5]. The results obtained are widely used even though it was recognized, right from the very first calculations, that the scaling of the solution is inconsistent with the hypothesis of the problem. In particular, when using the static closure the damping rate of the poloidal flow in a toroidal plasma becomes smaller than the ion-ion collision time which violates the static assumption used to derive the closure [6]. Because of this, a frequency dependent problem has been addressed [7]–[10] and recently an initial value problem has been considered [9, 10].

In this work, the time dynamic case is approached in a different way. The main objective is to describe the *time evolution* of the parallel viscous force and thus the parallel/poloidal flow. We are interested in an expression for the closure with an explicit time dependence that could be directly incorporated in numerical codes (such as NIMROD [11]). This calculation also includes initial conditions by taking a formal Laplace transform of the kinetic equation. By inverting the solution one obtains a time dependent closure that depends both on the fraction of trapped particles and on the initial conditions for short times. Previous authors solved the initial value problem by considering a boundary layer where collisions can be neglected [9, 10]. In this work, this assumption is not made and the full pitch-angle scattering (Lorentz) collision operator is considered.

The time-dependent closure obtained here is one step towards a complete description of the plasma dynamics. It allows a time dependent analysis of the evolution equation for the parallel flow including the dynamic viscous damping force. It could also provide a more complete picture of the transition to the steady state of the parallel/poloidal flow evolution and the determination of the time scale on which an equilibrium assumption is formally valid. The relaxation of the poloidal flow occurs on fast time scales ($\sim$ ion collision time) but is still relevant for some experiments, for example after a sawtooth crash [12]. In addition to providing a time dependent expression for the viscous damping force on the overall parallel flow, the time-dependent closure also introduces a frequency dependent viscous force term in the parallel Ohm's law and hence a correction to the electrical conductivity.

Some selected flux-surface-averaged phenomena can be addressed with the time dependent expression for $\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \rangle$— for example the two applications mentioned in the previous lines. However, in general not all phenomena of interest can be addressed with a flux-surface-averaged parallel viscous force.

In order to describe the dynamics of a plasma more accurately, the variation of the parallel viscous force within a flux surface is required. A part of this problem has been addressed in the steady state case by Wang and Callen [5]. Also, Hsu et al. [8] gave some insight into the spatial variation issue in their consideration of the time dependent problem. Recently, a generalized stress tensor in a slab geometry was formulated for arbitrary collisionality [13]. Here, we calculate the local closure for the steady state case in a slightly different way from what was done in Ref. [5]. The relevant drift-kinetic equation is directly integrated along field lines for a bumpy cylinder magnetic field. The stress moment of the resultant kinetic distortion yields the pressure anisotropy $p_{\|} - p_{\perp}$, the stress tensor $\mathbf{\Pi}_{\|}$, and the (un-averaged) parallel viscous force $\mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|}$.

The combined time and space dependent problem is addressed towards the end of this work. A formal solution of this problem is presented. However, recognizing its complexity, we only discuss some elements of the combined calculation.

The remainder of this chapter is devoted to formulating the problem by describing its motivation and context. Chapter II describes the Chapman-Enskog like approach used here and by other authors [5, 13, 14, 16] to calculate kinetic closures. The relevant drift-kinetic equation and magnetic field model are also introduced there.

The formal calculation starts in Chapter III by formulating a perturbation method for solving the drift-kinetic equation and obtaining its solution to lowest order. In order to clearly describe the procedure to obtain the frequency dependent closure, the static problem is solved first in Chapter IV using the procedure that will be employed in subsequent chapters. The dynamic case is addressed in Chapter V and the time dependent closure is obtained in a small field modulation approximation.

Chapters VI and VII address two relevant applications of the result: the trapped-particle corrections to the frequency dependence of the electrical conductivity (Chapter VI) and the temporal evolution of parallel and poloidal flows (Chapter VII) in a bumpy cylinder and toroidal plasmas, respectively. Heat flux effects are neglected in most of the calculation for simplicity; however, they are introduced and analyzed in Chapter VIII.

The spatially varying closure is calculated in Chapters IX and X. In Chapter IX the calculation is carried out in some detail in a steady state case and for small field modulations. Progress in the more complicated combined spatial and frequency-dependent closure is treated in Chapter X. The summary in Chapter XI includes concluding remarks as well as possible future work for which the results calculated in this research can be applied and extended.

A. Basic fluid moment balance equations, moment approach

The evolution of the distribution function for a system of charged particles is given by the plasma kinetic equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = C\left(f\right). \tag{1}$$

The source of the irreversible, dissipative evolution is the collision operator $C\left(f\right)$ which describes binary collisions between particles of the same or different species. For a magnetized plasma medium one can obtain the fluid moment balance equations by taking velocity-space moments of Eq. (1):

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V} = 0, \tag{2}$$

$$\frac{3}{2} n \frac{dT}{dt} + nT\left(\nabla \cdot \mathbf{V}\right) = -\nabla \cdot \mathbf{q} - \Pi{:}\nabla\mathbf{V} + \mathbf{Q}, \tag{3}$$

$$mn\frac{d\mathbf{V}}{dt} = nq(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \nabla p - \nabla \cdot \Pi + \mathbf{F}_0, \tag{4}$$

$$\frac{m}{T}\frac{d\mathbf{q}}{dt} = \frac{q}{T}\mathbf{q} \times \mathbf{B} - \frac{5}{2} n\nabla T - \nabla \cdot \Theta + \mathbf{F}_1. \tag{5}$$

Equations (2) to (5) describe the evolution of the macroscopic variables defined as

$$n\left(\mathbf{x}, t\right) = \int d^3v\, f \qquad \text{density}, \tag{6}$$

$$T\left(\mathbf{x}, t\right) = \int d^3v\, \frac{mv^2}{3}\frac{f}{n} \qquad \text{temperature}, \tag{7}$$

$$\mathbf{V}\left(\mathbf{x},t\right)=\int d^3v\,\mathbf{v}\frac{f}{n}\qquad\text{mass flow,}\tag{8}$$

$$\mathbf{q}\left(\mathbf{x},t\right)=\int d^3v\,\frac{mv^2}{2}\mathbf{v}f\qquad\text{heat flux,}\tag{9}$$

$$\mathbf{Q}\left(\mathbf{x},t\right)=\int d^3v\,\frac{mv^2}{2}C\left(f\right)\qquad\text{collisional energy exchange rate.}\tag{10}$$

The higher order moments drive the evolution of the flow and heat flux. The collision induced friction force terms are

$$\mathbf{F}_0\left(\mathbf{x},t\right)=\int d^3v\,m\mathbf{v}\,C\left(f\right)\qquad\text{frictional force,}\tag{11}$$

$$\mathbf{F}_1\left(\mathbf{x},t\right)=\int d^3v\,L_1^{3/2}m\mathbf{v}\,C\left(f\right)\qquad\text{heat frictional force.}\tag{12}$$

Finally, the viscous stress closure moments are defined by

$$\mathbf{\Pi}\left(\mathbf{x},t\right)=\int d^3v\,m\left(\mathbf{vv}-\frac{1}{3}v^2\mathbf{I}\right)f\qquad\text{anisotropic stress tensor,}\tag{13}$$

$$\mathbf{\Theta}\left(\mathbf{x},t\right)=\int d^3v\,L_1^{3/2}\left(\mathbf{vv}-\frac{1}{3}v^2\mathbf{I}\right)mf\qquad\text{anisotropic heat stress tensor.}\tag{14}$$

The set of coupled equations given by Eqs. (2)–(5) is not closed. The Maxwell equations relate electromagnetic fields to the current and charge densities $\mathbf{J}$ and ρ_q. However, closures for $\mathbf{\Pi}$ and $\mathbf{\Theta}$ have to be introduced in order to close the system. Kinetic theory can provide those closures by considering a lowest order distribution function that generates all moments from (6) to (8) plus a kinetic-based distortion which generates the heat flux and viscous stresses. This kinetic distortion has to include all relevant effects. In this work we will concentrate on the viscous stress closures; similar work on the heat flux closures has been carried out by Held et al. [14].

B. Neoclassical theory

Neoclassical theory differs from classical theory in that it considers the parallel dynamics drift-motion-induced transport instead of that due to the gyromotion of the charged particles in the magnetic field. The magnitude of the magnetic field in a toroidal geometry is modulated between B_{max} and B_{min} along its helical field lines mostly in the short way around the device (the poloidal direction). This gradient affects charged particles which, depending on their kinetic energy $E = mv^2/2$, can flow pass the potential hill created by this gradient or not. Those particles with $E > \mu B_{max}$ ($\mu = mv_\perp^2/2B$ is the magnetic moment) have enough energy to pass over the potential hill μB_{max}. These types of particles are usually called untrapped or circulating particles. Their orbits around the torus are just slightly shifted off flux surfaces due to the $\mathbf{B} \times \nabla B$ drift. On the other hand, particles with $\mu B_{min} < E < \mu B_{max}$ are trapped in the potential well and bounce back and forth between the turning points of the orbit where $v_\parallel \to 0$. The superposition of this bounce motion along field lines with the radial $\mathbf{B} \times \nabla B$ drift is called the "banana orbit"— because of the shape of their drift orbits in a constant toroidal angle plane.

In the lowest collisionality regime, the so-called "banana regime," the characteristic time between collisions is longer than the time in which a trapped particle can complete its orbit. In this regime the trapped particles complete their banana orbits and do not contribute to the flow. Thus, the entire parallel flow is carried by the untrapped (circulating) particles.

The pitch angle variable that will be mostly used throughout this work is $\lambda = 2\mu/v^2$ for which $v_\parallel^2 = v^2 (1 - \lambda B)$. For this variable the two types of particles are distinguished by

$$\lambda < \lambda_c \qquad \text{circulating,}$$

$$\lambda \geq \lambda_c \qquad \text{trapped,}$$

where $\lambda_c \equiv 1/B_{max}$. The dominant viscous stress in neoclassical theory is the parallel stress tensor since, in a small gyroradius $(\rho/\ell \ll 1)$ ordering,

$$\Pi_{\|} \sim \mathcal{O}\left(\rho^0\right), \quad \Pi_{\wedge} \sim \mathcal{O}\left(\rho^1\right), \quad \Pi_{\perp} \sim \mathcal{O}\left(\rho^2\right).$$

Thus, the critical closure moment is the parallel stress tensor. Also, since the relevant dynamics are in the direction parallel to the field, the parallel viscous force $\mathbf{B} \cdot \nabla \cdot \Pi_{\|}$ is of particular interest since this is the quantity that enters the parallel momentum balance, Eq. (4), and thus affects the evolution of the flow.

II. CHAPMAN-ENSKOG PROCEDURE

As discussed in the Introduction, the fluid moment equations require closures to form a complete set together with the Maxwell equations. The higher order moments in Eqs. (13)–(14) are required and can be obtained through a distribution function f given by kinetic theory. This distribution should include all relevant effects. In this case, for the regime considered, the trapped particle effects contained in the distribution function should yield nonzero viscous stresses and thus close the system given by Eqs. (2)–(5).

In this Chapter, a procedure to include kinetic effects in the distribution function, which is a Chapman-Enskog like procedure, is described.

A. Chapman-Enskog-like procedure

In the usual Chapman-Enskog procedure [15] the first step is showing that when collisional effects dominate the lowest order distribution is a Maxwellian. In our Chapman-Enskog-like procedure the distribution function of a system is assumed to be a dynamic, flow- and heat flow-shifted Maxwellian plus a small kinetic distortion. All neoclassical effects are then included in a distribution distortion, F, which is considered to be small compared to the equilibrium distribution (in a small gyro-radius approximation). In most of this work, heat flux effects will be neglected for simplicity; they will be addressed separately in Chapter VIII.

Thus, the distribution function for the system will be assumed to be [15]

$$f = f_M + F, \tag{15}$$

where f_M is a Maxwellian distribution function in the relative velocity $\mathbf{v} = \mathbf{v}' - \mathbf{V}$

i. e., the velocity of individual particles in the flow ($\mathbf{V}$) rest frame:

$$f_M(\mathbf{v}', \mathbf{x}, t) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{mv^2}{2kT}\right]. \tag{16}$$

The dependence on physical space variables, $\mathbf{x}$ and t, of the equilibrium f_M is given through the thermodynamic variables. Here $n(\mathbf{x}, t)$ is the number density, $\mathbf{V}(\mathbf{x}, t)$ the flow velocity and $T(\mathbf{x}, t)$ the temperature defined in Eqs. (6)–(8). The particle mass is denoted m and k is Boltzmann's constant.

By taking moments on both sides of Eq. (15) one can verify that the first three moments of the kinetic distortion should vanish. Thus, for the present problem, we consider as the Chapman-Enskog Ansatz the following conditions:

$$\int d^3v\, F = 0, \quad \int d^3v\, \mathbf{v} F = 0, \quad \int d^3v\, \frac{mv^2}{2} F = 0. \tag{17}$$

By including the first three moments of the distribution function in the Maxwellian part, the kinetic distortion does not add terms to the density, momentum and energy balance equations. Neoclassical effects will appear in this formulation through the higher order moments of F, in particular the stresses given in Eqs. (13) and (14).

B. Magnetic field model

For a general magnetic field we can write

$$B(\ell) = B_0\left[1 + 2\epsilon\tau(\ell)\right], \tag{18}$$

where $\tau(\ell)$ is some function that varies between zero and one. In toroidal geometries, the magnetic field lines have a helical twist. Field modulations along field lines create potential wells into which charged particles can get trapped. In the spirit of restricting the calculation to the effect of interactions between these trapped particles and those that flow freely along field lines, for most of this work we consider a

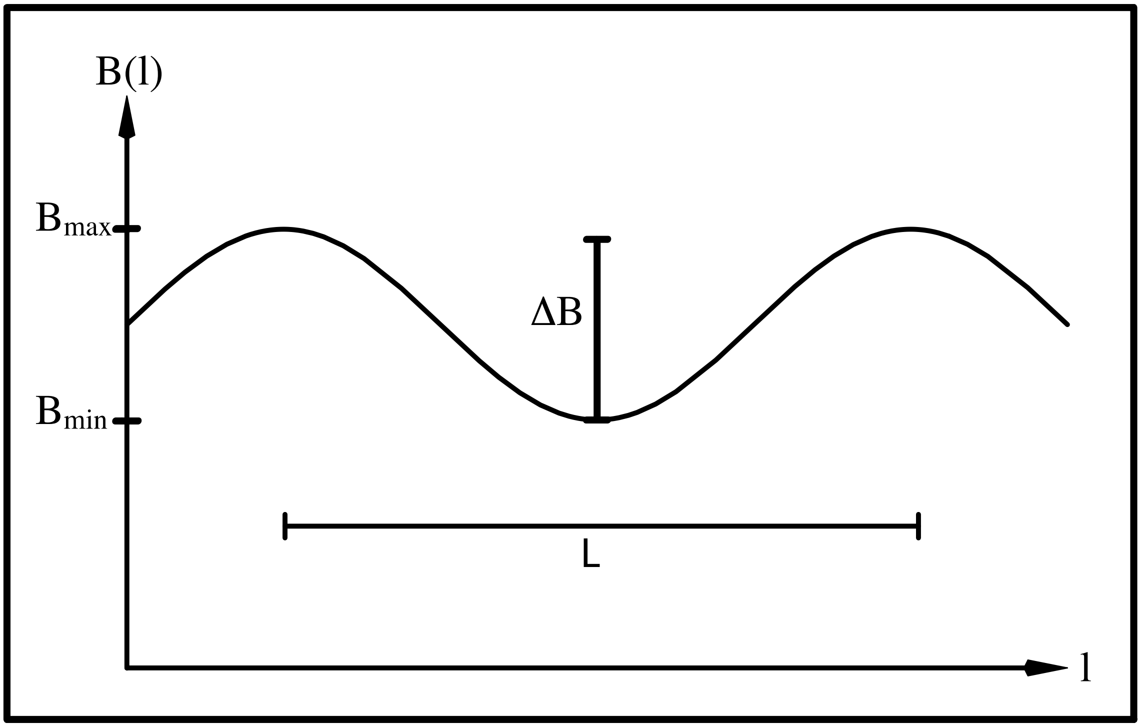

Fig. 1. Bumpy cylinder magnetic field with periodicity length L and modulation $\Delta B = 2\epsilon B_0$.

cylindrical magnetic field with periodic bumps as shown in Fig. 1. The modulations in this field simulate the field gradients in the poloidal direction in a torus. We write this field as

$$B = B_0 \left[1 - \epsilon \cos \left(\frac{2\pi\ell}{L} \right) \right]. \tag{19}$$

The field minima and maxima for this model are given by

$$B_{min} = B_0 \left(1 - \epsilon \right), \qquad \ell = 0, \tag{20}$$

$$B_{max} = B_0 \left(1 + \epsilon \right), \qquad \ell = \pm L/2. \tag{21}$$

Thus, the magnitude of the modulation is

$$\frac{\Delta B}{B_0} \equiv \frac{B_{max} - B_{min}}{B_0} = 2\epsilon. \tag{22}$$

Equation (19) can also be written as

$$B(\ell) = B_{\min} + 2\epsilon B_0 \sin^2\left(\frac{\pi\ell}{L}\right), \tag{23}$$

which corresponds to $\tau(\ell) = \sin^2(\pi\ell/L)$ in Eq. (18). These expressions will be used to connect the expressions for general fields to the sinusoidal bumpy cylinder magnetic field.

C. Drift-kinetic equation

Kinetic theory provides a framework for calculating closures from the distribution function f which evolves according to the plasma kinetic equation

$$\frac{df}{dt} = C(f). \tag{24}$$

For a system of charged particles and independent variables $\mathbf{v}$ (the relative velocity), $\mathbf{x}$, and t, the total time derivative on the left side of the plasma kinetic equation is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \left[\frac{q}{m}\left(\mathbf{E} + \frac{1}{c}(\mathbf{V}+\mathbf{v})\times\mathbf{B}\right) - \frac{\partial \mathbf{V}}{\partial t} - (\mathbf{V}+\mathbf{v})\cdot\nabla\mathbf{V}\right]\cdot\frac{\partial f}{\partial \mathbf{v}}. \tag{25}$$

The distribution function given by Eqs. (15)-(17) can be introduced in Eq. (25). Then, using the balance Eqs. (2)–(5), the total time derivative of the distribution function can be written in terms of F.

By considering a model collision operator that separates the effects on f_M and F, Wang and Callen [16] recast the kinetic equation into a formal gyro-averaged drift kinetic equation (DKE) for the kinetic distortion F. The full equation is not written here since only a simplified version of it will be needed. Neglecting all heat flux terms (treated in Chapter VIII) and higher order moments, Eq. (127) in Reference [16] reduces to

$$\frac{\partial F}{\partial t} + \left(v_\parallel \mathbf{b} + \mathbf{V}\right)\cdot\nabla\mathbf{F} - C(F) = \left\{-\left(v_\parallel^2 - \frac{v_\perp^2}{2}\right)G + \frac{v_\parallel}{p}\mathbf{b}\cdot\nabla\cdot\Pi\right\} f_M, \tag{26}$$

where the "flow drive" term is given by

$$G=\frac{2}{v_t^2}\left[\mathbf{V}\cdot\nabla\ln B-\frac{1}{B}\mathbf{b}\cdot\nabla\times\left(\mathbf{B}\times\mathbf{V}\right)+\frac{2}{3}\nabla\cdot\mathbf{V}\right]. \tag{27}$$

We consider the linearized, approximate Coulomb collision operator [17]

$$C\left(F\right)=\bar{\nu}\mathcal{L}\left(F\right)+\sum_l P_l\left[C^l\left(f_l\right)+\frac{l\left(l+1\right)}{2}\bar{\nu}f_l\right], \tag{28}$$

where P_l are Legendre polynomials and we have defined $\bar{\nu}\equiv\nu_\perp/2$. The collision operator in Eq. (28) is written in terms of the pitch angle variable $\lambda\equiv 2\mu/v^2$ introduced in Chapter I. With this variable, the parallel speed is $v_\parallel=\varsigma v\sqrt{1-\lambda B}$ and ς depends on the direction in which the particles circulate and is defined as $\varsigma=\left|v_\parallel\right|/v_\parallel=\mathrm{sign}\left(v_\parallel\right)$. Perpendicular diffusion in velocity space (pitch-angle scattering) is accounted for by the Lorentz scattering operator $\mathcal{L}$ given by

$$\mathcal{L}=\frac{2v_\parallel}{v^2}\frac{\partial}{\partial\lambda}\frac{\lambda}{B}v_\parallel\frac{\partial}{\partial\lambda}=2\sqrt{1-\lambda B}\frac{\partial}{\partial\lambda}\frac{\lambda}{B}\sqrt{1-\lambda B}\frac{\partial}{\partial\lambda}. \tag{29}$$

The second term in Eq. (28) contains the momentum restoring terms. These terms are required because $\mathcal{L}$ does not conserve momentum by itself; they are evaluated for

$$f_l=\frac{2l+1}{2}\int_{-1}^{1}P_l\left(\xi\right)f d\xi, \tag{30}$$

where $\xi=v_\parallel/v$. It can be also expressed in terms of λ as

$$f_l=\frac{2l+1}{4}\sum_\varsigma\int_0^{1/B}\frac{d\lambda B}{\sqrt{1-\lambda B}}P_l\left(\sqrt{1-\lambda B}\right)f. \tag{31}$$

The sum over $\varsigma=\pm 1$ is required in the pitch-angle integrals in order to include both counter- and co-passing particles. The speed-dependent variable f_l for $l=1$ relates to the velocity of a moving frame where momentum would be conserved by the first term in the collision operator alone.

Note that the operator in Eq. (29) is expanded in Legendre polynomials which are the eigenfunctions for the Lorentz collision operator [with $l\,(l+1)$ being the eigenvalue] in the case of rotational symmetry in velocity space. Because of this, the collision operator only affects the pitch-angle dependence of the first three harmonics of the distribution function. This yields two simplifications in the calculation. The speed v can be treated as a parameter in the DKE which simplifies its solution as will be discussed in Chapters III to V. Also, the series can be truncated at $l = 2$ because the higher order terms are smaller in the small mass ratio expansion [18]. Moreover, for the higher order terms $C^l \sim -l\,(l+1)\,\bar{\nu}/2$. Because of this, by expanding the distribution function in Legendre polynomials, only the first three terms will be required for the proposed collision operator to conserve density, momentum and energy [19].

The first three terms in the momentum restoring term, (for l = 0, 1, 2), correspond to the polynomials

$$P_0\left(\frac{v_\parallel}{v}\right) = 1, \tag{32}$$

$$P_1\left(\frac{v_\parallel}{v}\right) = \frac{v_\parallel}{v} = \varsigma\sqrt{1-\lambda B}, \tag{33}$$

$$P_2\left(\frac{v_\parallel}{v}\right) = \frac{1}{v^2}\left(v_\parallel^2 - \frac{1}{2}v_\perp^2\right) = 1 - \frac{3}{2}\lambda B. \tag{34}$$

Then, if the distribution f can be expanded in terms of the pitch angle eigenfunctions P_l, the second term in Eq. (28) clearly makes the moments in Eqs. (6)–(10) vanish. Thus, the collisional effects in f_1 will not introduce new terms in the balance equations and density, momentum and energy will be conserved in the scattering process.

Another simplification can be made at this point. The particle continuity equation Eq. (2), for bounce time scales is

$$0 \simeq \frac{\partial n}{\partial t} = -n\nabla\cdot\mathbf{V} - \mathbf{V}\cdot\nabla n, \tag{35}$$

which leads to an incompressible flow to lowest order for density constant along the magnetic field lines. As a consequence, by noting that

$$\nabla \cdot \mathbf{V} = (\mathbf{B} \cdot \nabla) \left(V_\| / B \right) = 0, \tag{36}$$

the incompressibility constraint can then be satisfied by defining a parallel flow variable

$$V_\| \left(\ell, t \right) / B \left(\ell \right) \equiv U \left(\psi, t \right), \tag{37}$$

which is constant on a given magnetic flux surface. Also, for the simple model given in Eq. (23)

$$\nabla \times (\mathbf{B} \times \mathbf{V}) = 0. \tag{38}$$

Introducing these simplifications in Eq. (27), the flow drive term reduces to $G = (m/T) \, \mathbf{V} \cdot \nabla \mathrm{ln} B$. Using the relation

$$v_\| \mathbf{b} \cdot \nabla \left(v_\| B \right) = \left(v_\|^2 - v_\perp^2 / 2 \right) \mathbf{B} \cdot \nabla \ln B, \tag{39}$$

the flow drive term reduces to

$$\left(v_\|^2 - \frac{v_\perp^2}{2} \right) G = \frac{2}{v_t^2} v_\| \mathbf{b} \cdot \nabla \left(v_\| B \right) U. \tag{40}$$

With these simplifications, the drift kinetic equation becomes

$$\frac{\partial F}{\partial t} + v_\| \mathbf{b} \cdot \nabla \left[F + \frac{2}{v_t^2} v_\| B U f_M \right] - C \left(F \right) = \frac{v_\|}{p} \left(\mathbf{b} \cdot \nabla \cdot \Pi_\| \right) f_M. \tag{41}$$

The first term represents the temporal evolution of the kinetic distortion while the second term reflects spatial variations of both the kinetic distortion and the free streaming flow of the circulating particles. The effects of collisions, which drive particles through a perpendicular diffusion process (in velocity space) into or out of trapped particle space, are reflected in the collision operator in the third term.

