Sheared flow effects on ballooning instabilities in three-dimensional equilibria

C. C. Hegna

Department of Engineering Physics

University of Wisconsin

Madison, WI 53706-1687

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Abstract

The stability of ideal MHD ballooning modes in the presence of sheared flow is investigated for three-dimensional equilibria. Application of ballooning formalism reduces the problem to a partial differential equation in three dimensions that can be solved in the limit of small flow. Analytic calculations demonstrate the stabilizing effect of shear flow. The derived stability criterion generalizes prior work related to axisymmetric equilibrium with sheared toroidal flow.

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I. INTRODUCTION

A distinctive property of many of the new stellarator experimental designs is the presence of a quasi-symmetry [1, 2, 3]. In these devices, neoclassical theory predicts weak damping of the plasma flows in the symmetry direction. Recent experimental observations of the Helically Symmetric Experiment (HSX) have demonstrated this effect [4] for quasi-helically symmetric stellarators. With weak viscous damping, plasma flows can be more prominent in this class of three-dimensional equilibria relative to conventional stellarators. These flows can then produce a stabilizing effect on magnetohydrodynamic (MHD) modes, microinstabilities and anomalous transport. In the following, an explicit demonstration of this stabilizating effect on ideal MHD ballooning modes in three-dimensional configurations is presented.

A number of authors have addressed the effect of sheared toroidal flow on the high-k stability properties of tokamak plasmas [5, 6, 7, 8, 9, 10, 11, 12]. These calculations rely on using a ballooning mode WKB-like formalism [13, 14]. In the static limit (no equilibrium flow), the ballooning transformation produces a lowest order solution in the form of an ordinary differential equation along the field line. Solutions of this equation yield local eigenvalues that depend upon the flux surface and normalized radial wavelength, $\gamma = \gamma(\Psi, \theta_k)$. Here, Ψ labels the flux surface and θ_k is the ballooning mode angle. Most studies on sheared flow's influence on ballooning stability use the Cooper form for the eikonal, which is explicity time dependent [5]. The theory is then formulated as an initial value problem that can be solved

numerically or analytically using asymptotic techniques. These calculations show that sheared flow has the effect of convecting the mode along θ_k arising from the relative effects of magnetic shear and rotation shear. Eventually, all points in θ_k are experienced and a long time averaged growth rate can be identified. This averaged growth rate was originally derived in Ref. [8] and given by the formula

$$\gamma = <\gamma> = \oint \frac{d\theta_k}{2\pi} \gamma