The parallel viscous force is a source for the evolution of the unknown distribution F and hence will be present in the solution. However, the Chapman-Enskog constraints given in Eq. (17) and the conservation properties of the collision operator in Eq. (28) will allow an expression to be obtained for $\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle$ that has no explicit dependence on F. In the next Chapters, Eq. (41) will be solved by a perturbation technique in the low collisionality (banana) regime.

III. LOWEST ORDER DISTORTION F_0

The drift-kinetic equation (DKE) obtained in the previous Chapter, Eq. (41), can be solved using a standard perturbation technique. In the low collisionality banana regime trapped particles can complete their orbits before being scattered by collisions. Then, the parameter $\nu_* = \nu/\epsilon^{3/2}\omega_b$ is small and will be used as an expansion parameter for F. In defining ν_*, ν is the (90° scattering) collision frequency and $\omega_b \sim v_{th}/L$ is the bounce frequency.

A. Lowest order solution: F_0

The kinetic distortion is expanded in terms of the parameter ν_*:

$$F = F_0 + \nu_* F_1 + \dots . \tag{42}$$

The smallness of this parameter, as mentioned before, defines the banana regime. That is, the time required for a particle to complete its orbit is less than the "effective" collision time (given by $\tau_{eff} = \nu_{eff}^{-1} = \epsilon^{3/2}\tau$) which is the characteristic time to scatter a particle out of the trapped region of velocity space. If the time derivative is assumed to be order ν_*, the lowest order DKE is

$$v_\parallel \mathbf{b} \cdot \nabla \left[F + \frac{2}{v_t^2} v_\parallel B U f_M \right] = 0. \tag{43}$$

Since for the model magnetic field considered $\mathbf{b} \cdot \nabla = \partial/\partial\ell$, the term in square brackets cannot depend on ℓ. Thus, the lowest order distortion is given by

$$F_0 = -\frac{m}{T} v_\parallel B U f_M + g(v, \lambda, \varsigma, t). \tag{44}$$

Here, g is an integration "constant," which is a function of all the variables of the system other than the spatial ℓ over which the integration was performed. The first term in Eq. (44) represents the free streaming part of F_0 while the second term is a collisional correction. Note that the only difference up to here with a usual static calculation (see Refs. [1]–[4]) is in the time dependence of the function g. The only variables left after the integration in the steady state case are speed v and pitch angle λ.

B. First order drift-kinetic equation

In order to solve for the integration constant g we consider the next order (in ν_*) DKE:

$$v_{||}\mathbf{b}\cdot\nabla F_1+\frac{\partial F_0}{\partial t}-C\left(F_0\right)=v_{||}\frac{1}{p}\mathbf{b}\cdot\nabla\cdot\Pi_{||}f_M. \tag{45}$$

Only the spatial variations of F_1 appear in the next order equation and thus an appropriate loop integration in ℓ can annihilate it. Since $v_{||}\mathbf{b}\cdot\nabla F_1=v_{||}\partial F_1/\partial\ell$, consider the integral

$$\oint\frac{d\ell}{v_{||}}=\begin{cases}\int_0^L d\ell/v_{||}, & \text{untrapped particles,}\\ \sum_{\varsigma}\int_{-\ell_c}^{\ell_c} d\ell/\left|v_{||}\right|, & \text{trapped particles.}\end{cases} \tag{46}$$

Here $\pm\ell_c$ are the turning points of the closed, trapped particle orbits where $v_{||}\rightarrow 0$. On the other hand, circulating particles can travel through the whole cylinder covering the distance L. These orbits are closed due to the periodicity of the problem. Then, if F_1 is a smooth distribution, its gradient vanishes upon integration over these closed orbits. Eliminating the first term in Eq. (45) makes it possible to obtain a solution for F_0 without having to explicitly solve for F_1. The bounce-averaged first order equation provides a constraint that can be solved for the integration constant in the lowest order solution.

For trapped particles, to take account of density conservation at the tip of the bounce orbits we must have $g_t\left(\varsigma\right)=-g_t\left(-\varsigma\right)$. Then, for g_t even, we have $g_t=0$ at

ℓ_c. Since g does not depend on ℓ, $g_t = 0$ for all values of ℓ.

For circulating particles, a partial differential equation for g is obtained from

$$\frac{\partial}{\partial t}\oint \frac{d\ell}{v_\|}F_0 - \oint \frac{d\ell}{v_\|}C\left(F_0\right) = f_M \frac{1}{p}\oint \frac{d\ell}{v_\|}\mathbf{b}\cdot\nabla\cdot\Pi_{||}. \tag{47}$$

The collisional term can be simplified by noting that for the free streaming term we have

$$C\left(\sqrt{1-\lambda B}\right) = -\bar{\nu}\, v_\| + \sum_l P_l\left[C^l\left(F_l\right) + \frac{l\left(l+1\right)}{2}\bar{\nu}F_l\right]. \tag{48}$$

Since $v_\| = v\,P_1$

$$v_{\|l} = \frac{3}{4}v\sum_l^3 \int_0^\lambda \frac{d\lambda B\,P_1\,P_l}{\sqrt{1-\lambda B}} = v\,\delta_{1l}, \tag{49}$$

and

$$C\left(v_\|\right) = -\bar{\nu}\,v_\| + \bar{\nu}\,v\,P_1 = 0. \tag{50}$$

Thus, the free streaming part does not contribute; all the collisional effects are included in g, as is desired.

Using this result, the DKE for circulating particles is reduced to a differential equation for g

$$\frac{\partial g}{\partial t}\left\langle\frac{B}{v_\|}\right\rangle - \frac{\oint \frac{d\ell}{v_\|}C_R\left(g\right)}{\oint \frac{d\ell}{B}} = \frac{1}{p}\left\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\|\right\rangle f_M + \frac{m}{T}f_M\frac{\partial U}{\partial t}\left\langle B^2\right\rangle, \tag{51}$$

in which the flux surface average is defined as

$$\langle A\rangle \equiv \oint \frac{d\ell}{B}A/\oint \frac{d\ell}{B}. \tag{52}$$

Equation (51) is an inhomogeneous partial differential equation in time (first order) and pitch angle (second order) since the collision operator in Eq. (28) does not affect the energy structure of g. Thus, the speed variable v is a parameter and Eq. (50) is to be understood as valid for each energy.

In order to solve for the λ-dependence, a Laplace transform in time can be taken. Then, frequency will be treated also as a parameter and the equation to be solved becomes

$$-i\omega\widehat{g}\left\langle\frac{B}{v_\|}\right\rangle - \frac{\oint \frac{d\ell}{v_\|} C\left(\widehat{g}\right)}{\oint \frac{d\ell}{B}} = \frac{1}{p} f_M \langle \widehat{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\|}\rangle + \left\langle\frac{B}{v_\|}\right\rangle g\left(0\right) - \frac{m}{T} f_M \left\langle B^2\right\rangle \left[i\omega\widehat{U} + U_0\right]. \tag{53}$$

Here hats denote Laplace-transformed quantities with transform variable $-i\omega$ defined by:

$$\hat{h}\left(\omega\right) = L\left\{h\left(t\right)\right\} \equiv \int_{0^-}^{\infty} dt\, e^{i\omega t} h\left(t\right). \tag{54}$$

The initial condition for the flow is $U_0 \equiv U(t=0)$. The λ dependence on the right side of Eq. (53) is only through the initial pitch-angle structure of the collisional correction g. The speed dependence in $g\left(0\right)$ is arbitrary since the collision operator does not operate on it; thus, it is conveniently chosen as [9]

$$g\left(v,\lambda,t=0\right) \equiv f_M\left(v\right) g_0\left(\lambda\right), \tag{55}$$

to match the separable structure of the rest of the terms in the source.

Morris et al. [9] were the first to recognize that this should be treated as an initial value problem. The collisional diffusion into trapped space of a distribution of untrapped particles will depend strongly (at least for early times) on how close the initial distribution is located relative to the boundary between the two types of particles. An initial distribution of particles is expected to damp more rapidly if it is peaked close to the trapped-circulating boundary than if particles are introduced far away from it since then the portion of phase space they have to diffuse through is larger.

Using Eq. (55), Eq. (53) can be written as

$$-i\omega\widehat{g}\left\langle\frac{B}{v_\|}\right\rangle-\frac{\displaystyle\oint\frac{d\ell}{v_\|}C\left(\widehat{g}\right)}{\displaystyle\oint\frac{d\ell}{B}} = \frac{1}{p}f_M\langle\widehat{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\|}\rangle-\frac{m}{T}f_M\left\langle B^2\right\rangle i\omega\widehat{U}$$
$$+\left\langle\frac{B}{v_\|}\right\rangle g_0f_M-\frac{m}{T}f_M\left\langle B^2\right\rangle U_0. \qquad (56)$$

For the bounce-averaged collision operator term we have

$$\frac{C\left(\widehat{g}\right)}{v_\|}=\frac{1}{\varsigma v}\left\{2\bar{\nu}\,\frac{\partial}{\partial\lambda}\frac{\lambda}{B}\sqrt{1-\lambda B}\,\frac{\partial\widehat{g}}{\partial\lambda}+\frac{1}{\sqrt{1-\lambda B}}\sum_l P_l\left[C^1\left(\widehat{g}_l\right)+\bar{\nu}\,\widehat{g}_l\right]\right\}. \qquad (57)$$

In order to calculate $\widehat{g}_l$, we note that $\widehat{g}$ only has a P_1 term if expanded in Legendre polynomials. This can easily seen by multiplying Eq. (44) by $\int_{-1}^{1}d\xi\,P_l\left(\xi\right)$ for $l=0,2$. Since the distortion F_0 must satisfy the conditions in Eq. (17) and $v_\|\sim P_1$, we have

$$\int_{-1}^{1}d\xi\,P_l\left(\xi\right)\widehat{g}=\int_{-1}^{1}d\xi\,P_l\left(\xi\right)F_0+\frac{m}{T}\int_{-1}^{1}d\xi\,P_l\left(\xi\right)v_\|B\widehat{U}f_M=0. \qquad (58)$$

The only nonzero moment of $\widehat{g}$ (with $l\leq 2$) is the parallel momentum moment; it is given by

$$\sum_\varsigma\int_0^{1/B}d\lambda BP_1\,\widehat{g}=\widehat{U}\frac{m}{T}\sum_\varsigma\int_0^{1/B}d\lambda B^2P_1v_\|f_M. \qquad (59)$$

This condition precisely yields the Chapman-Enskog constraint ($\int d^3v\,v_\|F_0=0$)

$$\int d^3v\,v_\|\widehat{g}=\frac{m}{T}B\widehat{U}\int d^3v\,v_\|^2f_M=B\widehat{U}n, \qquad (60)$$

which will be used in the next Section to solve for the parallel viscous force.

Defining a flow-like variable $\widehat{V}\left(\psi,v,\omega\right)\equiv\widehat{g}_1/B$ [8]

$$\widehat{V}\left(\psi,v,\omega\right)=\frac{3}{4}\sum_\varsigma\varsigma\int_0^{\lambda_c}d\lambda\,\widehat{g}, \qquad (61)$$

we obtain

$$\oint\frac{d\ell}{v_\|}C\left(\widehat{g}\right)=\frac{2}{\varsigma v}\bar{\nu}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial\widehat{g}}{\partial\lambda}+\frac{1}{v}\oint d\ell\,B\left[C^1\left(\widehat{V}\right)+\nu\widehat{V}\right]. \qquad (62)$$

Introducing this result in Eq. (56), the equation to solve for $\widehat{g}$ becomes

$$-i\omega\frac{1}{2}\left\langle\frac{B}{\sqrt{1-\lambda B}}\right\rangle\widehat{g}-\bar{\nu}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial\widehat{g}}{\partial\lambda}=\widehat{S}\left(v,\lambda,i\omega\right), \tag{63}$$

where the drives and initial conditions are included in the source term

$$\begin{aligned}\widehat{S}\left(v,\lambda,i\omega\right) \;=\; & \frac{v\varsigma}{2}\left\{\frac{1}{p}\langle\widehat{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}}\rangle f_M-i\omega\frac{m}{T}\widehat{U}f_M\left\langle B^2\right\rangle+\right.\\ & \left. g_0\left\langle\frac{B}{v_{\|}}\right\rangle f_M-\frac{m}{T}f_MU_0\left\langle B^2\right\rangle+\frac{1}{v}\left\langle B^2\right\rangle\left[C^1\left(\widehat{V}\right)+\nu\widehat{V}\right]\right\}.\end{aligned} \tag{64}$$

Equation (63) will be solved for the pitch angle dependence of $\widehat{g}\left(v,\lambda,\varsigma,t\right)$. The speed dependence will be determined by the equation itself, in which v is a parameter; thus, the solution obtained for $\widehat{g}$ will be valid for each v. The time dependence will be obtained by inverting the Laplace transform $-i\omega\rightarrow\partial/\partial t$. Since the drives for $\widehat{g}$ include $\langle\widehat{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}}\rangle$, which is a constant in λ and v, it will be part of the solution. Thus, the expression that will be obtained for $\widehat{F}_0$ will contain this term.

IV. STATIC CALCULATION

The problem given by Eqs. (63)–(64) consists of an ordinary, second order differential equation for the evolution of the function $g(v,\lambda,t)$, or $\widehat{g}(v,\lambda,i\omega)$, with an initial condition given by $f_M(v)\, g_0(\lambda)$. In general, the source in Eq. (64) depends on all variables except ℓ. While all terms depend on speed, only the second term depends on frequency and only the initial condition term on λ. These two terms make the calculation more involved than a steady state case in which $\omega = 0$ and initial conditions are not necessary. Because of this, we first treat the static problem to illustrate the methodology that will be used to solve the complete problem in Chapter V.

A. Pitch-angle solution

The steady state limit of Eqs. (63)–(64) is obtained by considering $\omega = 0$, i. e. $\partial/\partial t \to 0$ or $t \to \infty$. For long times, the initial conditions are damped away and the system is in an equilibrium state. This limit of Eqs. (63)–(64) yields

$$\bar{\nu}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial g}{\partial\lambda} = -\frac{v\varsigma}{2}\left\{\frac{1}{p}\left\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle f_M + \frac{1}{v}\left\langle B^2\right\rangle\left[C^1(V) + \bar{\nu}V\right]\right\}. \quad (65)$$

The solution of the homogeneous equation is easily obtained by integrating twice over λ; it is here defined as Λ:

$$\Lambda(\lambda) \equiv \int_{\lambda}^{\lambda_c}\frac{d\lambda\left\langle B^2\right\rangle^{1/2}}{\left\langle\sqrt{1-\lambda B}\right\rangle}H(\lambda_c-\lambda), \quad (66)$$

where the Heaviside step function is introduced in order to keep $g_t = 0$ for trapped particles. Then, the integral over all pitch angle space for g is effectively up to the critical value $\lambda_c \equiv 1/B_{max}$. The step-function will be omitted from now on to simplify the notation. Since all terms on the right side depend on speed, the solution can be written in the convenient form

$$g = \frac{\varsigma v}{2} Y(v)\, \Lambda(\lambda). \tag{67}$$

As was noted before, the speed dependence of g is not affected by the differential operator and can be included in the coefficient $\varsigma\, v\, Y(v)/2$ which is determined by the structure of the equation itself. Introducing the proposed expansion in Eq. (65) one obtains

$$Y(v) = \frac{1}{\bar{\nu} \left\langle B^2 \right\rangle^{1/2}} \left\{ \frac{1}{p} \left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle f_M + \frac{1}{v} \left\langle B^2 \right\rangle \left[C^1 (V) + \bar{\nu} V \right] \right\}. \tag{68}$$

This coefficient relates the energy dependence of the solution in terms of the (also speed dependent) drives. With this solution we can calculate the flow-like variable V as

$$V = \frac{3}{4} \sum_{\varsigma} \varsigma \int_0^{\lambda_c} d\lambda\, g = \frac{3}{4} v Y(v) \int_0^{\lambda_c} d\lambda\, \Lambda(\lambda). \tag{69}$$

Integrating by parts once, the coefficient in $V(v)$ is

$$\int_0^{\lambda_c} d\lambda\, \Lambda(\lambda) = - \int_0^{\lambda_c} d\lambda \int_{\lambda_c}^{\lambda} \frac{d\lambda \left\langle B^2 \right\rangle^{1/2}}{\left\langle \sqrt{1 - \lambda B} \right\rangle} = \int_0^{\lambda_c} \frac{\lambda\, d\lambda \left\langle B^2 \right\rangle^{1/2}}{\left\langle \sqrt{1 - \lambda B} \right\rangle}. \tag{70}$$

At this point it is convenient to define the fraction of circulating particles [4]

$$f_c = \frac{3}{4} \left\langle B^2 \right\rangle^{1/2} \int_0^{\lambda_c} d\lambda\, \Lambda(\lambda) = \frac{3}{4} \left\langle B^2 \right\rangle \int_0^{\lambda_c} \frac{\lambda\, d\lambda}{\left\langle \sqrt{1 - \lambda B} \right\rangle}, \tag{71}$$

and write

$$V = \frac{f_c}{\bar{\nu} \left\langle B^2 \right\rangle} \left\{ \frac{1}{p} \left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle v f_M + \left\langle B^2 \right\rangle \left[C^1 (V) + \bar{\nu} V \right] \right\}. \tag{72}$$

Then, the solution for F_0 is

$$F_0 = -\frac{m}{T} U B f_M + \frac{\varsigma}{2} \frac{\left\langle B^2 \right\rangle^{1/2}}{f_c} V(v) \Lambda(\lambda), \tag{73}$$

with V given by

$$\left\langle B^2\right\rangle \left(\frac{1}{f_c}-1\right)\bar{\nu}V=\left\{\frac{1}{p}\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle vf_M+\left\langle B^2\right\rangle C^1\left(V\right)\right\}. \tag{74}$$

We have thus obtained an expression for the kinetic distortion g in terms of the drives in its evolution equation. In the collisional drive for the evolution of g [square braket in Eq. (65)], the momentum restoring terms are calculated in terms of the energy dependence of the distribution itself contained in the quantity $V\left(v\right)$. Because of this, the result obtained in this section is not an explicit expression for g (or F_0). The solution given in Eq. (73) can be completely determined once the quantity V is calculated from Eq. (74), for which an expression for the viscous force would be needed. In the next subsection, we will illustrate how these equations can be combined with the conservation properties of the distribution function and the collision operator in order to obtain the parallel viscous force and thus the lowest order distortion.

B. Static closure

Equation (73) gives the solution for F_0 where the speed dependence is to be determined using Eq. (74). Because of the procedure followed in Section IV A [8], the parallel viscous force $\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle$ is part of the solution we obtained for F_0; it is precisely the term in the distribution function that is necessary to calculate the closure itself [see Eq. (13)]. This does not present an obstacle since we don't seek a solution for F_0 but instead a closure for $\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle$. To obtain it, we will use a conservation property of the collision operator and the constraint obtained in Eq. (60). By doing this, we construct an expression for the closure that is consistent with the Chapman-Enskog constraints and with the definition in Eq. (13) without having to explicitly solve for F_0.

As described above, to eliminate the differential term $C^1(V)$ in Eq. (74) and obtain an expression for the viscous force in terms of V, we use the momentum conserving property of the collision operator and require

$$\int d^3v\, v\, C^1(V) = 0. \tag{75}$$

Calculating this moment of both sides of Eq. (74) and rearranging terms we obtain

$$\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle = m\left\langle B^2\right\rangle \int d^3v\, \frac{v}{3}\bar{\nu}\, V\left(\frac{1}{f_c}-1\right). \tag{76}$$

Equation (60) implies a relation between the variable $V \sim \int d\lambda\, g$ and the parallel flow U. To obtain such a relation, we assume that V can be expanded in Laguerre polynomials ($L_0^{3/2}=1$, $L_0^{3/2}=5/2-mv^2/2T$,...) as

$$V(v) = \frac{m}{T}\frac{f_M}{n}v\sum_n V_n L_n^{3/2}. \tag{77}$$

This assumption is based on the fact that the dependence of the distribution function on energy is unaltered by the velocity-space angular effects considered in the problem. The coefficients in the expansion, obtained in Appendix A, are

$$V_n = \frac{n!\,\pi^{3/2}}{(n+3/2)!}\int_0^\infty dv\, v^3 V(v)\, L_n^{3/2}. \tag{78}$$

Equations 76 and 78 completely determine the solution for $\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle$ since the constraints in Eq. (17) relate the first three moments of g (and thus of V) with the parallel flow U. Thus, in general, the closure can be written as

$$\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle = m\left\langle B^2\right\rangle \int d^3v\, \frac{v^2}{3}\frac{\nu_\perp}{v_{th}^2}\frac{f_M}{n}\sum_n V_n L_n^{3/2}\left(\frac{1}{f_c}-1\right). \tag{79}$$

For consistency, only the $n=0,1,2$ moments of the distribution function are required since only these moments of the kinetic equation were calculated to obtain the set of fluid moment equations [Eqs. (2)–(4)]. Since the closure we want to obtain is going to be introduced in such a set, any higher order terms in the closure would be neglected at that point. Then, all terms $n > 2$ in the expansion Eq. (78) are neglected for consistency. In particular, for $n = 0$

$$V_0 = \frac{\pi^{3/2}}{(3/2)!} \int_0^\infty dv \, V \, v^3, \tag{80}$$

which, using Eq. (61) yields

$$V_0 = \sum_\sigma \sigma\pi \int_0^\infty dv \, v^3 \int_0^{\lambda_c} d\lambda \, g. \tag{81}$$

Using the condition $\int d^3v \, v_\| F_0 = 0$ given by Eq. (60) and

$$\int d^3v \, v_\|^2 f_M = \int d^3v \, \frac{v^2}{3} f_M = \frac{nT}{m}, \tag{82}$$

we obtain $V_0 = n \, U$.

By keeping only the first term $n = 0$ in Eq. (79), we obtain the static solution for the parallel viscous force

$$\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_\| \right\rangle \simeq mnU \left\langle B^2 \right\rangle \frac{f_t}{f_c} \int d^3v \, \frac{v^2}{3} \frac{\nu_\perp}{v_{th}^2} \frac{f_M}{n}, \tag{83}$$

or

$$\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_\| \right\rangle \simeq mn\mu_{00} U \left\langle B^2 \right\rangle. \tag{84}$$

Here, the viscosity coefficient μ_{nm} is defined as [4]

$$\mu_{mn} = \frac{f_t}{f_c} \int d^3v \, \frac{\nu_\perp}{v_{th}^2} \frac{v^2}{3} L_n^{3/2} L_m^{3/2} \frac{f_M}{n}, \tag{85}$$

and the fraction of trapped particles is $f_t \equiv 1 - f_c$. For electrons we have

$$\mu_{00e} \equiv \mu_e = \left[Z + \sqrt{2} - \ln\left(1 + \sqrt{2}\right) \right] \frac{f_t}{f_c} \nu_e \simeq 1.53 \frac{f_t}{f_c} \nu_e, \tag{86}$$

where the last approximate form is for $Z = 1$.

The fraction of trapped and circulating particles can be estimated for $\sqrt{\epsilon} = \sqrt{\Delta B/2B_{min}} << 1$. The smallness of this parameter represents the large aspect ratio limit in toroidal geometries. In this model it reflects the limit of small modulations of the magnetic field strength as one moves from the outboard to inboard side of the torus along the helical magnetic field lines and will be used in the following chapters as an expansion parameter. As one might expect, for small ΔB the fraction of trapped particles is small and almost all particles contribute to the flow. In this approximation, the fractions of trapped and circulating particles are given by [4]

$$f_c \approx 1 - 1.46\sqrt{\epsilon} + \mathcal{O}\left(\epsilon^{3/2}\right), \quad f_t \approx 1.46\sqrt{\epsilon}. \tag{87}$$

V. DYNAMIC CALCULATION

In the previous section we solved the static limit of Eqs. (63)–(64) in order to illustrate the procedure that is going to be carried out here while solving the dynamic case. As discussed before, the complication of the problem resides in the nature of the source terms on the right side of Eq. (63). Since a Laplace transform has been taken, the time dependence of g will not be an issue until we invert the transform. This permits us to solve for the pitch angle dependence first, keeping both frequency and speed as parameters, following the method in Section IV A. The differential equation for $\widehat{g}$ is still an ordinary differential equation but now the source on the right side has a λ-dependent term. In what follows will treat this term separately and extend the results obtained previously.

A. Expansion in Cordey eigenfunctions

To solve this initial value problem, it is convenient to separate the terms in the drives that were not treated in the static problem. We re-write Eqs. (63)–(64) as

$$-i\omega\frac{1}{2}\left\langle\frac{B}{\sqrt{1-\lambda B}}\right\rangle\widehat{g}_c-\bar{\nu}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial\widehat{g}_c}{\partial\lambda}=\frac{\varsigma v}{2}\left[\widehat{D}\left(v,i\omega\right)+f_M\left\langle\frac{B}{v_\parallel}\right\rangle g_0\left(\lambda\right)\right], \tag{88}$$

where the λ-independent drives, and thus "constants" in the equation, are

$$\widehat{D}\left(v,i\omega\right)\equiv\frac{1}{p}\langle\mathbf{B}\cdot\widehat{\nabla\cdot\mathbf{\Pi}_\parallel}\rangle f_M-\frac{m}{T}\left\langle B^2\right\rangle f_M\left(i\omega\widehat{U}+U_0\right)+\frac{1}{v}\left\langle B^2\right\rangle\left[C^1\left(\widehat{V}\right)+\nu\widehat{V}\right]. \tag{89}$$

As before, we seek a solution similar to Eq. (67) in which the λ dependence is given by the solution of the homogeneous equation. In this case, the function Λ has to be a solution of

$$\left\{i\omega\frac{d}{d\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle-\bar{\nu}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial}{\partial\lambda}\right\}\Lambda=0. \tag{90}$$

Equation (90) does not have an analytical solution. However, a numerical solution can be obtained by considering the eigenvalue problem

$$\frac{d}{d\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{d\Lambda_n}{d\lambda}=\kappa_n\frac{d}{d\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle\Lambda_n, \tag{91}$$

where Λ_n are the eigenfunctions and κ_n are the eigenvalues. The eigenfunctions that are solutions of such an equation, "Cordey" eigenfunctions [20], can be numerically generated and are usually employed to solve problems that involve the pitch-angle scattering term in combination with a linear term [13, 14, 17, 21] (in our case the time-derivative reflected in the coefficient $i\omega$). These functions form a complete, orthogonal set with an orthogonality condition given by

$$\int_0^{\lambda_c}\Lambda_n\Lambda_m\frac{\partial\left\langle\sqrt{1-\lambda B}\right\rangle}{\partial\lambda}d\lambda=\delta_{nm}\int_0^{\lambda_c}\Lambda_n^2\frac{\partial\left\langle\sqrt{1-\lambda B}\right\rangle}{\partial\lambda}d\lambda. \tag{92}$$

The functions Λ_n for n odd vanish between $-\ell_c$ and ℓ_c (i. e., for trapped particles) and are even in ς for circulating particles [13, 20]. Thus, they are appropriate for the complete description of the solution inside the trapped region where $\widehat{g_t}=0$ and in the untrapped region for $\widehat{g_c}\neq 0$. At the boundary, the functions are continuous. Hence, the subindex on $\widehat{g_c}$ is not necessary and will be dropped from now on. Thus, we propose to project the solution in Cordey eigenfunction as done in references [13, 14, 17, 20, 21]:

$$\widehat{g}=\frac{\varsigma v}{2}\sum_{n=1}^{\infty}\widehat{Y}_n(v,i\omega)\Lambda_n(\lambda). \tag{93}$$

The operator in Eq. (90) is a combination of "time-dependence" and pitch-angle scattering. Thus, the eigenfunctions are similar in structure and reduce to the Legendre polynomials for a homogeneous magnetic field. Introducing the proposed expansion in Eq. (93) and using the eigenfunction equation [Eq. (91)], one obtains

$$\sum_{n=1}^{\infty}\widehat{Y}_n\bar{\nu}\left(\kappa_n-\frac{i\omega}{\bar{\nu}}\right)\frac{\partial}{\partial\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle\Lambda_n=-\widehat{D}\left(v,i\omega\right)-f_M\left\langle\frac{B}{v_\parallel}\right\rangle g_0\left(\lambda\right). \tag{94}$$

In order to obtain $\widehat{Y}_n$, we multiply both sides by Λ_m and integrate over passing particle space. Then, the orthogonality condition Eq. (92) can be used and one obtains

$$\widehat{Y}_n = -\frac{1}{\bar{\nu}}\left[\frac{\eta_n \widehat{D}}{\kappa_n - i\omega/\bar{\nu}} + \frac{m}{T} f_M \frac{\alpha_n}{\kappa_n - i\omega/\bar{\nu}}\right]. \tag{95}$$

The coefficients η_n and α_n are calculated from

$$\eta_n = \frac{\int_0^{\lambda_c} \Lambda_n d\lambda}{\int_0^{\lambda_c} \Lambda_n^2 \frac{\partial}{\partial\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle d\lambda}, \tag{96}$$

$$\alpha_n = \frac{\int_0^{\lambda_c} d\lambda \left\langle \frac{B}{v_\parallel}\right\rangle g_0(\lambda)\,\Lambda_n}{\int_0^{\lambda_c} \Lambda_n^2 \frac{\partial}{\partial\lambda}\left\langle\sqrt{1-\lambda B}\right\rangle d\lambda}. \tag{97}$$

Note that even though these coefficients do not depend on λ, the initial structure of the distribution in pitch angle will enter as a drive for the viscosity through the α_n integrals.

With the expression for $\widehat{g}$ in Eq. (93) the flow variable $\widehat{V}$ is

$$\widehat{V} = \frac{3}{4}\sum_\varsigma \varsigma \int_0^{\lambda_c} d\lambda\, \widehat{g} = \frac{3}{4} v\, \widehat{Y}_n(v) \int_0^{\lambda_c} d\lambda\, \Lambda(\lambda), \tag{98}$$

which we write as

$$\widehat{V} = v\frac{1}{\bar{\nu}}\frac{1}{\langle B^2\rangle}\left(\widehat{f}_c\, \widehat{D} + \left\langle B^2\right\rangle^{1/2} f_M \frac{m}{T}\,\widehat{f}_g\right). \tag{99}$$

The fraction of trapped particles in this case depends on frequency and speed; it is defined as [8]

$$\widehat{f}_c \equiv \widehat{f}_c(\omega, v) = \sum \frac{\gamma_n}{\kappa_n - i\omega/\bar{\nu}}, \tag{100}$$

where

$$\gamma_n = -\frac{3}{4}\left\langle B^2\right\rangle \eta_n \int_0^{\lambda_c} d\lambda\, \Lambda_n. \tag{101}$$

The initial distribution is buried in the coefficient

$$\chi_n = -\frac{3}{4}\left\langle B^2\right\rangle^{1/2}\alpha_n\int_0^{\lambda_c} d\lambda\,\Lambda_n, \tag{102}$$

and we defined

$$\widehat{f}_g \equiv \widehat{f}_g\left(\omega, v\right) = \sum \frac{\chi_n}{\kappa_n - i\omega/\bar{\nu}}, \tag{103}$$

by analogy with the definition of $\widehat{f}_c$. Substituting the drive $\widehat{D}$ in Eq. (99) we obtain

$$\begin{aligned} \bar{\nu}\widehat{V}\left(\frac{1}{\widehat{f}_c} - 1\right)\left\langle B^2\right\rangle \; &= \; v f_M \frac{1}{p}\langle \widehat{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\|}\rangle - v\frac{m}{T}\left\langle B^2\right\rangle\left(i\omega\widehat{U} + U_0\right) f_M \\ &- v\frac{m}{T}\left\langle B^2\right\rangle\left\{\left\langle B^2\right\rangle^{-1/2} f_M\frac{\widehat{f}_g}{\widehat{f}_c} + \frac{1}{v}C^1\left(\widehat{V}\right)\right\}. \end{aligned} \tag{104}$$

The solution for the Laplace-transformed kinetic distortion is then given by

$$\widehat{g} = -\frac{\varsigma}{2}\sum_{n=1}^{\infty}\left[\frac{\eta_n}{\kappa_n - i\omega/\bar{\nu}}\frac{1}{\widehat{f}_c}\left(\widehat{V}\left\langle B^2\right\rangle - \left\langle B^2\right\rangle^{1/2} f_M\frac{v}{\bar{\nu}}\frac{m}{T}\widehat{f}_g\right) + \frac{m}{T}\frac{v}{\bar{\nu}}f_M\frac{\alpha_n}{\kappa_n - i\omega/\bar{\nu}}\right]\Lambda_n, \tag{105}$$

where the flow-like variable $\widehat{V}$ is calculated from Eq. (104).

Once again, the viscous force is part of the expression for $\widehat{g}$ through the variable $\widehat{V}$. In the next section, the closure is obtained following a procedure similar to the one introduced in Section IV B.

B. Dynamic closure for $\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_\|}\rangle$

As in the static case, the solution for the kinetic correction $\widehat{g}$ is in terms of the parallel viscous force through $\widehat{V}$. This relation is given by Eq. (104); to solve for it we proceed as before. We start by considering the property from Eq. (75)

$$\int d^3v\, v\, C^1\left(\widehat{V}\right) = 0, \tag{106}$$

to eliminate that term and at the same time introduce the velocity space integrals of $\widehat{V}$ which we can relate to the flow $\widehat{U}$ as in the steady state calculation. Calculating this moment of Eq. (104) we obtain an expression for the parallel viscous force in terms of the parallel flow, the "momentum restoring flow" $\widehat{V}$, and the initial conditions

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}} \right\rangle = mn\left\langle B^2\right\rangle \left\{ \int d^3v\, \frac{v}{3}\bar{\nu}\widehat{V}\left(\frac{1}{\widehat{f}_c}-1\right) + i\omega\widehat{U} \right.$$
$$\left. + U_0 - \left\langle B^2\right\rangle^{-1/2} \frac{m}{T}\int d^3v\, \frac{v^2}{3}\frac{f_M}{n}\frac{\widehat{f}_g}{\widehat{f}_c} \right\}. \tag{107}$$

From Section IV B, and Appendix A, we have a method for relating each term in an expansion of $\widehat{V}$ with the parallel flow. The expansion proposed in Eq. (77) is valid also in this dynamic case. If we keep only the first term in the Laguerre polynomial expansion ($V_0 \simeq Un$) as before, we obtain a frequency-dependent closure

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_{\|}} \right\rangle = nm\left\langle B^2\right\rangle \left[\widehat{U}\,\widehat{\upsilon}\left(\omega\right) + U_0 + \widehat{\Upsilon}\left(\omega\right)\right], \tag{108}$$

which coincides with the result in Ref. [8] if initial conditions are neglected. The first term shows the usual proportionality between the viscous force and the parallel flow. The coefficient $\widehat{\upsilon}\left(\omega\right)$ is defined as

$$\widehat{\upsilon}\left(\omega\right) = \frac{m}{T}\int d^3v\, \bar{\nu}\frac{v^2}{3}\frac{f_M}{n}\frac{\widehat{f}_t}{\widehat{f}_c}, \tag{109}$$

where the fraction of trapped particles is defined as

$$\widehat{f}_t = 1 - \left(1 - \frac{i\omega}{\bar{\nu}}\right)\widehat{f}_c. \tag{110}$$

The initial pitch angle distribution is contained in the last term which is defined as

$$\widehat{\Upsilon}\left(\omega\right) = \frac{m}{T}\int d^3v\, \frac{v^2}{3}\frac{f_M}{n}\frac{\widehat{f}_g}{\widehat{f}_c}. \tag{111}$$

Note that for long times, when initial conditions are completely damped and can thus be neglected, one can use the result [8]

$$f_c \equiv \widehat{f}_c\left(\omega = 0\right) = \sum \frac{\gamma_n}{\kappa_n} = \frac{3\left\langle B^2\right\rangle}{4}\int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle\sqrt{1-\lambda B}\right\rangle}, \tag{112}$$

and the steady state limit of Eq. (108) yields the standard result

$$\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle = nm\mu \left\langle B^2 \right\rangle U, \tag{113}$$

which is precisely the expression obtained in Eq. (84) in the previous chapter by solving the steady state problem directly.

In this dynamical case, the fraction of trapped and circulating particles, $\widehat{f}_t$ and $\widehat{f}_c$, depend on frequency but reduce to the usual ones in the $\omega \ll \bar{\nu}$ limit. No physical interpretation of the fact that $\widehat{f}_c + \widehat{f}_t$ is a frequency dependent function is necessary since these quantities are defined only to obtain a simple expression for $\left\langle \mathbf{B} \cdot \widehat{\nabla \cdot \Pi_{\|}} \right\rangle$ similar in structure to the steady state solution.

Equation (108) is valid for any time scale, since no approximations have been introduced so far. From it, a damping rate can be estimated numerically. For an explicit time-dependent expression, neither an analytical nor numerical Laplace inverse transform are trivial to perform since the expression involves infinite sums of terms that depend on integrals over the (numerically generated) eigenfunctions. Moreover, some of these infinite sums reside on the denominator and thus should be calculated to high accuracy if any poles of the response are to be found. However, it will be shown that in a small ϵ expansion the time-dependent closure can be obtained analytically.

C. Time dependent solution for small field modulations

In general, an analytical inversion of the Laplace transform in Eq. (108) cannot be obtained. However, for small field variations ($\Delta B \ll B_{min}$) an analytical solution can be obtained through a power series expansion in the small parameter $\sqrt{\epsilon}$. In order to make such an expansion, we invoke the $f_t \sim \sqrt{\epsilon} \ll 1$ result obtained in Section IV B. This ordering has also been obtained by various authors [4, 8] in both dynamic and static situations. For the dynamic case a similar behavior in the function f_t, which is now frequency dependent, can be assumed. Based on this argument, we propose an expansion in powers of $\widehat{f}_t$ and will expect the lowest order terms to dominate. The relevant factors to be inverted are $\widehat{f}_t/\widehat{f}_c$ and $\widehat{f}_g/\widehat{f}_c$, for which we propose the following expansions:

$$\frac{\widehat{f}_t}{\widehat{f}_c} \simeq \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f}_t + \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f}_t^{\,2} + ..., \tag{114}$$

$$\frac{\widehat{f}_g}{\widehat{f}_c} \simeq \left(1 - \frac{i\omega}{\bar{\nu}}\right) \left(1 + \widehat{f}_t\right) \widehat{f}_g + \dots . \tag{115}$$

Introducing Eqs. (114) and (115) in the closure given by Eq. (108), a much simpler expression for the parallel viscous force is obtained and the Laplace transform can be inverted term by term. In particular, to lowest order in $\widehat{f}_t$ we have

$$\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle \simeq nm \left\langle B^2 \right\rangle \frac{m}{T} \int d^3v \, \frac{v^2}{3} \frac{f_M}{n} \bar{\nu} L^{-1} \left\{ \left(1 - \frac{i\omega}{\bar{\nu}}\right) \widehat{f}_t \widehat{U} \right\}, \tag{116}$$

in which L^{-1} is the inverse Laplace transform and initial conditions have been neglected for simplicity but can be easily introduced using Eq. (B5) in Appendix B. After calculating the inverse Laplace transform of the term in curly brackets in Eq. (116) (see Appendix B), the time dependent closure for the parallel viscous force in this small ϵ limit is

$$\begin{aligned} \left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle \; = \; & \left\langle B^2 \right\rangle mn \int d^3v \, \frac{v^2}{3} \frac{f_M}{n} \bar{\nu} \frac{m}{T} \left\{ U\left(t\right)\left(1 - f_c\right) + \frac{1}{\bar{\nu}} \frac{\partial U\left(t\right)}{\partial t} \left(1 - \sum \gamma_n\right) \right. \\ & \left. + \sum \frac{\gamma_n}{\kappa_n} \left(\kappa_n - 1\right)^2 \int_0^t \frac{dU}{d\tau} e^{-\bar{\nu}\kappa_n (t-\tau)} d\tau \right\}. \end{aligned} \tag{117}$$

This equation exhibits the explicit behavior in time of $\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle$. The first term is proportional to the parallel flow and will be dominant in the long time asymptotic limit. The second term contains the time variation of the flow and is important only for times of the order of $1/\bar{\nu}$. The last term contains the time-history through the convolution integral of the time-dependent trapped particle fraction and the intrinsic time dependence of the flow of circulating particles. To higher order in this expansion, this last term develops a series in powers of $\bar{\nu}t$ inside the (convolution) time integral. For estimates of the coefficients in Eq. (117) see Eqs. (124)–(127) in Section V D.

The time dependent solution given in Eq. (117) can also be obtained following an alternate procedure. From the static calculation

$$\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{\|}} \rangle \sim f_t \sim \sqrt{\epsilon}. \tag{118}$$

Based on this, one can go back to Eqs. (63)–(64) and propose $g = g_c^0 + \sqrt{\epsilon}\, g_c^1 + \epsilon\, g_c^2 + \ldots$. Then, to lowest order, the term in the source $\widehat{S}$ due to the parallel viscous force can be neglected. The resulting differential equation for $\widehat{g}_c$ is

$$-i\omega \left\langle \frac{B}{v_{||}} \right\rangle \widehat{g}_c - C\left(\widehat{g}_c\right) = f_M \frac{m}{T} \left\langle B^2 \right\rangle \left(\bar{\nu} - i\omega\right) \widehat{U} \ , \tag{119}$$

which can be solved with the eigenfunction expansion introduced in Section V A to obtain a lowest order $\widehat{g_c^0}$. With the kinetic correction obtained in this way, the lowest order distortion $\widehat{F_0}$ is completely determined to this order. The viscous force is not present in this solution but can be obtained from the bounce averaged next order DKE as

$$\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{\|}} \rangle = \left\langle mB \int d^3v\, m v_{\|}^2 \mathbf{b} \cdot \nabla \widehat{F_1} \right\rangle, \tag{120}$$

or, in terms of the solution $\widehat{F_0}$

$$\langle \widehat{\mathbf{B} \cdot \nabla \cdot \Pi_{\|}} \rangle = \left\langle mB \int d^3v\, m v_{\|} \left\{ C(\widehat{F_0}) - i\omega \widehat{F_0} \right\} \right\rangle. \tag{121}$$

This solution can be introduced as a lowest order approximation in the DKE and used to solve for a next order result. This iterative procedure leads to the same results as in Eq. (117) when the inverse Laplace transform is calculated. It is described here because it provides an easy method to numerically obtain the closure to any order.

D. Numerical estimates for small ϵ

The sums in Eq. (117) can be estimated using the small ϵ approximation for the eigenfunctions in terms of Legendre functions given in Ref. [20]. Since in the $\epsilon = 0$ case the eigenfunctions are Legendre polynomials, one can consider for small ϵ [20]

$$\Lambda_n \sim P_{\nu_n}, \qquad \kappa_n \sim \nu_n \left(\nu_n + 1\right), \tag{122}$$

where the index ν_n is an integer plus a small correction proportional to $\sqrt{\epsilon}$:

$$\nu_n = n + \sqrt{2\epsilon}\,\frac{4}{\pi}\frac{\Gamma^2\left(1+n/2\right)}{\Gamma^2\left(1/2+n/2\right)}. \tag{123}$$

In this approximation, the relevant sums can be calculated numerically and are found to be

$$\sum \gamma_n \simeq 1 - 2.93\,\epsilon^{3/2} + \mathcal{O}\left(\epsilon^2\right), \tag{124}$$

$$\sum \gamma_n \kappa_n \simeq 1 + 1.48\sqrt{\epsilon} + \mathcal{O}\left(\epsilon\right), \tag{125}$$

$$\sum \frac{\gamma_n}{\kappa_n} \simeq 1 - 1.48\sqrt{\epsilon} + \mathcal{O}\left(\epsilon\right), \tag{126}$$

$$\sum \frac{\gamma_n}{\kappa_n^2} \simeq 1 - 2\times 1.48\sqrt{\epsilon} + \mathcal{O}\left(\epsilon\right). \tag{127}$$

These results will be useful in the following chapters where two relevant applications of the results obtained here are explored.

VI. ELECTRICAL CONDUCTIVITY

The parallel component of the momentum equation, Eq. (4), for electrons yields a "parallel Ohm's law" for the plasma. In neoclassical regimes, the parallel viscous force constitutes an extra drive on the right side of this equation. Having a kinetic closure for this force in terms of the flow introduces a correction in the electrical conductivity which relates the flow and the electric field. In this chapter this correction is calculated and estimated for small magnetic field modulations.

A. Moment approach to electrical conductivity.

To illustrate the calculation of the electrical conductivity, we begin by sketching the calculation using the moment approach [4]. For simplicity, we consider an infinite, homogeneous, unmagnetized plasma. A "weak" electric field is applied to the system and the electrical conductivity will be given by the relation

$$J_{\|} = \sigma E_{\|}. \tag{128}$$

The electric field accelerates electrons (ions are ignored due to their larger inertia) while Coulomb collisions tend to scatter them away from the field direction. This process is reflected in the proportionality constant σ which can be adjusted to account for the collisional effects. Phenomenologically, the conductivity σ can be roughly approximated considering a force balance

$$m\frac{dV_{\|}}{dt} \sim qE_{\|}. \tag{129}$$

Assuming the collisional effects result in a damping rate ν, we can write $dV_{\|}/dt \sim \nu V_{\|}$. Then

$$mV_{\|} \sim \frac{qE_{\|}}{\nu} \quad \Rightarrow \quad J_{\|} = \frac{n_e e^2}{m\nu}E_{\|}. \tag{130}$$

Thus, this rough approximation yields an electrical conductivity

$$\sigma_r = \frac{ne^2}{m\nu}, \tag{131}$$

which will be used as a reference value.

In order to account for the Coulomb collision effects in a more precise way, the (equilibrium) kinetic equation for the plasma is considered:

$$\frac{q}{m}\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f). \tag{132}$$

To solve this equation the distribution function is expanded in a parameter δ that reflects the smallness of the magnitude of the electric field compared with the collisional effects. Consider as a reference electric field the Dreicer field $E_D \sim n_e e^3/T_e$. It is the electric field strength at which the bulk of the electrons "run away." Electrons with kinetic energies that exceed the characteristic thermal energy of the plasma by a factor of more than E_D/E can beat the friction due to collisions and thus "run away" under the influence of the electric force. Thus, in order to only have "run away electrons" in the high energy tail of the distribution we need $E_D/E \gg 1$. Hence, to lowest order in the parameter $\delta \equiv E/E_D$, the left side of Eq. (132) is small and the solution is a local Maxwellian.

The next order distribution function is expanded in Legendre (pitch angle) and Laguerre (energy) polynomials. With this hypothesis [4], the moments of the kinetic equation yield a set of coupled equations relating fluxes and forces. Considering only the first two moments (electron flow and heat flux), this system can be expressed in matrix form as

$$n_e e \begin{bmatrix} E_{||} \\ 0 \end{bmatrix} = -\frac{n_e m_e}{\tau_{ee}} [\mathbb{L}] \begin{bmatrix} BJ_{||} \\ \dfrac{2}{5n_e T_e} q_{||e} B \end{bmatrix} + \begin{bmatrix} \mathbf{B} \cdot \nabla \cdot \Pi \\ \mathbf{B} \cdot \nabla \cdot \Theta \end{bmatrix}. \tag{133}$$

The entries in the friction matrix $[\mathbb{L}]$, the friction coefficients L^e_{ij}, are complicated integrals with Laguerre polynomials as weighting functions [4]. For the illustrative purposes of this section, we only need know that they can be calculated and the matrix inverted. If viscosities are neglected, the electrical conductivity can be obtained from

$$\begin{bmatrix} BJ_{||} \\ -\dfrac{2}{5n_eT_e}q_{||e}B \end{bmatrix} = \frac{n_e e^2}{m_e \nu_{ee}} [\mathbb{L}]^{-1} \begin{bmatrix} BE_{||} \\ 0 \end{bmatrix}. \tag{134}$$

For the case of a linearized collision operator we have [4]

$$\left[\mathrm{L}_{ij}^{e}\right]^{-1} = \frac{1}{Z(\sqrt{2}+Z)} \begin{bmatrix} \sqrt{2}+\frac{13}{4}Z & -\frac{3}{2}Z \\ \frac{3}{2}Z & Z \end{bmatrix}. \tag{135}$$

Thus, to this level of approximation the parallel electrical conductivity is

$$\sigma = \frac{n_e e^2}{m_e \nu_{ee}} \frac{1}{\alpha_{Sp}}, \tag{136}$$

where the Spitzer electrical conductivity factor is

$$\alpha_{Sp} = \frac{\sqrt{2}+Z}{\sqrt{2}+13Z/4}. \tag{137}$$

This result is accurate relative to the exact results [22, 23] to within about 5% for all Z; it is thus well within the $1/\ln\Lambda \sim 7\%$ intrinsic accuracy of the Fokker-Planck operator. If closures are provided as expressions for the stresses in terms of the flows (or the currents), their effect can be included. In such a case, the matrix to invert will contain a second term corresponding to the contributions of these closures; this results in a neoclassical correction to the electrical conductivity. This is done in terms of the 2×2 moment approach above in Chapter VIII where heat flux effects are included. In the following section only the viscous stress is considered and thus only one equation is used—the parallel momentum equation.

B. Frequency-dependent electrical conductivity

To obtain the modification of the electrical conductivity caused by the frequency-dependent parallel viscous force $\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle$, the bounce-averaged parallel component of the complete momentum equation for electrons is considered:

$$-\frac{d}{dt}\left\langle J_{\|}B\right\rangle = \frac{n_e e^2}{m_e}\left\langle E_{\|}B\right\rangle + \frac{e}{m_e}\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|e}\right\rangle - \nu_e\left\langle J_{\|}B\right\rangle, \tag{138}$$

which is the dynamic, "1×1 matrix" version of Eq. (133). In this section initial conditions are neglected for simplicity. Introducing Eq. (108) in Eq. (138) and rearranging terms the solution for the ω-dependent electrical conductivity including viscous effects can be obtained. Defining

$$\widehat{\alpha}_e(\omega) = 1 + \frac{1}{\nu_e}\left[\widehat{v}(\omega) - i\omega\right], \tag{139}$$

the electrical conductivity can be written as

$$\widehat{\sigma}(\omega) = \frac{n_e e^2}{m_e\nu_e}\frac{1}{\widehat{\alpha}_e(\omega)}. \tag{140}$$

The $i\omega$ term in $\widehat{\alpha}_e(\omega)$ arises from the inertia term in the momentum equation. Here, the ion parallel flow and its dynamics have been neglected for time scales $t\sim 1/\nu_e$.

The dynamic conductivity in Eq. (140) is valid for any frequency (time). Once again, the analytical process cannot be carried out further for the same reasons discussed in the end of Section V B. However, a numerical computation could give the frequency dependence and upon taking the inverse Laplace transform, the time evolution of σ to some appropriate accuracy.

Here, in order to get some insight into the frequency behavior of $\widehat{\sigma}$, the low and high frequency limits of Eq. (140) in the small ϵ approximation will be treated analytically. In particular, we begin by checking the static limit, which should be obtained for $\omega\to 0$. For Eq. (139) we have

$$\alpha_e(0) = 1 + \mu_e/\nu_e. \tag{141}$$

Recalling that the fraction of trapped and circulating particles can be estimated for $\sqrt{\epsilon} << 1$ as $f_t \simeq 1.46\sqrt{\epsilon}$ [4] and considering the value for μ_e ($\simeq 1.53\, \nu_e f_t / f_c$ for hydrogenic ions) in the small $\sqrt{\epsilon}$ approximation, the usual trapped-particle correction to the static electrical conductivity is obtained:

$$\sigma = \frac{n_e e^2 / m_e \nu_e}{1 + \mu_e / \nu_e} = \frac{n_e e^2}{m_e \nu_e} \frac{1}{1 + 2.24\sqrt{\Delta B / 2B_{\min}}}. \tag{142}$$

For $\omega \neq 0$ we consider the low and high frequency limits of

$$\widehat{\alpha}_e(\omega) \simeq 1 + \frac{\widehat{f}_t(\omega, \nu_e)}{\widehat{f}_c(\omega, \nu_e)} - \frac{i\omega}{\nu_e}, \tag{143}$$

where we are approximating $\bar{\nu} = \nu_e$ for simplicity. Using Eq. (100) and the expansion in Eq. (114), for $\omega \ll \nu_e$ we obtain

$$\widehat{\alpha}_e(\omega) \simeq 1 + f_t - \frac{\omega^2}{\nu_e^2}\left(2\sum \frac{\gamma_n}{\kappa_n^2} - f_c\right) - \frac{i\omega}{\nu_e}\left(2f_t + \sum \frac{\gamma_n}{\kappa_n^2}\right). \tag{144}$$

Introducing this factor in Eq. (139), the real and imaginary parts of the electrical conductivity in this limit are approximately

$$Re[\widehat{\sigma}(\omega)] \simeq \frac{\sigma_r}{1 + f_t}\left\{1 - \frac{1}{1 + f_t}\left[f_c - 2\sum \frac{\gamma_n}{\kappa_n^2} + \frac{\left(2f_t + \sum \gamma_n / \kappa_n^2\right)^2}{1 + f_t}\right]\frac{\omega^2}{\nu_e^2}\right\}, \tag{145}$$

and

$$Im[\widehat{\sigma}(\omega)] \simeq \frac{\omega}{\nu_e}\sigma_r \frac{2f_t + \sum \gamma_n / \kappa_n^2}{(1 + f_t)^2}. \tag{146}$$

Thus, for small ω the real part of the conductivity will decay from the static limit as $-(\omega/\nu_e)^2$ while the imaginary ("reactive") part grows as ω/ν_e. The sums in Eqs. (145) and (146) can be estimated using the small ϵ approximation for the eigenfunctions in terms of Legendre functions given in Ref. [20] and illustrated in Subsection V D. Figure 2 shows the frequency-dependence of the electrical conductivity using this expansion.

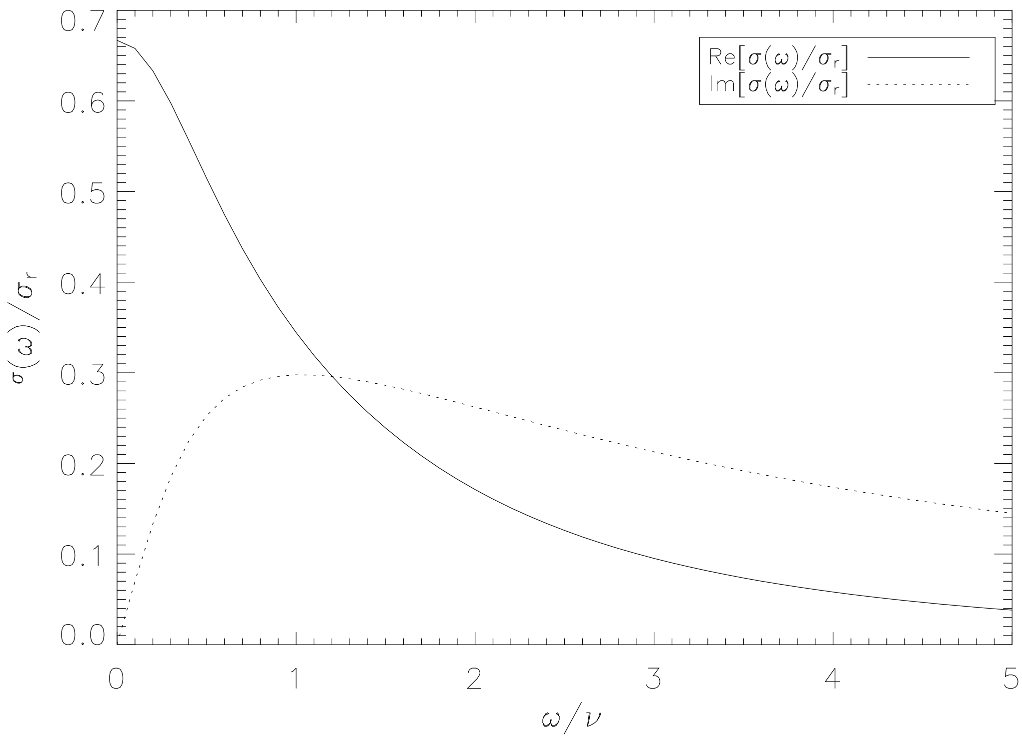

Fig. 2. Real and imaginary parts of $\hat{\sigma}/\sigma_r$ in the low frequency range for $\epsilon = 0.1$.

For high frequencies $\omega > \nu_e$, the ω-dependent factor in the conductivity can be expressed as

$$\begin{aligned}\widehat{\alpha}_e(\omega) \;\simeq\; & 2 + \sum \gamma_n (\kappa_n - 2) - \frac{\nu_e^2}{\omega^2} \sum \gamma_n \kappa_n \\ & - \frac{i\omega}{\nu_e}\left[2 - \sum \gamma_n + \frac{\nu_e^2}{\omega^2} \sum \gamma_n (1 - 2\kappa_n)\right]. \end{aligned} \tag{147}$$

The real and imaginary parts of the conductivity for $\omega \gg \nu_e$ are

$$Re[\widehat{\sigma}(\omega)] \simeq \frac{\nu_e^2}{\omega^2}\left[\frac{2 + \sum \gamma_n (2 + \kappa_n)}{\left(2 - \sum \gamma_n\right)^2}\right], \tag{148}$$

and

$$Im[\widehat{\sigma}(\omega)] \simeq \frac{\nu_e}{\omega}\left(\frac{1}{2 - \sum \gamma_n}\right). \tag{149}$$

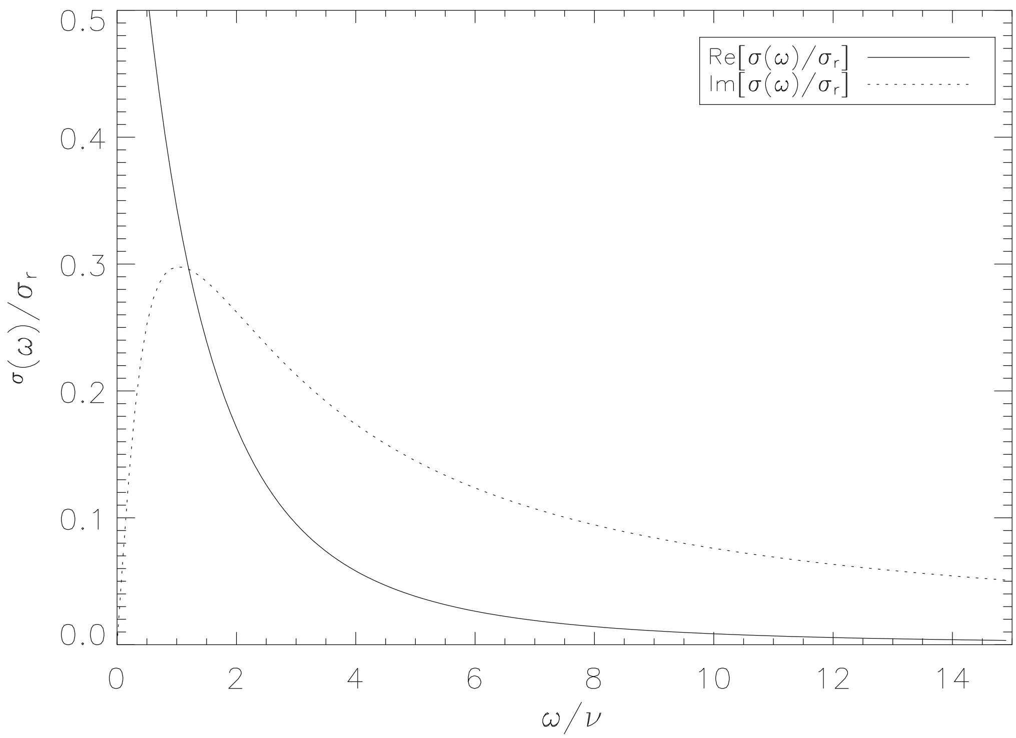

Fig. 3. Asymptotic decay of $\hat{\sigma}/\sigma_r$ in the high frequency range for $\epsilon = 0.1$.

In this limit, the real part of $\widehat{\sigma}(\omega)$ decreases as $(\nu_e/\omega)^2$. The imaginary part also decreases, but at a slower rate ($\sim \nu_e/\omega$). Both asymptote to zero for $\omega \to \infty$ as shown in Fig. 3.

Since in Eqs. (144)–(149) a small (or large) ω/ν_e is considered after a small $\sqrt{\epsilon}$ assumption, what is considered a small (or large) frequency is to be compared with the magnitude of the field modulation. That is, the high frequency asymptotic behavior is expected to be seen only for $\omega/\nu_e \gg 1/\sqrt{\epsilon}$ for a given modulation ϵ.

Note that in the static, low frequency limit, the effects of trapped particles are present for $\omega = 0$. On the other hand, as $\omega \to \infty$ (initial times) there are no significant trapped particle effects. Figures 4 and 5 show the effects of ϵ on the real

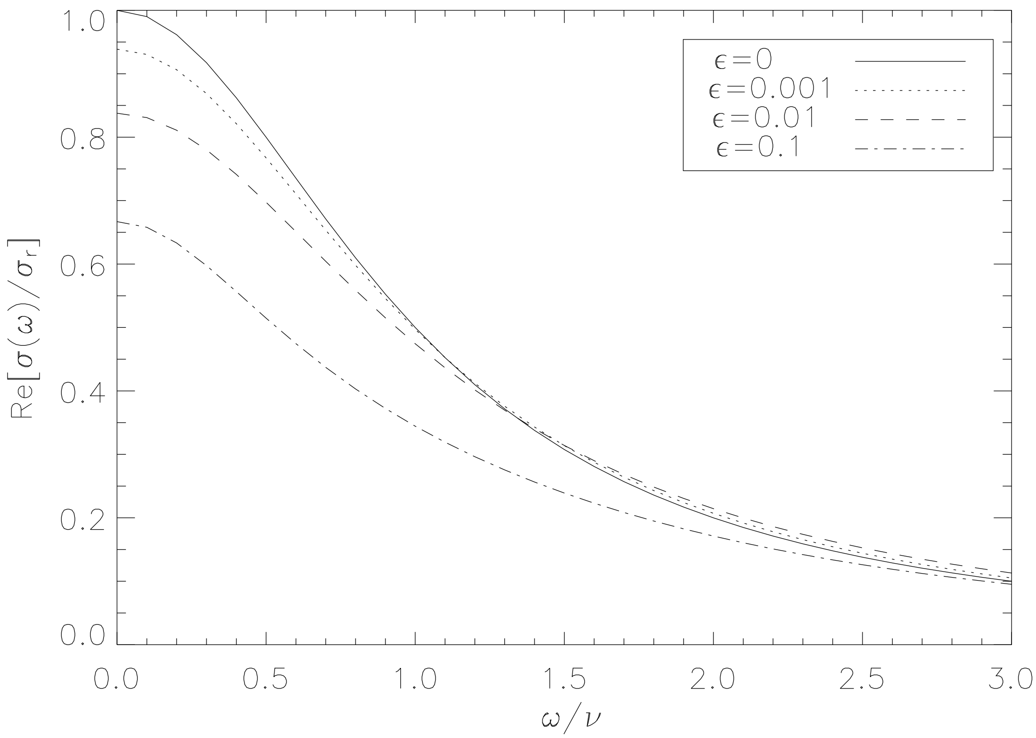

Fig. 4. Real part of $\hat{\sigma}/\sigma_r$ for various values of ϵ as a function of ω.

and imaginary parts of $\widehat{\sigma}\,(\omega)$.

The results in Eqs. (145)–(146) and (148)–(149) can be summarized as

$$Re\left[\widehat{\sigma}\,(\omega)\right] \sim \begin{cases} 1 - \mathcal{O}\left(\sqrt{\epsilon}\right) - \mathcal{O}\left(\sqrt{\epsilon}\right)\left(\omega/\nu_e\right)^2, & \omega \ll \nu_e, \\ \left[1 + \mathcal{O}\left(\sqrt{\epsilon}\right)\right]\left(\nu_e/\omega\right)^2, & \omega \gg \nu_e, \end{cases} \tag{150}$$

$$Im\left[\widehat{\sigma}\,(\omega)\right] \sim \begin{cases} \left[1 + \mathcal{O}\left(\sqrt{\epsilon}\right)\right]\omega/\nu_e, & \omega \ll \nu_e, \\ \left[1 + \mathcal{O}\left(\epsilon^{3/2}\right)\right]\nu_e/\omega, & \omega \gg \nu_e. \end{cases} \tag{151}$$

which is qualitatively consistent with the behavior shown in Figs. 4-5.

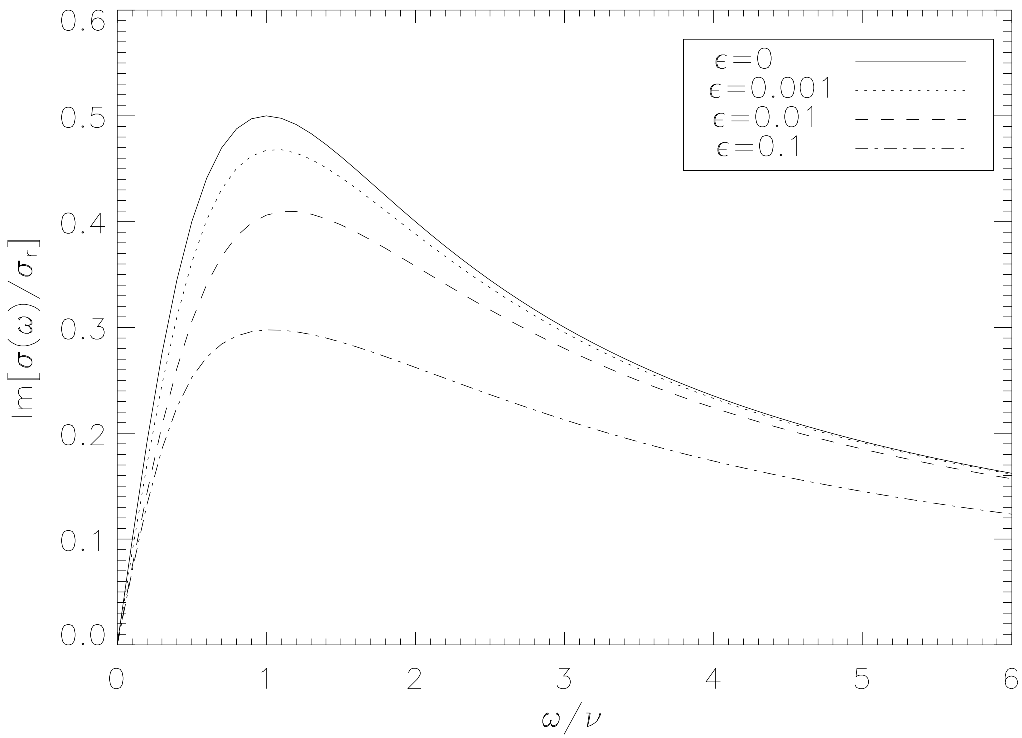

Fig. 5. Imaginary part of $\hat{\sigma}/\sigma_r$ for various values of ϵ as a function of ω.

VII. FLOW EVOLUTION

With the closure for the parallel viscous force obtained previously in Eq. (108), the evolution of the parallel flow can be calculated as an initial value problem. The temporal evolution of the flow is obtained by inverting the Laplace transform in the small ϵ case.

Two schemes are considered: the parallel flow damping within the bumpy cylinder model, and an extension to an axisymmetric geometry and thus to the time dynamics of the poloidal flow in a magnetically-confined toroidal plasma. In the poloidal flow case, some estimates of the damping rates are presented and compared with results obtained in Ref. [8]. The “ambipolarity paradox” [6] is briefly discussed also in this context.

A. Parallel flow

The parallel flow damping can be expressed, in terms of the flow variable U, as

$$mn \left\langle B^2 \right\rangle \frac{\partial U(t)}{\partial t} = -\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\parallel} \right\rangle . \tag{152}$$

Clearly, the result in Eq. (37) justifies taking a flux surface average of the full momentum equation which yields Eq. (152). Then, $U(t)$ is damped by the flux-surface averaged parallel viscous force and the closure obtained in Chapter V can be introduced.

The simplicity of the one-dimensional, bumpy cylinder magnetic field model permits a full calculation of the parallel flow dynamics including the effects of initial conditions. Since the time dependence of the left side of Eq. (152) is of interest, we start by again taking a Laplace transform and work from the full frequency-dependent closure in Eq. (108) as follows

$$\left[\widehat{\upsilon}\left(\omega\right)-i\omega\right]\widehat{U}\left(t\right)=-\widehat{\Upsilon}\left(\omega\right). \tag{153}$$

Considering the small $\sqrt{\epsilon}$ approximation in Eqs. (114) and (115) for $\widehat{\upsilon}$ and $\widehat{\Upsilon}$, we have

$$\widehat{\upsilon}\simeq\frac{m}{T}\int d^3v\,\bar{\nu}\frac{v^2}{3}\left(1-\frac{i\omega}{\bar{\nu}}\right)\widehat{f}_t\frac{f_M}{n}, \tag{154}$$

$$\widehat{\Upsilon}\simeq\frac{m}{T}\int d^3v\,\frac{v^2}{3}\frac{f_M}{n}\left(1-\frac{i\omega}{\bar{\nu}}\right)\left(1-\widehat{f}_t\right)\widehat{f}_g. \tag{155}$$

Solving for $\widehat{U}$ in Eq. (153) at this point would lead again (as in Section V B) to infinite sums in the denominator and thus complicate the inverse transform. Instead, the inverse Laplace transform can be taken on both sides of Eq. (153) and after some manipulation (see Appendix B) one obtains an inhomogeneous integral equation for $U\left(t\right)$:

$$U\left(t\right)=h\left(t\right)+\int_0^t K\left(t;\tau\right)U\left(\tau\right)d\tau. \tag{156}$$

This integral equation gives the time evolution of the parallel flow and has "memory" of the localization of the initial distribution relative to the boundary with trapped particle space. The inhomogeneous term $h\left(t\right)$ and the integration kernel $K\left(t;\tau\right)\sim\sum_n\exp\left[-\bar{\nu}\kappa_n\left(t-\tau\right)\right]$ are given in terms of Cordey eigenfunctions in Appendix B.

B. Poloidal flow evolution

An analysis similar to that for the bumpy cylinder can also be employed for a toroidal magnetic field. For the dynamic evolution of the "parallel" flow in an axisymmetric configuration the initial distribution in the pitch angle variable is not taken into account for simplicity. That is, the initial perturbation introduced in the system is only composed of untrapped particles that will contribute to the flow. In this configuration, the magnetic field can be written as

$$\mathbf{B} = B_T\hat{\zeta} + B_P\hat{\theta}, \tag{157}$$

where B_T and B_P are the components of the magnetic field in the toroidal (ζ) and poloidal (θ) directions respectively.

To apply the model developed in the previous sections to this geometry some modifications have to be introduced. In such a configuration, the field modulations along a field line are not unidimensional. That is, the relevant flow variable to be considered is the poloidal flow defined by [4, 5]

$$U_\theta\left(\psi, t\right) \equiv \frac{\mathbf{V}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\theta} = \frac{V_\parallel}{B} + \frac{\mathrm{I}}{B^2}\left(\frac{d\phi}{d\psi} + \frac{1}{nq}\frac{dp}{d\psi}\right), \tag{158}$$

where the first term is due to the parallel flow velocity and the second is due to the perpendicular flows in the plasma, which to lowest order in gyroradius are a combination of the $\mathbf{E}\times\mathbf{B}$ and diamagnetic flows:

$$\mathbf{V}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}\times\nabla p}{nqB^2}. \tag{159}$$

In Eq. (158) we have used $\mathbf{E} = -\nabla\phi$ and ψ is the poloidal magnetic flux function defined by $B_P\hat{\theta} = \nabla\zeta\times\nabla\psi$.

From the parallel (to $\mathbf{B}$) momentum balance, an evolution equation for U_θ will include a contribution ($\propto q^2$) from the toroidal flow [6]:

$$nm\left(1+2q^2\right)\frac{\partial U_\theta}{\partial t} = -\frac{\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel\rangle}{\langle B_P^2\rangle} + 4\pi\langle\frac{\partial\phi'}{\partial t}\rangle, \tag{160}$$

where the safety factor is defined as $q = \epsilon B_T/B_p$ in which now $\epsilon \equiv r/R_0 \ll 1$ is the inverse aspect ratio.

The closure in Eq. (108) may now be introduced. In this problem the parallel viscous force damps the poloidal component of the parallel flow. Because of toroidal axisymmetry there is no parallel viscous damping in the toroidal direction. Hence the toroidal momentum is only damped by the (higher order) perpendicular stress. With the flux surface average in this case being defined as

$$\langle A\rangle \equiv \oint \frac{d\theta A(\theta)}{\mathbf{B}\cdot\nabla\theta} / \oint \frac{d\theta}{\mathbf{B}\cdot\nabla\theta}, \tag{161}$$

one can write

$$\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_{\|}}\rangle = nm\left\langle B^2\right\rangle\left[\widehat{U_\theta}\,\widehat{v}(\omega) + U_{\theta 0}\right]. \tag{162}$$

Equations (154) and (155) can be used again to obtain an approximate closure to lowest order in $\widehat{f}_t$. Introducing this closure in Eq. (160) yields

$$\left[\frac{\langle B^2\rangle}{\langle B_P^2\rangle}\widehat{v}(\omega) - i\omega\left(1+2q^2\right)\right]\widehat{U_\theta} = \left[\left(1+2q^2\right) - \frac{\langle B^2\rangle}{\langle B_P^2\rangle}\right]U_{\theta 0} + \frac{4\pi}{nm}\langle\frac{\partial\phi'}{\partial t}\rangle. \tag{163}$$

The inverse Laplace transform can be calculated on both sides which yields the integral equation

$$U_\theta(t) = h_\theta(t) + \int_0^t K_\theta(t;\tau)\,U_\theta(t)\,d\tau, \tag{164}$$

where $h_\theta(t)$ and $K_\theta(t;\tau)$ are defined in Appendix B.

Using the approximation described in Section V D, the asymptotic limits of the poloidal flow evolution for short and long times can be inspected. For short times $t \ll \bar{\nu}$, Eq. (164) can be roughly approximated by

$$U_\theta(t) \simeq t\,K_\theta(t;t)\,U_\theta(t), \tag{165}$$

where we have assumed the initial condition is already damped. Then, the characteristic damping time is initially (after the initial perturbation is damped) given by $\tau_p \simeq 1/K_\theta(t;t)$, which yields (see Appendix B)

$$1/\tau_p \simeq \frac{0.51}{\epsilon}\frac{m}{T}\int d^3v\,\bar{\nu}\frac{v^2}{3}\frac{f_M}{n}. \tag{166}$$

The numerical result in Eq. (166) is similar to the estimate for the damping rate in Ref. [8] for this limit. This result is also obtained in Ref. [10] when taking a time average including the transient behavior induced by an initial perturbation.

In the present $\sqrt{\epsilon} \ll 1$ expansion, in addition to the damping rate ν_p there is a small oscillatory (ω_r) response [8]. It can be recovered by writing $i\omega = \nu_p + i\omega_r$ and expanding $\widehat{\upsilon}(\omega)$ about the damping rate assuming $\nu_p \gg \omega_r$. By taking account of the Landau-type pole [8] in the denominator of $\widehat{f}_c$ and equating the imaginary parts one obtains an imaginary component of the frequency ω_r. This effect contributes slight oscillatory responses in $\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_{\|} \right\rangle$ and in the kernels of the time-history integrals in Eq. (164).

As mentioned before, the result for the damping rate obtained assuming steady state is inconsistent. Here, we will obtain such result only to show that this limit is also included in Eq. (164). For long times and assuming a steady state ($U_\theta(t) \rightarrow U_\theta$), one can rewrite the flow evolution as

$$U_\theta \simeq U_\theta \int_0^t K_\theta(t;\tau)\, d\tau. \tag{167}$$

Thus, the characteristic damping rate (in the time-independent case) for $t \rightarrow \infty$ can be obtained from

$$\int_0^t K_\theta(t;\tau)\, d\tau = 1. \tag{168}$$

In the small ϵ approximation the damping rate for the static calculation is obtained (see Appendix B):

$$\nu_p \simeq \frac{0.31}{\sqrt{\epsilon}} \frac{m}{T} \int d^3v\, \bar{\nu} \frac{v^2}{3} \frac{f_M}{n}. \tag{169}$$

This result clearly violates the static assumption, as has been pointed out by other authors [6–8]. In Ref. [6], this inconsistency is tied to the so-called “Ambipolarity Paradox.” Conservation of classical (gyromotion-induced) toroidal angular momentum guarantees ambipolar diffusion of fluxes across the magnetic field lines. However, for neoclassical (drift-orbit induced) processes, there is a transient phase in which the poloidal flow is damped by the parallel viscous force according to Eq. (162), or its time-dependent version Eq. (164).

For the cross-field fluxes we consider the radial ion force balance. For times longer than the compressional Alfvén time ($\sim 0.1\mu s$), the inertia term can be neglected and thus

$$\hat{e}_r \cdot \left(\mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{1}{n_i q_i} \nabla p_i \right) = 0. \tag{170}$$

From this expression, the radial electric field can be related to the toroidal and poloidal flows by

$$E_r = V_\zeta B_\theta - V_\theta B_\zeta + \frac{1}{n_i Z_i e} \frac{dp_i}{dr}. \tag{171}$$

Assuming an exponential damping of the poloidal flow V_θ with the damping rate in Eq. (169), i. e. the static result, leads to inconsistencies. The static calculation neglects the left side of Eq. (160) under the assumption that $\partial/\partial t \ll \nu_i$. However, the result at the end of the calculation is the damping rate in Eq. (169) which yields $\partial/\partial t \sim \nu \gg \nu_i$ for large aspect ratio. Because of this contradiction, a time dependent evolution for the poloidal flow V_θ must be considered. In particular, Eq. (164) gives the explicit time-dynamics of the poloidal flow variable U_θ.

In a time scale τ_i (typically $\sim 200\,\mu s$ - $1\,ms$) the poloidal flux relaxes to its equilibrium value, which for an inhomogeneous plasma is determined by the poloidal ion flow induced by the ion temperature gradient and is given by [4, 24]

$$U_\theta \sim \frac{1.17}{q_i} \frac{\partial T_i}{\partial \psi}. \tag{172}$$

Once the poloidal flow is in equilibrium the ambipolar phase begins and the radial particle fluxes become ambipolar. Then, the radial force balance equation in

Eq. (171) relates toroidal flow to radial electric field [4, 25]

$$\mathbf{V}\cdot\nabla\zeta = -\left(\frac{\partial\phi_0}{\partial\psi} + \frac{1}{n_i q_i}\frac{\partial p_{i0}}{\partial\psi} - \frac{1.17}{q_i}\frac{\partial T_i}{\partial\psi}\right). \tag{173}$$

The consequent damping of the toroidal flow V_ζ (or equivalently electric field E_r) occurs, phenomenologically, on a radial transport time scale, presumably due to microturbulence-induced anomalous transport processes.

The discussion in the previous paragraphs does not solve any paradox. Actually, there is no such paradox. The problem discussed in Ref. [6] concerns semantic differences between authors on whether to consider the cross-field fluxes ambipolar or not. Thus, having a dynamic flow evolution equation provides a good formalism for determining whether (according to the time scales considered) the ambipolarity assumption with static closure relations is adequate for a particular problem.

VIII. HEAT FLUX EFFECTS

Heat flux effects were neglected in the previous chapters for simplicity. This chapter focuses on how the equations are modified when heat flux effects are included in the formulation and shows how the generalization is made. This can be extended to other applications besides the electrical conductivity and the evolution of the parallel flow that are treated here.

Since the energy eigenfunctions are Laguerre polynomials, the difference between the viscous stress and the heat viscous stress moments is only reflected in the index of the corresponding polynomial. This is the basis of the moment approach illustrated in Section VI A and allows a simultaneous treatment for the evolution of $V_\|$ and $q_\| \equiv \mathbf{q}\cdot\mathbf{b}$. Because of this, including heat flux effects in the solution for the kinetic distortion and in the parallel viscous force closure does not significantly complicate the calculation and the generalization becomes straightforward.

A. Drift-kinetic equation including heat flux

In order to include heat flux effects in the problem, the distribution function to be considered as the Chapman-Enskog Ansatz is a flow and *heat-flux* shifted Maxwellian plus a small kinetic distortion F:

$$f = f_M\left[1 + \frac{m}{T}\mathbf{v}\cdot\left(\frac{2}{5nT}\mathbf{q}\right)L_1^{3/2}\right] + F, \tag{174}$$

where f_M is defined in Eq. (16). The functional form of the second term in brackets is justified by the moments of the distribution function Eqs. (9)–(10) since in this case the constraint for the kinetic distortion given in Eq. (17) needs to be extended to these moments. In particular

$$\int d^3v\,\left(L_0^{3/2},\, L_1^{3/2}\right)v_\| F = 0. \tag{175}$$

This condition is consistent with the definition in Eq. (174). The Laguerre polynomials $L_i^{3/2}$ are evaluated in the usual variable $x = v^2/v_t^2$ and, for Eq. (175), are given by

$$L_0^{3/2} = 1, \qquad L_1^{3/2} = \frac{5}{2} - x. \tag{176}$$

The equilibrium distribution function now contains all the moments in Eqs. (6)–(10) and hence their evolution through the plasma kinetic equation. The kinetic distortion will not add terms to the full set of equations (2)–(5).

When Eq. (174) is introduced in the plasma kinetic equation and the procedure in Ref. [16] carried out, the recast DKE includes two extra terms. A parallel heat flux variable, $Q(\psi) = \mathbf{q} \cdot \mathbf{B}/B^2$ is defined, in order to satisfy heat flux incompressibility, and the recast DKE can be written in terms of the flow and heat flux variables as [5]:

$$\begin{aligned} \frac{dF}{dt} + v_\parallel \mathbf{b} \cdot \nabla \left[F + \frac{m}{T} v_\parallel B \left(U L_0^{3/2} - \frac{2}{5p} Q L_1^{3/2} \right) f_M \right] &= C(F) \\ &+ \frac{v_\parallel}{p} f_M \left(L_0^{3/2} \mathbf{b} \cdot \nabla \cdot \Pi_\parallel + \frac{2}{5} L_1^{3/2} \mathbf{b} \cdot \nabla \cdot \Theta_\parallel \right) \end{aligned} \tag{177}$$

where the heat stress tensor is given by

$$\Theta = \int d^3v \, m \left(\mathbf{vv} - \frac{1}{3} v^2 \mathbf{I} \right) L_1^{3/2} F. \tag{178}$$

Solving Eq. (177) to lowest order, as done previously in Chapter III, the lowest order distortion is obtained in terms of the flow U and the heat-flux Q

$$F_0 = -\frac{m}{T} v_\parallel B \left(U L_0^{3/2} - \frac{2}{5p} Q L_1^{3/2} \right) f_M + g(v, \lambda, \varsigma, t). \tag{179}$$

Once again, in order to solve for the integration constant g, the annihilator given in Eq. (46) operates on both sides of the next order DKE. By doing this and taking a Laplace transform as before, one obtains a differential equation for the $\widehat{g}$ representing circulating particles effects similar to Eq. (53). The operator acting on $\widehat{g}$ does not change since the process considered is not altered by the inclusion of Q in the problem. The heat-flux term in the lowest order solution introduces a new term in the energy expansion of the problem (Laguerre polynomials) but the free streaming term still only has a P_1 component in the pitch-angle expansion. Then, the equation for the integration constant $\widehat{g}$ is the same in form as Eq. (88) in Chapter V:

$$-i\omega\frac{1}{2}\left\langle\frac{B}{\sqrt{1-\lambda B}}\right\rangle\widehat{g}-\bar{\nu}\frac{\partial}{\partial\lambda}\lambda\left\langle\sqrt{1-\lambda B}\right\rangle\frac{\partial\widehat{g}}{\partial\lambda}=\frac{\varsigma v}{2}\left[\widehat{D}\left(v,i\omega\right)+f_M\left\langle\frac{B}{v_\parallel}\right\rangle g_0\left(\lambda\right)\right]. \tag{180}$$

However, the λ-independent drives now include extra terms due to the heat-flux contribution

$$\begin{aligned}\widehat{D}\left(v,i\omega\right) \equiv & \left(\langle\widehat{\mathbf{B}\cdot\nabla\cdot\Pi_\parallel}\rangle L_0^{3/2}+\frac{2}{5}\langle\widehat{\mathbf{B}\cdot\nabla\cdot\Theta_\parallel}\rangle L_1^{3/2}\right)f_M+\frac{1}{v}\left\langle B^2\right\rangle\left[C^1\left(\widehat{g}_1\right)+\nu\widehat{g}_1\right]\\ & -\frac{m}{T}\left\langle B^2\right\rangle f_M\left[i\omega\left(\widehat{U}L_0^{3/2}-\frac{2}{5p}\widehat{Q}L_1^{3/2}\right)+\left(U_0L_0^{3/2}-\frac{2}{5p}Q_0L_1^{3/2}\right)\right].\end{aligned} \tag{181}$$

B. Dynamic closure including heat flux effects

From Eqs. (180)–(181), a solution for $\widehat{g}$ and closures for each of the viscous forces can be obtained. In order to do this, we will proceed as before and expand the solution in pitch-angle eigenfunctions. Note that the restoring terms in the collision operator are the same as before since the heat-flux only introduces a new energy-dependent term. Since the collision operator (as well as the distribution function) are expanded in Legendre polynomials up to the same order, the difference in energy weighting of the flow and heat-flux terms does not alter its form; hence we define the flow-like variable $\widehat{V}$ as before

$$\widehat{V}\left(\psi,v\right)=\widehat{g}_1/B=\frac{3}{4}\sum_{\varsigma}\varsigma\int_0^{\lambda_c}d\lambda\,\widehat{g}. \tag{182}$$

Following the procedure in Chapter V, we obtain an expression for the function $\widehat{g}$ in terms of Cordey eigenfunctions:

$$\widehat{g} = -\frac{\varsigma}{2}\left[\frac{\eta_n}{\kappa_n - i\omega/\bar{\nu}}\frac{1}{\widehat{f}_c}\left(\widehat{V}\left\langle B^2\right\rangle - \left\langle B^2\right\rangle^{1/2} f_M \frac{v}{\bar{\nu}}\frac{m}{T}\widehat{f}_g\right) + \frac{m}{T}\frac{v}{\bar{\nu}} f_M \frac{\alpha_n}{\kappa_n - i\omega/\bar{\nu}}\right]\Lambda_n, \tag{183}$$

where all coefficients are defined in Chapter V A. For $\widehat{V}$ we obtain

$$\widehat{V} = v\frac{1}{\bar{\nu}}\frac{1}{\left\langle B^2\right\rangle}\left(\widehat{f}_c\,\widehat{D} + \left\langle B^2\right\rangle^{1/2} f_M \frac{m}{T}\,\widehat{f}_g\right), \tag{184}$$

which is now expressed in terms of both the flow and the heat-flux terms in $\widehat{D}$. Also, the expression for the function $\widehat{g}$ depends on heat-flux through this same driving term.

The solution up to now is equivalent in form as the one in Section V B. The heat flux effects enter the problem when the energy integrals are calculated in order to obtain the closure. That is, in this case, the equation to determine $\widehat{V}$ is

$$\begin{aligned}\bar{\nu}\widehat{V}\left(\frac{1}{\widehat{f}_c} - 1\right)\left\langle B^2\right\rangle \;=\; & v f_M \frac{1}{p}\left(\langle \mathbf{B}\cdot\widehat{\nabla\cdot\Pi_\parallel}\rangle L_0^{3/2} + \frac{2}{5}\langle \mathbf{B}\cdot\widehat{\nabla\cdot\Theta_\parallel}\rangle L_1^{3/2}\right) - \left\langle B^2\right\rangle^{-1/2} f_M \frac{\widehat{f}_g}{\widehat{f}_c} \\ & - C^1\left(\widehat{V}\right) - v\frac{m}{T}\left\langle B^2\right\rangle\left[i\omega\left(\widehat{U}L_0^{3/2} - \frac{2}{5p}\widehat{Q}L_1^{3/2}\right) + \left(U_0 L_0^{3/2} - \frac{2}{5p}Q_0 L_1^{3/2}\right)\right] f_M\end{aligned} \tag{185}$$

As was done previously, we take the $\int d^3v\, v$ integral and use the condition in Eq. (75). This yields an equation for the parallel viscous force

$$\begin{aligned}\left\langle \mathbf{B}\cdot\widehat{\nabla\cdot\Pi_\parallel}\right\rangle \;=\; & \left\langle B^2\right\rangle\left[\int d^3v\,\frac{v}{3}\bar{\nu}\widehat{V}\left(\frac{1}{\widehat{f}_c} - 1\right) + \widehat{U}\int d^3v\,\frac{v^2}{3}\frac{\nu_\perp}{v_{th}}\frac{i\omega}{\bar{\nu}}\right. \\ & \left. + U_0 + \left\langle B^2\right\rangle^{-3/2}\int d^3v\,\frac{v}{3} v f_M \frac{\widehat{f}_g}{\widehat{f}_c}\right]\end{aligned} \tag{186}$$

To obtain the closure for the heat viscous stress, the next order condition is used, i. e.

$$\int d^3v\, v L_1^{3/2} C^1\left(\widehat{V}\right) = 0. \tag{187}$$

By taking the $\int d^3v\, v L_1^{3/2}$ moment of Eq. (185), all terms proportional to $L_0^{3/2} = 1$ vanish and we obtain

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Theta_\parallel} \right\rangle = \left\langle B^2 \right\rangle \left[\int d^3v\, \frac{v}{3} L_1^{3/2} \bar{\nu} \widehat{V} \left(\frac{1}{\widehat{f}_c} - 1 \right) - \frac{2}{5p} \widehat{Q} \int d^3v\, \frac{v^2}{3} L_1^{3/2} \frac{\nu_\perp}{v_{th}} \frac{i\omega}{\bar{\nu}} \right.$$
$$\left. - \frac{2}{5p} Q_0 + \left\langle B^2 \right\rangle^{-3/2} \int d^3v\, \frac{v}{3} L_1^{3/2} f_M \frac{\widehat{f}_g}{\widehat{f}_c} \right] \tag{188}$$

In this case, the expansion in Eq. (77) (that ultimately relates $\widehat{V}$ to the flows) has to be calculated to order $n = 1$. This additional term in the expansion is related to the heat flux. The first two coefficients in the expansion are (see Appendix A)

$$\widehat{V}_0 = \widehat{U}\, n, \qquad \widehat{V}_1 = -\frac{2}{5p} \widehat{Q}\, n. \tag{189}$$

Introducing these relations in Eqs. (186) and (188), we obtain both closures:

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_\parallel} \right\rangle = nm \left\langle B^2 \right\rangle \left(\widehat{U}\, \widehat{\upsilon}_{00} - \frac{2}{5p} \widehat{Q}\, \widehat{\upsilon}_{01} + U_0 + \widehat{\Upsilon}_0 \right), \tag{190}$$

and

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Theta_\parallel} \right\rangle = nm \left\langle B^2 \right\rangle \left(\widehat{U}\, \widehat{\upsilon}_{01} - \frac{2}{5p} \widehat{Q}\, \widehat{\upsilon}_{11} - \frac{1}{p} q_0 + \widehat{\Upsilon}_1 \right). \tag{191}$$

Here $\widehat{\upsilon}_{ij}(\omega)$ are the "viscosity coefficients" [for $\widehat{\upsilon}_{ij}(0) = \mu_{ij}$ numerical values see Reference [5]]. Both $\widehat{\upsilon}_{ij}(\omega)$ and $\widehat{\Upsilon}_i(\omega)$ are obvious generalizations of the coefficients defined in Eqs. (109) and (111):

$$\widehat{\upsilon}_{ij}(\omega) = \frac{m}{T} \int d^3v\, \bar{\nu} \frac{v^2}{3} \frac{f_M}{n} L_i^{3/2} L_j^{3/2} \frac{\widehat{f}_t}{\widehat{f}_c}, \tag{192}$$

$$\widehat{\Upsilon}_i(\omega) = \frac{m}{T} \int d^3v\, \frac{v^2}{3} \frac{f_M}{n} L_i^{3/2} \frac{\widehat{f}_g}{\widehat{f}_c}. \tag{193}$$

Thus, the frequency dependent closure, including heat flux effects, can be expressed as a matrix equation:

$$\begin{bmatrix} \left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_\parallel} \right\rangle \\ \left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Theta_\parallel} \right\rangle \end{bmatrix} = nm \left\langle B^2 \right\rangle \begin{bmatrix} \widehat{\upsilon}_{00} & \widehat{\upsilon}_{01} \\ \widehat{\upsilon}_{01} & \widehat{\upsilon}_{11} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ -\frac{2}{5p}\widehat{Q} \end{bmatrix} + \begin{bmatrix} U_0 + \widehat{\Upsilon}_0(\omega) \\ -\frac{1}{p} Q_0 + \widehat{\Upsilon}_1(\omega) \end{bmatrix}. \tag{194}$$

In Eq. (194) both closures are stated. In particular, for the parallel viscous force we have

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_{\|}} \right\rangle = nm \left\langle B^2 \right\rangle \left[\widehat{U}\,\widehat{\upsilon}_{00} - \frac{2}{5p}\widehat{Q}\,\widehat{\upsilon}_{01} + U_0 + \Upsilon_0\left(\omega\right) \right]. \tag{195}$$

Clearly, the diagonal terms $\widehat{\upsilon}_{ii}$ relate each viscous force with its corresponding flow. The non-diagonal terms $\widehat{\upsilon}_{01} = \widehat{\upsilon}_{10}$ give the cross-effects in which the closure for the viscous force is in terms of both the parallel flow and the parallel heat flux. In a similar way, the parallel heat viscous force depends on both quantities. However, note that both the initial flow and the initial pitch angle distribution only enter the closure for their corresponding flow drive.

C. Corrections to the electrical conductivity and flow evolution

In order to obtain the heat flux correction to expression (139), we use the moment approach described in Section VI A. This is, taking the momentum and heat flux moments (i. e., $\int d^3v\, v_{\|} L_i^{3/2}$ for $i = 1, 2$) of the kinetic equation yields a matrix equation for the evolution of $V_{\|}$ and $q_{\|}$. The flux-surface-averaged parallel components of these balance equations for electrons can be written in matrix form as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \left\langle V_{\|} B\right\rangle \\ -\left\langle \mathbf{q}\cdot\mathbf{B}\right\rangle/p \end{bmatrix} &= -\frac{e}{m_e}\begin{bmatrix} \left\langle E_{\|}\, B\right\rangle \\ 0 \end{bmatrix} \\ &-\nu_e\left[\mathbb{L}\right]\begin{bmatrix} \left\langle V_{\|} B\right\rangle \\ -2\left\langle q_{\|} B\right\rangle/5p \end{bmatrix} + \frac{1}{n_e m_e}\begin{bmatrix} \left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|e}\right\rangle \\ \left\langle \mathbf{B}\cdot\nabla\cdot\Theta_{\|e}\right\rangle \end{bmatrix}, \end{aligned} \tag{196}$$

where $[\mathbb{L}]$ denotes the 2×2 matrix containing the usual electron friction coefficients [4] L_{ij}^{e} introduced in Section VI A, where the equilibrium version of Eq. (196) was considered [Eq. (133)].

When the closure in Eq. (194) is introduced, the Laplace transform of Eq. (196)

can be written as

$$\left[\begin{array}{c} \langle E_{||}B\rangle \\ 0 \end{array}\right] \simeq \frac{m_e\nu_e}{n_e e^2}\left[\mathbb{F}\right]\left[\begin{array}{c} \left\langle J_{||}\ B\right\rangle \\ \dfrac{2e}{5T_e}\left\langle q_{||e}B^2\right\rangle \end{array}\right], \tag{197}$$

where we defined a "friction-viscosity" matrix $[\mathbb{F}]$ as

$$[\mathbb{F}] = [\mathbb{L}] + \left[\begin{array}{cc} \left(\widehat{v}_{00} - i\omega\right)/\nu_e & \widehat{v}_{01}/\nu_e \\ \widehat{v}_{01}/\nu_e & \left(\widehat{v}_{11} - 5i\omega/2\right)/\nu_e \end{array}\right]. \tag{198}$$

The first component of Eq. (197) is an Ohm's law for the plasma and from it a conductivity, as defined by Eq. (128), can be obtained. This expression includes an extra correction due to heat flux effects and can be calculated as

$$\hat{\sigma}\left(\omega\right) \simeq \left[\mathbb{F}^{-1}\right]_{00} \simeq \left(\frac{\mathbb{F}_{11}}{\mathbb{F}_{00}\mathbb{F}_{11} - \mathbb{F}_{10}^2}\right)\sigma_r. \tag{199}$$

For the parallel flow damping correction, a similar generalization is obtained. Additional to the evolution for $U\left(t\right)$, an integral equation for the evolution of the parallel heat flux is obtained when the second component of Eq. (194) is introduced in the flux-surface averaged heat balance equation. These results, can be written in matrix form as

$$\left[\begin{array}{c} U\left(t\right) \\ -\frac{2}{5p}Q\left(t\right) \end{array}\right] = \left[\begin{array}{c} h_0 \\ h_1 \end{array}\right] + \int_0^t \left[\begin{array}{cc} K_{00} & K_{01} \\ K_{10} & K_{11} \end{array}\right]\left[\begin{array}{c} U\left(\tau\right) \\ -\frac{2}{5p}Q\left(\tau\right) \end{array}\right] d\tau, \tag{200}$$

where the generalized inhomogeneous terms and integration kernels are obtained by introducing weighting functions in the energy integral. For h_i we use

$$h\left(t\right) \sim \int d^3v\,\frac{m}{T}\frac{v^2}{3}\frac{f_M}{n}\left(...\right) \quad \Longrightarrow \quad h_i\left(t\right) \sim \int d^3v\,\frac{m}{T}\frac{v^2}{3}\frac{f_M}{n}L_i^{3/2}\left(...\right). \tag{201}$$

and for the integration kernels

$$K\left(t;\tau\right) \sim \int d^3v\,\frac{\nu_\perp}{2}\frac{m}{T}\frac{v^2}{3}\frac{f_M}{n} \quad \Longrightarrow \quad K_{ij}\left(t;\tau\right) \sim \int d^3v\,\frac{\nu_\perp}{2}\frac{m}{T}\frac{v^2}{3}\frac{f_M}{n}L_i^{3/2}L_j^{3/2}. \tag{202}$$

The cross-effects that are given in the closure [Eq. (194)] by the presence of non-diagonal terms υ_{ij} appear at this level through the non-diagonal K_{ij}. These effects modify the integration kernel of the particle flow evolution equation; the inhomogeneous term h in Eq. (156) is not altered ($L_0^{3/2} = 1$).

IX. STATIC PRESSURE ANISOTROPY

In previous chapters, the lowest order solution for the kinetic distortion, F_0, was used to calculate a closure for the flux-surface-averaged parallel viscous force. The spatial dependence of F_0 was determined early in the calculation to be proportional to $v_\parallel$ (in the free-streaming term) since the integration constant g is independent of ℓ. To solve for it, we took a bounce average through the annihilator in Eq. (46). In this operation all spatial dependence in the first order DKE is hidden in the flux surface average and F_1 is eliminated. The result is a lowest order distortion with a simple ℓ dependence through the free streaming term and a kinetic correction g expressed in terms of flux-surface-averaged quantities. This term in the expansion for f was useful in obtaining the closure for $\langle \mathbf{B} \cdot \nabla \cdot \Pi_\parallel \rangle$.

As will be seen, in order to obtain an expression for the pressure anisotropy $p_\parallel - p_\perp \equiv \Pi_\parallel$ and a closure for the *unaveraged* viscous force $\mathbf{B} \cdot \nabla \cdot \Pi_\parallel$, the ℓ-dependence of the distribution function through its third term, F_1, will be needed. In this chapter, the first order DKE will be solved retaining the spatial dependence. For simplicity, the time derivative will be neglected. That is, we will obtain a closure valid only in steady state: $\partial/\partial t \ll \nu$. Also, heat flux effects will be neglected in this Chapter. Time dependence for this problem is addressed in the next chapter.

A. Stress tensor and pressure anisotropy

We start the calculation by recalling the definition of the viscous stress tensor as a higher moment of the distribution function:

$$\mathbf{\Pi} = \int d^3v \, m \left(\mathbf{vv} - \frac{1}{3} v^2 \mathbf{I} \right) F. \tag{203}$$

Since the Maxwellian lowest order solution is isotropic in velocity space, the only part of the distribution that contributes to the stress tensor is the kinetic distortion F.

For the simple magnetic field model we are considering [Eq. (23)], the stress tensor can be written as

$$\mathbf{\Pi}_{\parallel} = \left(p_{\parallel} - p_{\perp}\right)(\mathbf{bb} - \mathbf{I}/3) \equiv \Pi_{\parallel}(\mathbf{bb} - \mathbf{I}/3), \tag{204}$$

where the last equality defines the "scalar" stress, i. e., the pressure anisotropy. This quantity reflects the interaction between charged particles and the magnetic field. The parallel gradient of the magnetic field affects the motion of charged particles and consequently the "parallel pressure," in a kinetic sense.

Considering again the expansion proposed for the banana regime with $\nu_* \ll 1$, we can write

$$\Pi_{\parallel} = m \int d^3v \left(v_{\parallel}^2 - \frac{v_{\perp}^2}{2}\right)(F_0 + F_1 + ...). \tag{205}$$

The first term in the distortion, F_0, was calculated before. Since both terms in Eq. (63) are proportional to ς, i. e. odd in $v_{\parallel}$, when adding particles in both directions the contribution of F_0 to Eq. (205) vanishes. Then, the first nonzero term in the P_2 moment of the distribution function is

$$\Pi_{\parallel} \equiv p_{\parallel} - p_{\perp} = m \int d^3v \left(v_{\parallel}^2 - \frac{v_{\perp}^2}{2}\right) F_1. \tag{206}$$

Changing to phase space variables λ and v, we have

$$\Pi_{\parallel} = -m \sum_{\varsigma} \sigma\pi \int dv\, v^5 \int \frac{B d\lambda}{v_{\parallel}} \left(\frac{3}{2}\lambda B - 1\right) F_1. \tag{207}$$

Since the variation within a flux surface of $\Pi_{\parallel}(\ell)$ is through F_1, we will consider once again the first order DKE. This time F_1 is the term to be solved for and thus all spatial dependence will be retained.

B. Magnitude of $\Pi_{\parallel}$

Most calculations involving parallel viscous damping introduce the flux-surface-averaged closure in the parallel force balance equation. This fact is due principally to the lack of an expression for the viscous force that varies within a flux surface. Because of this, in analytical calculations one has to take one of two paths. A flux surface average (or a bounce average) can be taken and the system of fluid equations closed with the known closure, or one can work with the averaged closure assuming the variation within a flux surface is negligible. Formally, the first case is correct, but all information about possible inhomogeneities is lost. The second case, as will be shown below, seems to be quite far from reality.

The parallel viscous force, at least within the model considered in previous chapters, is inhomogeneous in a flux surface and its constant part is smaller than this variation. To prove this statement we start by calculating the divergence of the pressure anisotropy stress. Recalling that, $\mathbf{\Pi}_\| = \Pi_\| \left(\mathbf{bb} - \mathbf{I}/3\right)$ we can write

$$\nabla \cdot \mathbf{\Pi}_\| = \left(\nabla \Pi_\|\right) \cdot \left(\mathbf{bb} - \mathbf{I}/3\right) + \Pi_\| \nabla \cdot \mathbf{bb}. \tag{208}$$

Using the tensor identities

$$\nabla \cdot \mathbf{bb} = \left(\nabla \cdot \mathbf{b}\right)\mathbf{b} + \left(\mathbf{b} \cdot \nabla\right)\mathbf{b}, \tag{209}$$

$$\left(\mathbf{b} \cdot \nabla\right)\mathbf{b} = \frac{1}{2}\nabla\left(\mathbf{b} \cdot \mathbf{b}\right) - \mathbf{b} \times \left(\nabla \times \mathbf{b}\right), \tag{210}$$

and

$$\nabla \cdot \mathbf{b} = -\mathbf{b} \cdot \nabla \ln \mathbf{B}, \tag{211}$$

the viscous force for this model is calculated from

$$\nabla \cdot \mathbf{\Pi}_\| = \mathbf{b}\left(\mathbf{b} \cdot \nabla \Pi_\|\right) - \nabla \Pi_\|/3 - \Pi_\| \left[\mathbf{b}\left(\frac{\mathbf{B} \cdot \nabla B}{B^2}\right) + \mathbf{b} \times \left(\nabla \times \mathbf{b}\right)\right]. \tag{212}$$

Projecting the previous result in the magnetic field's direction we obtain a simple expression for the parallel viscous force:

$$\mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_\| = \frac{2}{3}\left(\mathbf{B} \cdot \nabla\right)\Pi_\| - \Pi_\| \mathbf{b} \cdot \nabla B = \frac{2}{3} B \frac{\partial \Pi_\|}{\partial \ell} - \Pi_\| \frac{\partial B}{\partial \ell}, \tag{213}$$

were the second equality is for the bumpy cylinder magnetic field modulated within linear lengths ℓ.

Taking the flux surface average on both sides of Eq. (213) and using $B(\ell) = B_{\min} + 2\epsilon B_0 \sin^2(\pi\ell/L)$ we obtain

$$\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle = 2\epsilon B_0 \left\langle \Pi_{\|} \sin\left(\frac{2\pi\ell}{L}\right)\right\rangle . \tag{214}$$

From Eq. (118) we know that, for small ϵ, $\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle \sim \mathcal{O}\left(\sqrt{\epsilon}\right)$; thus, we can state

$$\left\langle \Pi_{\|} \sin\left(\frac{2\pi\ell}{L}\right)\right\rangle = \int_{-L/2}^{L/2} \Pi_{\|} \sin\left(\frac{2\pi\ell}{L}\right) d\ell \sim \mathcal{O}\left(1/\sqrt{\epsilon}\right). \tag{215}$$

Because of the symmetry of the problem, we have assumed the stress tensor to have the cylinder's periodicity. Based on this, and using Eq. (215), one concludes that for $\left\langle \Pi_{\|} \sin\left(2\pi\ell/L\right)\right\rangle \neq 0$ the pressure anisotropy has to be an odd function of $\vartheta \equiv 2\theta = 2\pi\ell/L$ and can thus be written as an odd Fourier series:

$$\Pi_{\|}(\vartheta) = \sum_{n=1}^{\infty} a_n \sin(n\vartheta), \tag{216}$$

with n odd. The coefficients in the series are calculated from

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \Pi_{\|}(\vartheta)\sin(n\vartheta)\, d\vartheta. \tag{217}$$

Since

$$\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle \simeq 2\epsilon\frac{B_0 L}{2\pi}\int_{-\pi}^{\pi} \Pi_{\|}(\vartheta)\sin(\vartheta)\, d\vartheta, \tag{218}$$

the first coefficient of the series is

$$a_1 = \frac{1}{2\epsilon}\frac{2}{B_0}\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle \sim \frac{1.46}{\sqrt{2\epsilon}}\frac{2}{B_0} mnU\left\langle B^2\right\rangle . \tag{219}$$

The complete series can then be written as

$$\Pi_{\|}(\vartheta) = \frac{1}{\sqrt{2\epsilon}}\frac{2.92}{B_0} mnU\left\langle B^2\right\rangle \sin\vartheta + \sum_{n=3}^{\infty} a_n \sin(n\vartheta), \tag{220}$$

and the derivative is

$$\frac{\partial}{\partial\theta}\Pi_{\|} = \frac{1}{\sqrt{2\epsilon}}\frac{2.92}{B_0} mnU \left\langle B^2 \right\rangle \cos\vartheta + \sum_{n=3}^{\infty} a_n n \, \cos\left(n\vartheta\right). \tag{221}$$

Introducing these expressions in the second line of Eq. (213) and rearranging terms we obtain

$$\begin{aligned}\mathbf{B}\cdot\nabla\cdot\Pi_{\|} &= \frac{2.92}{B_0} mnU \left\langle B^2 \right\rangle \left[\frac{1}{\sqrt{2\epsilon}}\frac{2}{3}\cos\vartheta - \sqrt{2\epsilon}\sin^2\vartheta\right] \\ &+ \sum_{n=3}^{\infty} a_n \left[\frac{2}{3} n \, \cos\left(n\vartheta\right) - 2\epsilon \, \sin\theta \sin\left(n\vartheta\right)\right]. \end{aligned} \tag{222}$$

Taking a closer look at the sum in the second term one can approximate

$$n \, \cos\left(n\vartheta\right) - 2\epsilon \, \sin\vartheta \sin\left(n\vartheta\right) \sim n \, \cos\left(n\vartheta\right), \tag{223}$$

since for large aspect ratio we can clearly consider $n \geq 3 \gg 2\epsilon$. Thus, we obtain

$$\mathbf{B}\cdot\nabla\cdot\Pi_{\|} \simeq \frac{2}{3}\frac{1}{\sqrt{2\epsilon}}\frac{2.92}{B_0} mnU \left\langle B^2 \right\rangle \cos\vartheta + \frac{2}{3}\sum_{n=3}^{\infty} a_n n \, \cos\left(n\vartheta\right) + \mathcal{O}\left(\sqrt{\epsilon}\right). \tag{224}$$

Thus, to lowest order (in $\epsilon \ll 1$) we have

$$\mathbf{B}\cdot\nabla\cdot\Pi_{\|} \simeq \frac{2B}{3}\frac{\partial}{\partial\ell}\Pi_{\|} = \frac{2B}{3}\frac{\partial}{\partial\ell}\left(p_{\|} - p_{\perp}\right). \tag{225}$$

Since the magnitude of $\mathbf{B}\cdot\nabla\cdot\Pi_{\|}$ is larger than $\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|} \right\rangle \sim \sqrt{\epsilon}$, the variations in the parallel viscous force are larger than the average. Note also that the dominant contribution to the spatially varying $\mathbf{B}\cdot\nabla\cdot\Pi_{\|}$ is due to the spatial variations in the pressure anisotropy [Eq. (220)] and not that due to the parallel gradient of the magnetic field magnitude [Eq. (221)]. The term in Eq. (225) is precisely the one that vanishes when flux-surface-averaged. Thus, when one examines the dynamics of the plasma flows with the flux-surface-averaged viscous force as a drive, only the effects of the smallest component of the stress is accounted for. Based on this, we can predict that, by considering spatially dependent quantities in the parallel force balance equation, the local (in θ) dynamics of the flows could differ significantly from what was obtained in Chapters VI and VII.

In the lines above, we showed that the variations of the magnetic field magnitude are negligible compared to the variations of the pressure anisotropy when calculating the parallel viscous force in this model and for large aspect ratio. We cannot infer from the calculation the order (in ϵ) of $\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}$. However, we can conclude that it has at least one term $\sim \mathcal{O}\left(1/\sqrt{\epsilon}\right)$ and that Eq. (225) is valid within our approximations.

C. Distribution function

The calculation of F_1 was not needed in determining $\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle$ since the annihilator integrated its spatial gradient along a closed trajectory. Thus, for consistency, we should expect to obtain F_1 as a continuous functional of distance along a field line, periodic and well defined at the extremes of the orbits for both trapped and circulating particles.

In the model we are considering, all gradients are in the direction of the magnetic field $\mathbf{B}$ and in terms of the length ℓ along a field line. Thus, we can write the first order DKE [in this case the equilibrium version of Eq. (45)] as

$$\frac{\partial F_1}{\partial \ell} - \frac{C(F_0)}{v_{\|}} = \frac{1}{pB}\left(\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right) f_M. \tag{226}$$

For the second term, the collisional drive, an expression for F_0 is required. In the steady state case, it can be shown [5] that

$$F_0 = U\frac{m}{T} f_M \left\{ -v_{\|}B + \frac{v\sigma}{2}\frac{1}{f_c}\int_{\lambda}^{\lambda_c} \frac{\left\langle B^2\right\rangle \mathrm{d}\lambda'}{\left\langle\sqrt{1-\lambda B}\right\rangle} H(\lambda_c - \lambda) \right\}, \tag{227}$$

where $H(\lambda_c - \lambda)$ is the Heaviside step function and f_c is defined in Eq. (71). This solution for F_0 can be obtained by introducing Eq. (66) in Eq. (73) (with $V(v) \simeq$

$m\, v\, f_M\, U/T)$. To simplify the calculation, a small ϵ approximation is introduced at this point and thus we use the small field modulations limit Eq. (227) which, since $f_c \sim 1 + \mathcal{O}\left(\sqrt{\epsilon}\right)$, can be written as

$$F_0 \simeq U \frac{m}{T} f_M \left\{ -v_\| B + \frac{v^2}{2} \left\langle B^2 \right\rangle \int_\lambda^{\lambda_c} \frac{\mathrm{d}\lambda'}{\left\langle v_\| \right\rangle} H(\lambda_c - \lambda) \right\} + \mathcal{O}\left(\sqrt{\epsilon}\right). \tag{228}$$

Multiplying Eq. (226) by $v_\|$, using Eq. (225) and integrating along field lines we can calculate F_1 from

$$F_1 \simeq \int_0^\ell \frac{C(F_0)}{v_\|} d\ell + \frac{2}{3B} \frac{f_M}{p} \Pi_\| + h\left(\psi, \lambda, v\right) + \mathcal{O}\left(\sqrt{\epsilon}\right). \tag{229}$$

Once again, since we integrated over ℓ, a new integration constant h has to be introduced. In Appendix C this function is shown to be small compared to the rest of the terms. We will also neglect the viscous drive on the right side, following a procedure equivalent to the one described in the last paragraph of Section V C.

Using Eq. (228) for F_0, the relevant integral to obtain the next order distortion is

$$\begin{aligned} \int_0^\ell \frac{d\ell}{v_\|} C_R(F_0) \; &= \; -U f_M \frac{m}{T} \frac{\nu_\perp}{2} \left\{ \zeta\left(B^2\right) - \left\langle B^2 \right\rangle \frac{\zeta\left(v_\|\right)}{\left\langle v_\| \right\rangle} \right. \\ &\left. + \; \left\langle B^2 \right\rangle \frac{\lambda}{\left\langle v_\| \right\rangle} \frac{v^2}{2} \left[\zeta\left(\frac{B}{v_\|}\right) - \frac{\zeta\left(v_\|\right)}{\left\langle v_\| \right\rangle} \left\langle \frac{B}{v_\|} \right\rangle \right] \right\}, \end{aligned} \tag{230}$$

where we have defined the operator

$$\zeta\left[A\left(\theta\right)\right] = \int_0^\ell \frac{d\ell'}{B\left(\ell'\right)} A\left(\ell'\right) = \frac{L}{2\pi} \int_0^{L\theta/2\pi} \frac{d\theta'}{B\left(\theta'\right)} A\left(\theta'\right). \tag{231}$$

In order to progress further, an explicit form of $B\left(\theta\right)$ has to be specified. As noted in Chapter II B, for any magnetic field that varies in its own direction we can write

$$B\left(\theta\right) = B_0 \left[1 + 2\epsilon\tau\left(\theta\right)\right], \tag{232}$$

where $\tau\left(\theta\right)$ is continuous, periodic and varies between 0 and 1. Also, as before, the magnitude of the modulation is given by $\epsilon = \left(B_{max} - B_{min}\right)/B_0$. Then, the spatial variation of the parallel velocity can be approximated as (see Appendix C)

$$v_{\parallel} = \frac{\sqrt{2\epsilon}}{\left(2\epsilon + s^2\right)^{1/2}} \upsilon\varsigma\sqrt{1 + s^2\tau\left(\theta\right)}. \tag{233}$$

The new pitch-angle variable s is defined as

$$s^2 = \frac{\lambda\Delta B}{1 - \lambda B_{min}} \simeq \frac{2\epsilon\lambda B_0}{1 - \lambda B_0}, \tag{234}$$

and varies between $s = 0$ (strongly passing particles) for $\lambda = 0$ and $s = 1$ at the trapped-passing boundary were $\lambda = \lambda_c$. Using this variable, the result for F_1 in the static, small ϵ limit, can be written as

$$\begin{aligned} F_1 \simeq{}& \frac{1}{2\epsilon}\left\langle B^2\right\rangle U f_M \frac{\nu_\perp}{v_{th}^2} \frac{s^2}{2\left\langle\sqrt{1 + s^2\tau\left(\theta\right)}\right\rangle} \times \\ & \left[\frac{\zeta\left(\sqrt{1 + s^2\tau\left(\theta\right)}\right)}{\left\langle\sqrt{1 + s^2\tau\left(\theta\right)}\right\rangle}\left\langle\frac{1}{\sqrt{1 + s^2\tau\left(\theta\right)}}\right\rangle - \zeta\left(\frac{1}{\sqrt{1 + s^2\tau\left(\theta\right)}}\right)\right]. \end{aligned} \tag{235}$$

As shown in Appendix C, both the free streaming term and the direct λ-derivative term from the collision operator F_0 do not contribute to the result in Eq. (235). The terms in brackets correspond to the kinetic correction g. For the bumpy cylinder magnetic field model [considering $\tau\left(\theta\right) = \sin^2\theta$], F_1 is expressed in terms of elliptic functions as follows

$$F_1 \simeq \frac{1}{2\epsilon}\frac{\left\langle B^2\right\rangle}{B}\frac{L}{2} U f_M \frac{\nu_\perp}{v_{th}^2}\frac{s^2}{E\left(s^2\right)}\left[\frac{E\left(\theta, s^2\right)}{E\left(s^2\right)} K\left(s^2\right) - F\left(\theta, s^2\right)\right]. \tag{236}$$

The complete elliptic integrals $E\left(s^2\right)$ and $K\left(s^2\right)$ relate to the flux-surface-averaged quantities as

$$K\left(s^2\right) \simeq \frac{\pi}{2B_0}\left\langle\frac{B}{\sqrt{1 - s^2\sin^2\theta}}\right\rangle, \tag{237}$$

$$E\left(s^2\right) \simeq \frac{\pi}{2}\left\langle\sqrt{1 - s^2\sin^2\theta}\right\rangle, \tag{238}$$

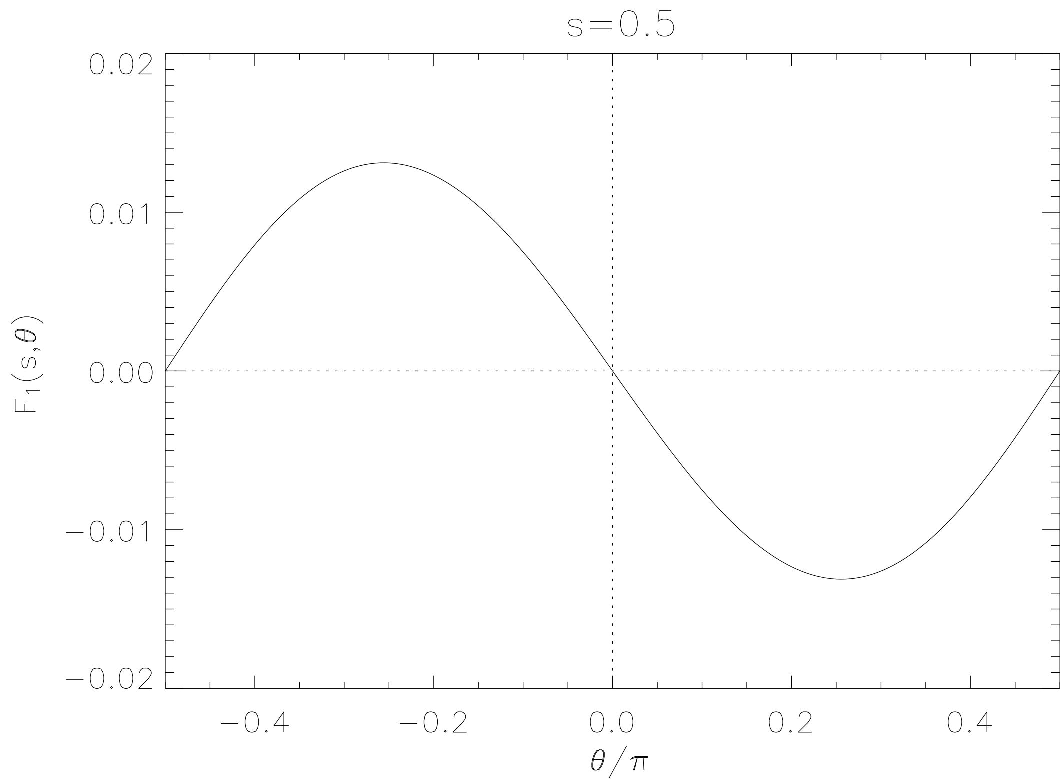

Fig. 6. Variation of $F_1(s,\theta)$ as a function of θ for $s = 0.5$.

and the incomplete integrals $E\left(\theta, s^2\right)$and $F\left(\theta, s^2\right)$ are

$$F\left(\theta, s^2\right) \simeq \frac{\pi}{L}\zeta\left(\frac{B}{\sqrt{1+s^2\sin^2\theta}}\right), \tag{239}$$

$$E\left(\theta, s^2\right) \simeq B_0\frac{\pi}{L}\zeta\left(\sqrt{1+s^2\sin^2\theta}\right). \tag{240}$$

Figure 6 shows the variation of F_1 as a function of θ for an intermediate $s = 0.5$. The function is continuous, finite and has the right periodicity. Thus, the requirements mentioned in the first paragraph of this section are met and the hypothesis in Section III B satisfied. However, as can be seen in Fig. 7, as $s \to 1$ the derivative $\partial/\partial\theta$ becomes infinite at $\theta = 0, \pm\pi/2$. These singular "points" in phase space, corresponding to particles close to the pitch-angle boundary and at field maxima, are problematic

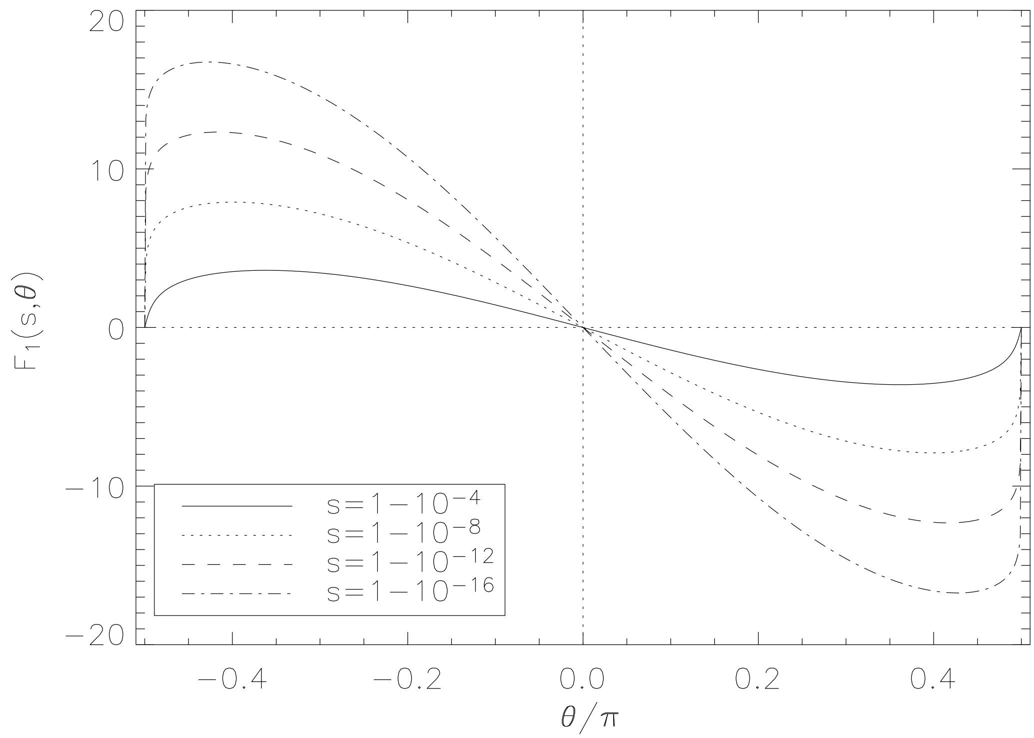

Fig. 7. Variation of $F_1(s,\theta)$ as a function of θ for $s \to 1$.

throughout the calculation. Particles right at the boundary are not either trapped or circulating. When they reach their turning point (B_{max} for λ_c) they cannot bounce back and neither can they move past this point. Thus, the time they spend at B_{max}, which corresponds to $1/\omega_b$ ($\omega_b \to 0$ for these particles), becomes infinite and the approximation $\nu_* = \nu/\epsilon^{3/2}\omega_b \ll 1$ is not valid anymore. A boundary layer treatment for this problem is suggested in Section IX E.

The singularity can also be seen in Fig. 8 which shows the dependence of F_1 on pitch angle, through s, for various angles θ. This singular character does not interfere with the assumption made in Sect. III B when taking the bounce average to obtain Eq. (63). The function F_1 remains continuous and periodic in θ for all $s < 1$ and

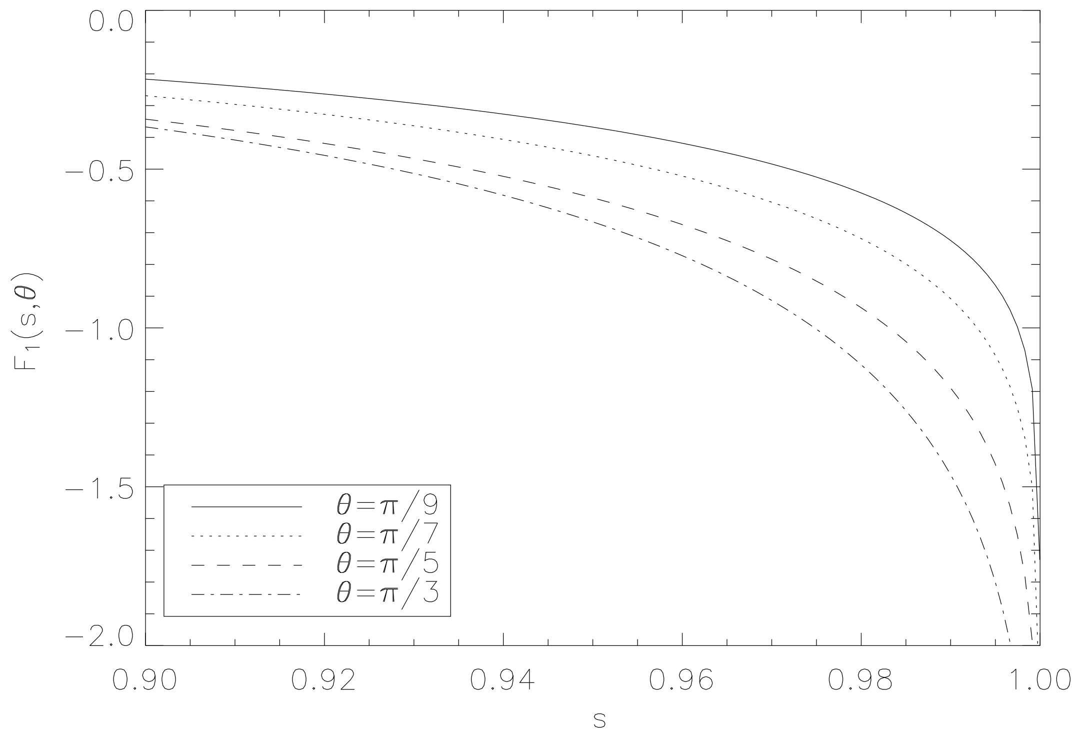

Fig. 8. Variation of $F_1(s, \theta)$ as a function of s for various values of θ.

since it vanishes identically at the field extrema, we have $F_1 = 0$ for the problematic points.

D. Pressure anisotropy

The pressure anisotropy, or $\Pi_\parallel \equiv p_\parallel - p_\perp$, is calculated using Eq. (207). With the result obtained for F_1, Eq. (235) yields, to lowest order in the $\epsilon \ll 1$ expansion,

$$\Pi_\parallel \simeq \frac{1}{\sqrt{2\epsilon}} \frac{f_c}{f_t} mn\mu U \left\langle B^2 \right\rangle \int \frac{\left(s^2+2\epsilon\right)^{-3/2}}{\sqrt{1+s^2\tau(\theta)}} \frac{s^3}{\left\langle \sqrt{1+s^2\tau(\theta)} \right\rangle} \times$$
$$\left[\frac{\zeta\left(\sqrt{1+s^2\tau(\theta)}\right)}{\left\langle \sqrt{1+s^2\tau(\theta)} \right\rangle} \left\langle \frac{B}{\sqrt{1+s^2\tau(\theta)}} \right\rangle - \zeta\left(\frac{B}{\sqrt{1+s^2\tau(\theta)}} \right) \right] ds, \quad (241)$$

in which the operator $\zeta[A(\theta)]$ is the operator defined in Eq. (231). For the bumpy cylinder case, we obtain (See Appendix C)

$$\Pi_\parallel = \frac{1}{\sqrt{2\epsilon}} \frac{3\pi}{8} \frac{f_c}{f_t} mn\mu U \left\langle B^2 \right\rangle \int_0^1 \frac{ds}{\sqrt{1-s^2\sin^2\theta}} \left\{ \frac{F\left(\theta,s^2\right)}{E\left(s^2\right)} - \frac{K\left(s^2\right)}{E^2\left(s^2\right)} E\left(\theta,s^2\right) \right\}. \quad (242)$$

Thus, the pressure anisotropy can be expressed in a compact way and, for the bumpy cylinder magnetic field model, the result is in terms of complete and incomplete elliptic functions.

The term in brackets in Eq. (242) may seem singular for $s=1$ since both $F\left(\theta,s^2\right)$ and $K\left(s^2\right)$ diverge at this point. Also, the square root in the denominator that multiplies the bracket vanishes at $\theta = \pi/2$. In order to examine the behavior of the integrand in Eq. (242) in the vicinity of $s=1$ an integration by parts can be performed and yields

$$\Pi_\parallel = \frac{1}{\sqrt{2\epsilon}} \frac{f_c}{f_t} mn\mu U \left\langle B^2 \right\rangle \left\{ \int_0^1 \frac{ds}{E\left(s^2\right)} \frac{E\left(\theta,s^2\right)}{\left(1-s^2\sin^2\theta\right)^{3/2}} - \frac{E(\theta,1)}{\sqrt{1-\sin^2\theta}} \right\}. \quad (243)$$

Indeed, both terms in brackets are singular at $\theta = (2n+1)\pi/2$. However, the function remains finite and continuous at these points because both singularities are of the same order and cancel.

This cancellation can be seen by inspecting the integral in Eq. (243). For $\theta = \pi/2$ we have

$$\int_0^1 \frac{ds}{E\left(s^2\right)} \frac{E\left(\theta, s^2\right)}{\left(1 - s^2 \sin^2\theta\right)^{3/2}} = \int_0^1 \frac{ds}{\left(1 - s^2\right)^{3/2}} \sim \left.\frac{s}{\sqrt{1 - s^2}}\right|_{s\to 1}. \tag{244}$$

This singularity is exactly the same singularity as that in the second term

$$\frac{E\left(\theta, 1\right)}{\sqrt{1 - \sin^2\theta}} = \left.\tan\theta\right|_{\theta\to\pi/2}. \tag{245}$$

This can be easily seen by changing the variable $s \to \sin\varphi$ in Eq. (244). Both singularities are in real space even though the first one seems to be in velocity space. As we approach $\theta = \pi/2$ one can integrate in s and the singularity is equivalent to the spatial one in Eq. (245). Figure 9 shows the behavior of $\Pi_{\|}$ as a function of θ. There it can clearly be seen that while $\Pi_{\|}$ is a continuous function of θ it has an infinite derivative at $\theta = \pm\pi/2$.

Both Eq. (242) and Eq. (243) give well behaved expressions for the variation of the pressure anisotropy as a function of θ. The continuity of $\Pi_{\|}$ is thus guaranteed and no singularity problems should interfere with any numerical evaluation of the expressions obtained here.

E. Parallel viscous force

In this section, the parallel viscous force is calculated using Eq. (213). Both terms will be included even though the first one has been shown to dominate in Sect. IX B.

The first term in Eq. (213), which is the largest in the small ϵ approximation, is calculated using the result in Eq. (243), and can be written as

$$\begin{aligned}\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|} \simeq\ & -\frac{1}{\sqrt{2\epsilon}}\frac{\pi}{8}\frac{f_c}{f_t} mn\mu\left\langle B^2\right\rangle U \left\{1 + \frac{1}{2}\frac{\sin\left(2\theta\right)E\left(\theta, 1\right)}{\left(1 - \sin^2\theta\right)^{3/2}} - \right.\\ & \left.\int_0^1 \frac{2ds}{\left|1 - s^2\sin^2\theta\right| E\left(s^2\right)}\left[1 + \frac{3}{2}\frac{s^2\sin\left(2\theta\right)E\left(\theta, s^2\right)}{\left(1 - s^2\sin^2\theta\right)^{3/2}}\right]\right\}.\end{aligned} \tag{246}$$

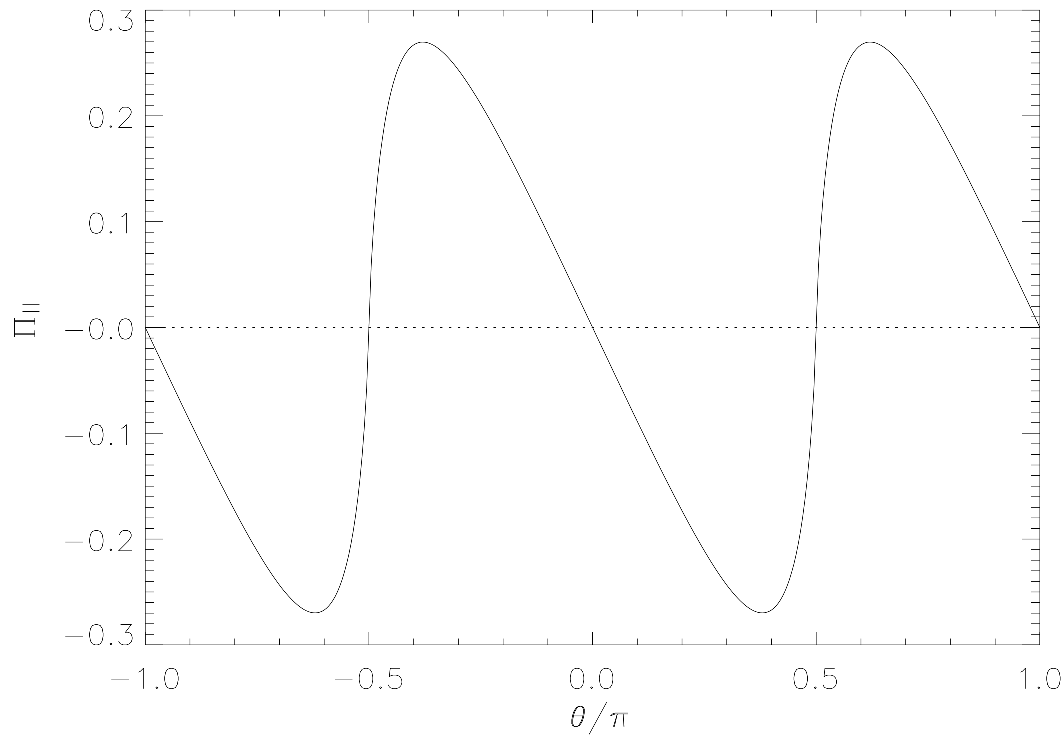

Fig. 9. Variation of the pressure anisotropy $\Pi_{\|}$ as a function of θ.

In the previous section, it was shown that $\Pi_{\|}$ in this model is a continuous function of θ. However, its derivative in θ is singular at $\theta = \pm\pi/2$. In the pressure anisotropy each term in the expressions obtained is singular but the order of the singularities (in the variable θ) is the same and they cancel. When one takes the derivative, this balance at $\theta = \pm\pi/2$ is lost and the terms do not cancel anymore. The singularities in the derivatives of each term in Eq. (243) have different order:

$$\frac{\partial}{\partial\theta}\frac{E\left(\theta, s^2\right)}{\left(1 - s^2\sin^2\theta\right)^{3/2}} \sim \frac{1}{\left|1 - s^2\sin^2\theta\right|} + \frac{s^2\sin\left(2\theta\right)E\left(\theta, s^2\right)}{\left(1 - s^2\sin^2\theta\right)^{5/2}} \Rightarrow \ln\left(x\right)\big|_{x\to 0}\,, \qquad (247)$$

and

$$\frac{\partial}{\partial\theta}\frac{E\left(\theta, 1\right)}{\sqrt{1 - \sin^2\theta}} \sim \frac{1}{\cos^2\theta} \Rightarrow \left.\frac{1}{x^2}\right|_{x\to 0}. \qquad (248)$$

Thus, the boundary term [Eq. (248)] grows faster as $\theta \to \pm\pi/2$ and thus the viscous force diverges as $1/x^2$.

This $1/\cos^2\theta \sim 1/x^2$ singularity arises from integrating up to the boundary at $s = 1$. Analyzing the procedure that was used to obtain Eq. (246), it can be seen that the first term in

$$C_R(g) = \nu_\perp \frac{v_\parallel}{v^2}\left\{\frac{\partial}{\partial\lambda}\frac{v_\parallel\lambda}{B}\frac{\partial g}{\partial\lambda} + \frac{\lambda}{B}v_\parallel\frac{\partial^2 g}{\partial\lambda^2}\right\}, \tag{249}$$

is responsible for this singularity. Thus, a standard boundary layer procedure [3, 10, 26] at $\lambda \simeq \lambda_c$ should smooth the function at the singular points. Within the boundary layer, the factor $v_\parallel\lambda$ does not vary significantly and thus can be treated as a constant (i. e, evaluated at λ_c). Because of this, in a boundary layer treatment the problematic first term in Eq. (249) vanishes and the function should remain finite.

The second term in Eq. (213) is trivial to obtain using

$$\frac{\partial B}{\partial\ell} = 2\epsilon\frac{\pi}{L}\sin\left(\frac{2\pi\ell}{L}\right) = 2\epsilon\frac{\pi}{L}\sin\theta, \tag{250}$$

and the solution already obtained for $\Pi_\parallel$. Recalling that $\left\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel\right\rangle \simeq mn\mu\left\langle B^2\right\rangle U$, the parallel viscous force can then be written as

$$\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel = \left\langle\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel\right\rangle\left[f_1(\theta) + f_2(\theta)\right]. \tag{251}$$

The geometric factors are given by

$$\begin{aligned} f_1(\theta) \;\propto\; & \frac{1}{\sqrt{2\epsilon}}\frac{f_c}{f_t}\left\{\int_0^1 \frac{ds}{\left|1 - s^2\sin^2\theta\right| E(s^2)}\left[1 + \frac{3}{2}\frac{s^2\sin(2\theta)E(\theta,s^2)}{\left(1 - s^2\sin^2\theta\right)^{3/2}}\right] - \right. \\ & \left. 1 - \frac{\sin(2\theta)E(\theta,1)}{2\left(1-\sin^2\theta\right)^{3/2}}\right\} \sim \mathcal{O}(1/\epsilon), \end{aligned} \tag{252}$$

and

$$f_2(\theta) \propto \sqrt{2\epsilon}\frac{f_c\sin(2\theta)}{f_t}\left\{\frac{E(\theta,1)}{\sqrt{1-\sin^2\theta}} - \int_0^1 \frac{ds}{E(s^2)}\frac{E(\theta,s^2)}{\left(1 - s^2\sin^2\theta\right)^{3/2}}\right\} \sim \mathcal{O}(1). \tag{253}$$

Clearly, f_1 is the biggest term, the one which has been accounted for in the previous sections of this chapter, and f_2 corresponds to the smallest term. The θ-variation in these functions is roughly

$$f_1(\theta) \sim \frac{1}{\epsilon}\left(\frac{1}{\cos^2\theta} - \int A(s,\theta)\,ds\right), \tag{254}$$

$$f_2(\ell) \sim \sin^2\theta, \tag{255}$$

and when flux-surface-averaged they yield

$$\langle f_1(\theta)\rangle = 0, \tag{256}$$

$$\langle f_2(\theta)\rangle = 1. \tag{257}$$

The cancellation of the singular terms in $\Pi_{\|} \sim \tan\theta$ results in a similar cancellation of the flux-surface-average of f_1 where $\int \langle A(s,\theta)\rangle\,ds = \langle 1/\cos^2\theta\rangle$. Actually, the continuity of $p_{\|} - p_{\perp}$ is enough to obtain Eq. (256) since $f_1 \sim \partial\Pi_{\|}/\partial\theta$.

Figures 10 and 11 show both terms in Eq. (251) separately for $\epsilon = 0.1$. For all practical purposes, it is a good approximation for the variation of $\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}$ within a flux surface to consider only f_1 in the parallel viscous force; however its flux-surface-average vanishes. The singularities shown in Fig. 11 at $\pm\pi/2$ can be smoothed out by a boundary layer procedure.

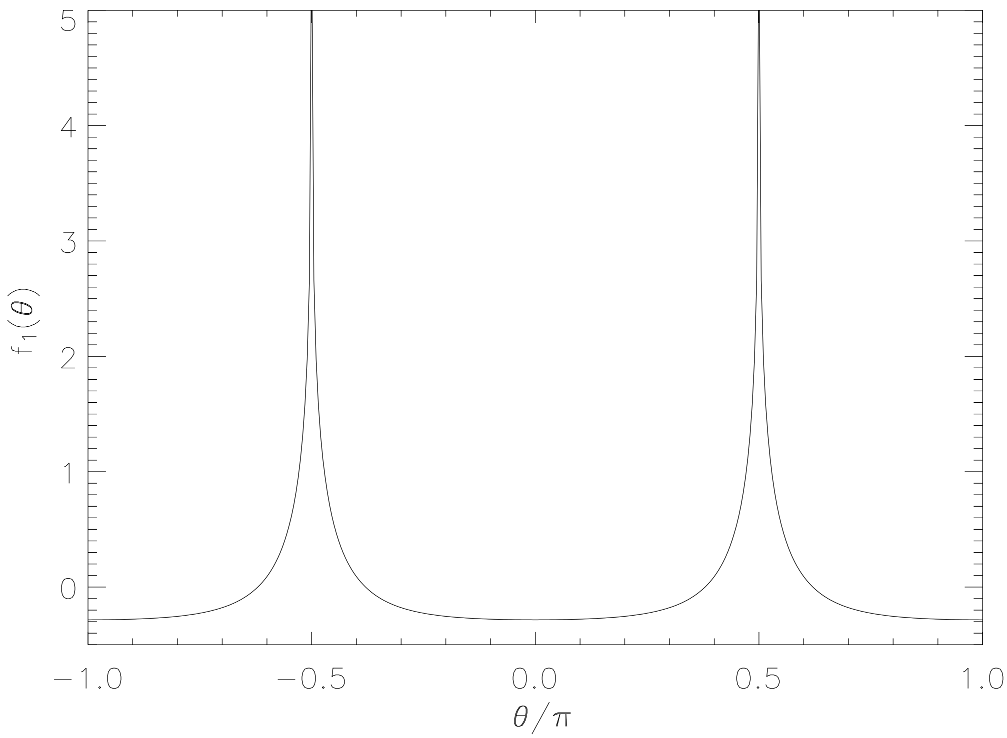

Fig. 10. Variation f_1 as a function of θ.

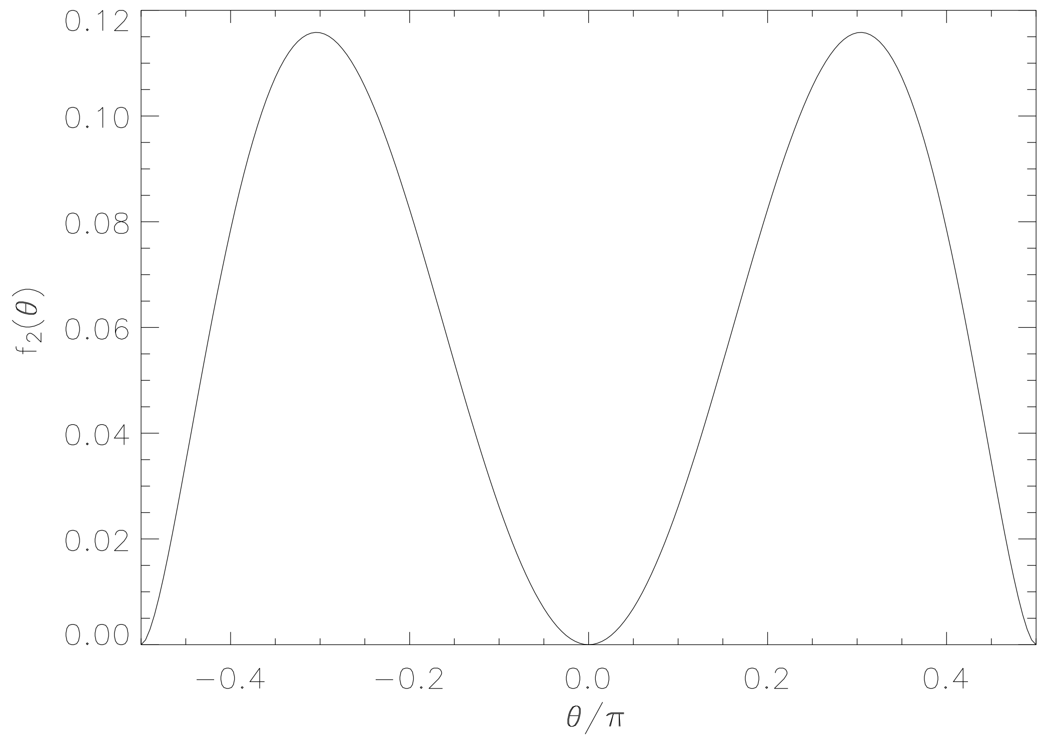

Fig. 11. Variation of f_2 as a function of θ.

X. DYNAMIC PRESSURE ANISOTROPY

In this Chapter, the pressure anisotropy calculation is carried out as a frequency-dependent problem by retaining the time derivative in the DKE. Cordey eigenfunctions are introduced once again through the solution for the lowest order distribution function. Even though the expression for the stress tensor obtained here is more complicated than in the static case, it can be evaluated numerically once the eigenfunctions are generated. The procedure in this time-dependent calculation follows closely the calculation in the previous chapter and thus it will be carried out here without further justification unless necessary.

The time-dependent DKE is

$$\frac{\partial F}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\left(F + \frac{m}{T}v_{\|}BUf_M\right) - C\left(F\right) = \frac{v_{\|}}{p}f_M\mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_{\|}, \tag{258}$$

and can be solved to lowest order as done before. Recall that in solving the differential equation for $\hat{g}_c$ Eq. (88), an expansion of the pitch-angle dependence in Cordey eigenfunctions Λ_n is introduced. The solution for the lowest order distribution function given by Eq. (105) is

$$\widehat{F_0} = \left\{-v_{\|}B + \frac{\bar{\nu}}{\widehat{f}_c}\frac{v\varsigma}{2}\sum_{1}^{\infty}\frac{\eta_n\Lambda_n}{\left(\bar{\nu}\kappa_n - i\omega\right)}\right\}\frac{m}{T}f_M\widehat{U}, \tag{259}$$

where, for simplicity, the source $\widehat{S}$ is assumed to be independent of λ by neglecting the initial distribution in pitch angle $\widehat{g}\,(t=0)$. All quantities in Eq. (259) have been defined in Chapter V.

Having the solution for the lowest order distortion, the next order unaveraged DKE is integrated along field lines as was done before:

$$\widehat{F_1} = \int_0^\theta d\theta\,\frac{C(\widehat{F_0})}{v_{||}} + i\omega\int_0^\theta d\theta\,\frac{1}{v_{||}}\widehat{F_0}. \tag{260}$$

The second term in Eq. (260) is the direct time derivative after a Laplace transform is calculated as in Section III B. Then, the pressure anisotropy can be calculated using Eq. (207) as

$$\widehat{\Pi}_{\parallel} = -m \sum_{\sigma} \sigma\pi \int_0^{\infty} dv\, v^5 \int_0^{\lambda_c} \frac{B d\lambda}{v_{\parallel}} \left(\frac{3}{2}\lambda B - 1\right) \int_0^{\theta} \frac{d\theta}{v_{\parallel}} \left[C(\widehat{F_0}) + i\omega \widehat{F_0}\right]. \tag{261}$$

Introducing the $\widehat{F_0}$ given by Eq. (259) one can obtain, for small ϵ

$$\widehat{\Pi}_{\parallel} \simeq \frac{1}{2\epsilon} nm\widehat{U} \sum_{1}^{\infty} \eta_n I_n(\theta) \int d^3v \frac{\bar{\nu}}{v_{th}^2} v^2 \frac{f_M}{n} \frac{1}{\widehat{f_c}} \frac{1}{\kappa_n - i\omega/\bar{\nu}}, \tag{262}$$

where the geometric coefficient $I_n(\theta)$ is given by

$$I_n(\theta) = \frac{1}{4}\left\{ \int_0^1 \frac{s^2 E(\theta, s)}{\left(1 - s^2 \sin^2\theta\right)^{3/2}} \frac{\partial \Lambda_n}{\partial s} ds - \frac{E(\theta, 1)}{\sqrt{1-\sin^2\theta}} \left.\frac{\partial \Lambda_n}{\partial s}\right|_{s=1} \right\}. \tag{263}$$

The first term can be expressed in a more convenient form in terms of the eigenfunctions (and not their derivatives) by integrating by parts. This yields

$$\begin{aligned} I_n(\theta) &= -\frac{1}{4} \frac{E(\theta, 1)}{\sqrt{1-\sin^2\theta}} \left.\frac{\partial \Lambda_n}{\partial s}\right|_{s=1} \\ &\quad \frac{1}{4} \int_0^1 \frac{s\Lambda_n}{\left(1 - s^2\sin^2\theta\right)^{3/2}} \left[\frac{3E(\theta, s)}{\left(1 - s^2 \sin^2\theta\right)} - F(\theta, s)\right] ds. \end{aligned} \tag{264}$$

In this case, we cannot proceed any further. The eigenfunction equation that Λ_n satisfies is for a bounce-averaged collision operator. In this frequency dependent case the DKE is not bounce-averaged and the spatial dependence remains present in the problem. Because of this, the differential operator on $\hat{g}_c$ is not flux-surface averaged. Thus, there is no trivial way to introduce the eigenvalues or to use the orthogonality condition to simplify the expression for $I_n(\theta)$. However, as pointed out before, the eigenfunctions can be generated numerically and introduced in Eq. (264) to obtain a numerical solution for the pressure anisotropy.

XI. SUMMARY

A simple inhomogeneous magnetic field model has been used to calculate the kinetic closure for the parallel viscous force $\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\|}\right\rangle$ in the banana collisionality regime. This simple bumpy cylinder magnetic field model retains the effect of trapped particles and can be readily extended to more complicated three-dimensional geometries.

In carrying out the calculation a formal Laplace transform was introduced which retains the initial value character of the problem. The closure is obtained by using the conservation properties of the distribution function and the collision operator. In frequency space and including heat flux effects, the closure obtained for the parallel viscous force [Eq. (190)] can be expressed in a compact form

$$\left\langle \widehat{\mathbf{B}\cdot\nabla\cdot\Pi_{\|}}\right\rangle = nm\left\langle B^2\right\rangle\left(\widehat{U}\,\widehat{v}_{00}-\frac{2}{5p}\widehat{Q}\,\widehat{v}_{01}+U_0+\widehat{\Upsilon}_0\right), \tag{265}$$

were all the coefficients are given in terms of integrals of the (numerically generated) Cordey eigenfunctions. Using this expression (neglecting heat flux effects), the parallel flow evolution was obtained as a Volterra-type integral equation [Eq. (156)]. The result was also extended to an axiaxisymmetricsymmetric geometry [Eq. (164)] where a similar behavior was found. In general, the evolution for the parallel/poloidal flow was expressed as

$$U(t)=h(t)+\int_0^t K(t;\tau)\,U(\tau)\,d\tau. \tag{266}$$

The inhomogeneous term and the integration kernel are given in terms of Cordey eigenfunctions and were calculated numerically in the small inverse aspect ratio $\epsilon \equiv r/R_0 \ll 1$ approximation.

By considering the parallel Ohm's law, the trapped-particle corrections to the electrical conductivity were obtained in frequency space in the form [Eq. (140)]

$$\widehat{\sigma}(\omega)=\frac{n_e e^2}{m_e\nu_e\widehat{\alpha}_e}\frac{1}{(\omega)}. \tag{267}$$

This expression was plotted in Figures 2 and 3 for both large and small frequency asymptotic limits using an expansion in terms of Legendre polynomials for $\epsilon \ll 1$. The general results from the numerical and analytical calculations coincide and can be summarized as

$$\omega \ll \nu_e \begin{cases} Re\left[\widehat{\sigma}\left(\omega\right)\right]/\sigma_r \sim 1 + \mathcal{O}\left(\sqrt{\epsilon}\right)\left(\omega/\nu_e\right)^2, \\ Im\left[\widehat{\sigma}\left(\omega\right)\right]/\sigma_r \sim \omega/\nu_e, \end{cases} \tag{268}$$

$$\omega \gg \nu_e \begin{cases} Re\left[\widehat{\sigma}\left(\omega\right)\right]/\sigma_r \sim \left(\nu_e/\omega\right)^2, \\ Im\left[\widehat{\sigma}\left(\omega\right)\right]/\sigma_r \sim \nu_e/\omega. \end{cases} \tag{269}$$

The complexity of the expressions obtained for the closure and the conductivity [Eq. (140)] call for an expansion for small magnetic field modulations in order to calculate a time dependent closure. With this expansion, the inverse Laplace transform can be calculated to any order. The time dependent closure obtained in this small ϵ limit, to lowest order, relates to the time dependent flow as follows [Eq. (117)]

$$\begin{aligned} \left\langle \mathbf{B}\cdot\nabla\cdot\Pi_{\parallel}\right\rangle &= \left\langle B^2\right\rangle mn\int d^3v\,\frac{v^2}{3}\frac{f_M}{n}\bar{\nu}\frac{m}{T}\left\{U\left(t\right)\left(1-f_c\right)+\frac{1}{\bar{\nu}}\frac{\partial U\left(t\right)}{\partial t}\left(1-\sum\gamma_n\right)\right. \\ &\left.+\sum\frac{\gamma_n}{\kappa_n}\left(\kappa_n-1\right)^2\int_0^t\frac{dU}{d\tau}e^{-\bar{\nu}\kappa_n(t-\tau)}d\tau\right\}. \end{aligned} \tag{270}$$

This explicit time-dependent closure may be useful in numerical codes as well as for the theoretical modeling of fast phenomena ($t \lesssim 1/\nu$).

The variation of the parallel viscous was shown, by a simple argument (in Section IX B), to be larger than the averaged component. To calculate the unaveraged closure, the first order drift-kinetic equation was integrated along field lines to obtain the lowest order contribution to the stress tensor moment Eq. (13). With this result, the pressure anisotropy $p_{\parallel} - p_{\perp}$ was obtained, in steady state case and for small ϵ, as a continuous functional of the distance along a field line [Eq. (243)]. Taking the divergence and projecting the pressure anisotropy in the magnetic field direction, the parallel viscous force was obtained in the form

$$
\begin{array}{ccccc}
\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|} & = \left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle & \{f_1(\theta) & + & f_2(\theta)\} \\
1/\sqrt{\epsilon} & \sqrt{\epsilon} & \sim 1/\epsilon cos^2\theta & & \sim sin^2\theta \\
 & & \langle f_1(\theta)\rangle = 0 & & \langle f_2(\theta)\rangle = 1.
\end{array}
\tag{271}
$$

As expected, the largest term is the one that averages to zero [Eq. (252)]. This term could modify the plasma dynamics significantly compared with what is obtained with the averaged result. The extension to the dynamic case is not simple since the solution for the lowest order $\widehat{F_0}$ is given in terms of infinite sums over Cordey eigenfunctions which are solutions of the flux-surface-averaged drift-kinetic equation.

The results obtained here have various applications that are yet to be exploited. The time or frequency dependent closure can be used to explore the dynamics of the transition of the perpendicular dielectric from regular Alfvenic to enhanced neoclassical regimes [27] and for the effects of flow dynamics on neoclassical tearing modes [28]. The spatially varying closure can also be applied to neoclassical tearing modes where the neoclassical contribution to the perpendicular current is calculated from $J_\perp \propto \mathbf{B}\times\nabla\cdot\mathbf{\Pi}/B^2$.

In addition to the applications mentioned above, there are various improvements that could be made to the results here obtained. Of particular interest is the solution for the parallel viscous force varying along a field line in the dynamic case. Although a solution was sketched here, its form cannot be explored in general without the use of a numerical code. It could be worthwhile pursuing the analytical closure in terms of some other eigenfunction expansion that could be more easily handled to complete the calculation. The extension of the entire calculation to toroidal or other geometries is also an interesting and useful generalization still to be done.

Appendix A: RELATION OF $\widehat{V}$ TO $\widehat{U}$ AND $\widehat{Q}$

To obtain the coefficients in the expansion for $\widehat{V}$ given in Eq. (77), multiply the equation by $v\,L_m^{3/2}$ on both sides to obtain

$$\widehat{V}\,L_i^{3/2} v = \frac{m}{T}\frac{f_M}{n}\sum_j \widehat{V}_j\, v^2 L_i^{3/2} L_j^{3/2}. \tag{A1}$$

The orthogonality condition for the Laguerre polynomials is given by

$$\int d^3v\, x^k f_M L_i^{k+1/2} L_j^{k+1/2} = n\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(i+k+3/2\right)}{\Gamma\left(i+1\right)}\delta_{ij}, \tag{A2}$$

where the polynomials are evaluated in $x \equiv mv^2/2T$ and for this case $k=1$. Then, for the expansion in Eq. (77), the relevant integrals are

$$\int d^3v\, v^2 f_M L_i^{3/2} L_j^{3/2} = \frac{nT}{m}\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(i+5/2\right)}{\Gamma\left(i+1\right)}\delta_{ij}. \tag{A3}$$

Integrating Eq. (A1) and using this relation, the coefficients can be written in terms of parallel momentum moments of $\widehat{V}$

$$\begin{aligned} \widehat{V}_j &= \sqrt{\pi}\frac{\Gamma\left(i+1\right)}{\Gamma\left(i+5/2\right)}\int d^3v\,\widehat{V}\,L_j^{3/2} v \\ &= \frac{3\sqrt{\pi}}{2}\frac{\Gamma\left(i+1\right)}{\Gamma\left(i+5/2\right)}\int d^3v\,\widehat{V}\,L_j^{3/2} v_\parallel. \end{aligned} \tag{A4}$$

The pitch-angle part of the previous equation can be written in terms of the solution for $\widehat{g}$ by using the definition:

$$\widehat{V}\left(\psi, v\right) = \frac{3}{4}\sum_\varsigma \varsigma \int_0^{\lambda_c} d\lambda\,\widehat{g}. \tag{A5}$$

Then, the coefficients are calculated as

$$\begin{aligned} \widehat{V}_j &= \frac{3\sqrt{\pi}}{2}\frac{\Gamma\left(i+1\right)}{\Gamma\left(i+5/2\right)}\int d^3v\,L_j^{3/2} v_\parallel \int_0^{\lambda_c} d\lambda\,\widehat{g} \\ &= \sqrt{\pi}\frac{\Gamma\left(i+1\right)}{\Gamma\left(i+5/2\right)}\int d^3v\,L_j^{3/2} v \int_0^{\lambda_c} d\lambda\,\widehat{g}. \end{aligned} \tag{A6}$$

The first two coefficients are required since the expansion of the distribution function is up to $j = 1$. The constraint that the parallel momentum moment of $\widehat{F}_0$ has to vanish can be used to relate these coefficients with $\widehat{U}$ and $\widehat{Q}$. That is, the integral

$$\int d^3v\, v_\parallel \left(L_0^{3/2}, L_1^{3/2}\right) \widehat{F_0} = 0, \tag{A7}$$

gives the following relation:

$$\int d^3v\, v_\parallel \left(L_0^{3/2},\, L_1^{3/2}\right) \widehat{g} = \frac{m}{T} B \int d^3v\, \left(\widehat{U} L_0^{3/2} - \frac{2}{5p}\widehat{Q}\, L_1^{3/2}\right) \left(L_0^{3/2},\, L_1^{3/2}\right) f_M v_\parallel^2. \tag{A8}$$

By using the orthogonality condition in Eq. (A3) we obtain from the first two energy moments

$$\int d^3v\, v_\parallel \widehat{g} = \frac{2m}{T} B\widehat{U} \int d^3v\, \left(L_0^{3/2}\right)^2 \frac{v^2}{3} f_M = B\widehat{U}\, n, \tag{A9}$$

and

$$\int d^3v\, v_\parallel L_1^{3/2} \widehat{g} = -\frac{4}{5p}\frac{m}{T}\widehat{Q}\, B \int d^3v\, \left(L_1^{3/2}\right)^2 \frac{v^2}{3} f_M = -\frac{2}{5p} B\widehat{Q}\, n. \tag{A10}$$

Thus, the first two coefficients in the expansion are

$$V_0 = \widehat{U}\, n, \qquad V_1 = -\frac{2}{5p}\widehat{Q}\, n. \tag{A11}$$

Appendix B: TIME DEPENDENT CLOSURE AND FLOW EVOLUTION

In this appendix the inverse Laplace transform of the frequency-dependent closure and the parallel and poloidal flow evolution equations are developed in some detail. For the parallel stress, the term in curly brackets in Eq. (116) can be inverted term by term:

$$L^{-1}\left\{\widehat{f}_t\right\} = \sum \left(\kappa_n - 1\right) \bar{\nu}\gamma_n e^{-\kappa_n \bar{\nu} t} + \left(1 - \sum \gamma_n\right) \delta\left(t\right). \tag{B1}$$

Defining the integral $I_n(t) = \int_0^t d\tau e^{-\kappa_n \bar{\nu}(t-\tau)} U(\tau)$, the convolution with the parallel flow variable is

$$L^{-1}\left\{\widehat{f}_t \widehat{U}\right\} = \left(1 - \sum \gamma_n\right) U(t) + \bar{\nu} \sum \gamma_n (\kappa_n - 1) I_n(t), \tag{B2}$$

and the triple convolution can be calculated as follows:

$$\begin{aligned} L^{-1}\left\{-\frac{i\omega}{\bar{\nu}} \widehat{f}_t \widehat{U}\right\} &= \frac{1}{\bar{\nu}}\left(1 - \sum \gamma_n\right)\left[\frac{\partial U(t)}{\partial t} + U_0 \delta(t)\right] \\ &+ \sum \gamma_n (\kappa_n - 1)(U(t) - \kappa_n \bar{\nu} I_n(t)). \end{aligned} \tag{B3}$$

Putting these results together we have

$$\begin{aligned} L^{-1}\left\{\frac{\widehat{f}_t}{\widehat{f}_c} \widehat{U}\right\} &\simeq \frac{1}{\bar{\nu}}\left(1 - \sum \gamma_n\right)\left(\frac{\partial U(t)}{\partial t} + U_0 \delta(t)\right) \\ &+ \left[1 + \sum \gamma_n (\kappa_n - 2)\right] U(t) - \sum \gamma_n (\kappa_n - 1)^2 \bar{\nu} I_n(t), \end{aligned} \tag{B4}$$

which, after some manipulation leads to the dynamic closure in Eq. (117) for $t > 0$.

To obtain the time evolution of the parallel flow variable, the inverse Laplace transform is calculated on both sides of Eq. (153). For the right side, using Eq. (115) and the previous results one can obtain

$$\begin{aligned} L^{-1}\left\{\frac{\widehat{f}_g}{\widehat{f}_c}\right\} &\simeq \sum_{m,n} \{\delta(t) \gamma_n \chi_m - \bar{\nu}(\kappa_m - 1) e^{-\kappa_m \bar{\nu} t} [(\chi_m \gamma_n + \chi_n \gamma_m) \\ &- \bar{\nu} \chi_m \gamma_n (\kappa_n - 1) \int_0^t e^{\bar{\nu}\tau(\kappa_m - \kappa_n)} d\tau. \end{aligned} \tag{B5}$$

To eliminate the delta functions, a time integral is taken on both sides and after a simple calculation one can solve for $U(t)$. Equation (156) is obtained for which we define the inhomogeneous term and the integration kernel as follows:

$$\begin{aligned} h(t) &= a \int d^3 v \frac{m}{T} \frac{v^2}{3} \frac{f_M}{n} \sum_{m,n} \left\{\frac{\chi_n \gamma_m}{\kappa_m \kappa_n} - (\kappa_m - 1) \frac{\chi_m \gamma_n}{\kappa_m} e^{-\kappa_m \bar{\nu} t}\right. \\ &- \left.(\kappa_m - 1) \frac{\chi_m \gamma_n}{\kappa_m} e^{-\kappa_m \bar{\nu} t}\left[1 + \frac{\chi_n \gamma_m}{\kappa_n \chi_m \gamma_n} - \bar{\nu}(\kappa_n - 1) \int_0^t e^{\bar{\nu}\tau(\kappa_m - \kappa_n)} d\tau\right]\right\}, \end{aligned} \tag{B6}$$

and

$$K\left(t;\tau\right)=a\int d^{3}v\bar{\nu}\frac{m}{T}\frac{v^{2}}{3}\frac{f_{M}}{n}\left\{1+\sum\frac{\gamma_{m}}{\kappa_{m}}\left[\left(\kappa_{m}-1\right)^{2}e^{-\kappa_{m}\bar{\nu}\left(t-\tau\right)}-1\right]\right\}. \tag{B7}$$

The common factor for both expressions is $a=\left(\sum\gamma_{m}-2\right)^{-1}$.

For toroidal geometry, the inverse Laplace transform of each term in Eq. (163) is calculated as before. A time integral has to be calculated in this case also and one can obtain Eq. (164) where the kernel and inhomogeneous term are defined by

$$h_{\theta}\left(t\right)=b\left[\left(1+2q^{2}-\frac{\langle B^{2}\rangle}{\langle B_{P}^{2}\rangle}\right)U_{\theta 0}-\frac{4\pi}{nm}\langle\phi'\left(t\right)-\phi_{0}'\rangle\right], \tag{B8}$$

and

$$K_{\theta}\left(t;\tau\right)=-b\frac{\langle B^{2}\rangle}{\langle B_{P}^{2}\rangle}\frac{m}{T}\int d^{3}v\,\bar{\nu}\frac{v^{2}}{3}\frac{f_{M}}{n}\left\{1-\sum\frac{\gamma_{n}}{\kappa_{n}}\left[1+\left(\kappa_{n}-1\right)^{2}e^{-\kappa_{n}\bar{\nu}\left(t-\tau\right)}\right]\right\}, \tag{B9}$$

and the factor b for both expressions is here

$$b=\left[1+2q^{2}+\frac{\langle B^{2}\rangle}{\langle B_{P}^{2}\rangle}\left(1-\sum\gamma_{n}\right)\right]^{-1}\simeq\frac{\langle B^{2}\rangle}{\langle B_{P}^{2}\rangle}\left(1-\sum\gamma_{n}\right), \tag{B10}$$

where the last equality is valid for small ϵ.

Using Eq. (B10) and the values in Eqs. (124)–(127), the kernel of integration $K_{\theta}\left(t;t\right)\sim 1/\tau_{p}$ for the $t\ll\bar{\nu}$ case in Section VII B is

$$K_{\theta}\left(t;t\right)\simeq\left(\frac{0.51}{\epsilon}-2.69\right)\frac{m}{T}\int d^{3}v\,\bar{\nu}\frac{v^{2}}{3}\frac{f_{M}}{n}. \tag{B11}$$

In the long time case, for the static limit at the end of Section VII B we have, for $\int_{0}^{t}K_{\theta}\left(t;\tau\right)d\tau=1$,

$$t-\sum\frac{\gamma_{n}}{\kappa_{n}}\left[t+\int_{0}^{t}\left(\kappa_{n}-1\right)^{2}e^{-\kappa_{n}\bar{\nu}\left(t-\tau\right)}d\tau\right]=1-\sum\gamma_{n}, \tag{B12}$$

or, solving for $t=\tau_{p}$,

$$\tau_{p}\simeq\left(\frac{\sqrt{\epsilon}}{0.31}\right)/\frac{m}{T}\int d^{3}v\,\bar{\nu}\frac{v^{2}}{3}\frac{f_{M}}{n}, \tag{B13}$$

which yields the damping rate in Eq. (169).

Appendix C: PRESSURE ANISOTROPY

The pitch-angle variable introduced in Chapter IX is

$$s^2 = \frac{\lambda \Delta B}{1 - \lambda B_{min}}. \tag{C1}$$

Using the general form for the magnetic field $B(\theta) = B_{min}[1 + 2\epsilon\tau(\theta)]$ with $\epsilon = \Delta B/2B_{min}$, we have

$$\lambda = \frac{1}{B_{min}} \frac{s^2}{s^2 + 2\epsilon}, \tag{C2}$$

and thus

$$d\lambda = \frac{2\epsilon}{(s^2 + 2\epsilon)^2} \frac{2s\, ds}{B_{min}}. \tag{C3}$$

The parallel velocity can be calculated from $v_\parallel = \varsigma v \sqrt{1 - \lambda B}$ as

$$v_\parallel = \varsigma v \sqrt{(1 - \lambda B_{min}) \left(1 + \frac{2\epsilon \lambda B_{min}}{1 - \lambda B_{min}} \tau(\theta)\right)}. \tag{C4}$$

Rearranging terms and using Eq. (C1), we can write $v_\parallel$ in the following convenient way

$$v_\parallel = \varsigma v \frac{\sqrt{2\epsilon}}{(2\epsilon + s^2)^{1/2}} \sqrt{1 + s^2 \tau(\theta)}, \tag{C5}$$

which is Eq. (233). To calculate F_1, consider Eq. (230) with $B(\theta) = B_0[1 + 2\epsilon\tau(\theta)]$. Changing the pitch-angle variable to s, we obtain

$$\begin{aligned} \int_0^\theta \frac{d\theta}{v_\parallel} C_R(F_0) \;\simeq\; & \left\langle B^2 \right\rangle U f_M \frac{\nu_\perp}{v_{th}^2} \left\{ \frac{\zeta\left(\sqrt{1 + s^2\tau(\theta)}\right)}{\left\langle \sqrt{1 + s^2\tau(\theta)} \right\rangle} - \zeta\left([1 + 2\epsilon\tau(\theta)]^2\right) \right. \\ & - \frac{1}{2\epsilon} \frac{s^2/2}{\left\langle \sqrt{1 + s^2\tau(\theta)} \right\rangle} \left[\zeta\left(\frac{1}{\sqrt{1 + s^2\tau(\theta)}}\right) \right. \\ & \left.\left. - \frac{\zeta\left(\sqrt{1 + s^2\tau(\theta)}\right)}{\left\langle \sqrt{1 + s^2\tau(\theta)} \right\rangle} \left\langle \frac{1}{\sqrt{1 + s^2\tau(\theta)}} \right\rangle \right] \right\}. \end{aligned} \tag{C6}$$

The operator ζ is defined in Eq. (231). Clearly, the free streaming term and the direct λ-derivative term [first two terms in Eq. (C6)] are smaller in the ϵ expansion and were thus neglected in the expression for F_1 in Section IX C [Eq. (235)].

The pressure anisotropy can be calculated as

$$\begin{aligned}\Pi_\parallel &= -m\sum_\sigma \varsigma\pi\int_0^\infty dv\, v^5\int_0^{1/B}\frac{Bd\lambda}{v_\parallel}\left(\frac{3}{2}\lambda B-1\right)F_1\\ &\sim -m\sum_\sigma \pi\int_0^\infty dv\, v^4\int_0^1\frac{s\,ds\sqrt{2\epsilon}}{\left(s^2+2\epsilon\right)^{3/2}}\frac{1}{\sqrt{1+s^2\tau\left(\theta\right)}}\int_0^\theta\frac{d\theta}{v_\parallel}C(F_0),\end{aligned} \tag{C7}$$

from which the expression for the general case [Eq. (241)] is trivial to obtain assuming the first two terms in Eq. (C6) can be neglected as before. This is not trivial to see and is here verified for the bumpy cylinder case. The contribution of the free streaming term to $\Pi_\parallel$ is

$$\int_0^{1/B}\frac{Bd\lambda}{v_\parallel}\zeta\left(B^2\right)\simeq\frac{2}{\sigma v}B_0\theta+\mathcal{O}\left(\sqrt{2\epsilon}\right), \tag{C8}$$

which is $\mathcal{O}\left(\epsilon^0\right)$. The second term is

$$\int_0^{\lambda_c}\frac{d\lambda}{v_\parallel}\frac{\zeta\left(v_\parallel\right)}{\left\langle v_\parallel\right\rangle}=-\frac{4\sqrt{2\epsilon}}{\sigma v B_0^2}\int_0^1\frac{s\,dk}{\left(s^2+2\epsilon\right)^{3/2}}\frac{1}{\sqrt{1-s^2\sin^2\left(\theta\right)}}\frac{E\left(\theta,s^2\right)}{2E\left(s^2\right)}. \tag{C9}$$

In this case the integrand is singular at $s=0$. However, adding and subtracting the term

$$\int_0^1\frac{s\,ds}{\left(s^2+2\epsilon\right)^{3/2}}\simeq\frac{1}{\sqrt{2\epsilon}}-1+\frac{2\epsilon}{2}, \tag{C10}$$

one can write Eq. (C9) as

$$\int_0^{\lambda_c}\frac{d\lambda}{v_\parallel}\frac{\zeta\left(v_\parallel\right)}{\left\langle v_\parallel\right\rangle}\simeq\frac{2\theta}{\varsigma v B}\left\{1-\sqrt{2\epsilon}\left[1-\int_0^1\frac{ds}{s^2}\left[\frac{L/\ell}{\sqrt{1-s^2\sin^2\left(\theta\right)}}\frac{E\left(\theta,s^2\right)}{2E\left(s^2\right)}-1\right]\right]\right\}, \tag{C11}$$

which is also $\mathcal{O}\left(1\right)$. Thus, the combination of the first two terms is $\mathcal{O}\left(\sqrt{2\epsilon}\right)$ and can be neglected. Keeping only the last two terms in Eq. (C6), we obtain

$$\Pi_{\|} \sim \frac{1}{\sqrt{2\epsilon}} m \frac{\langle B^2 \rangle}{4} UL \sum_{\varsigma} \pi \int dv\, v^4 f_M \frac{\nu_\perp}{v_{th}^2} \times \int_0^1 \frac{ds}{\sqrt{1+s^2\tau(\theta)}} \left[\frac{F(\theta,s^2)}{E(\theta)} - \frac{E(\theta,s^2)}{E^2(s^2)} K(s^2) \right], \tag{C12}$$

from which Eq. (242) follows.

To calculate the integration constant $h(\psi,\lambda,v)$, consider the flux surface average of Eq. (229)

$$\oint \frac{d\theta}{B} F_1 \simeq \oint \frac{d\theta}{B} \int_0^\theta \frac{d\theta'}{v_{\|}} C_R(F_0) + \frac{2}{3B} \frac{f_M}{p} \oint \frac{d\theta}{B} \Pi_{\|} + \oint \frac{d\theta}{B} h(\psi,\lambda,v). \tag{C13}$$

Since the distribution function is periodic in θ and from Section IX B $\oint d\theta \Pi_{\|}/B \sim \mathcal{O}(\sqrt{\epsilon})$, we have

$$\oint \frac{d\theta}{B} h(\psi,\lambda,v) = -\oint \frac{d\theta}{B} \int_0^\theta \frac{d\theta'}{v_{\|}} C_R(F_0) + \mathcal{O}(\sqrt{\epsilon}). \tag{C14}$$

Using Eq. (C6), with $\tau(\theta) = \sin^2(\theta/2)$, we obtain

$$\oint \frac{d\theta}{B} h(\psi,\lambda,v) = \frac{U\langle B^2\rangle L}{B} f_M \frac{\nu_\perp}{v_{th}^2} \oint \frac{d\theta}{B} \left\{ 1 - \frac{E(\theta,s^2)}{2E(s^2)} + \frac{1}{2\epsilon}\frac{k^2}{2} \left[\frac{1}{2E(s^2)} F(\theta,s^2) - \frac{E(\theta,s^2)}{2E^2(s^2)} K(s^2) \right] \right\}, \tag{C15}$$

where all terms have been included. It can be easily seen that the second line vanishes when flux-surface averaged and thus

$$h(\psi,\lambda,v) = \frac{\langle B^2\rangle L}{2B} U f_M \frac{\nu_\perp}{v_{th}^2} \sim \mathcal{O}(1). \tag{C16}$$

Then, since $\int_0^\theta d\theta\, C_R(F_0)/v_{\|} \sim \mathcal{O}(\epsilon^{-1})$, the integration constant $h(\psi,\lambda,v)$ can be neglected and

$$F_1 = \int_0^\theta \frac{d\theta}{v_{\|}} C_R(F_0) + h(\psi,\lambda,v) \sim \int_0^\theta \frac{d\theta}{v_{\|}} C_R(F_0). \tag{C17}$$

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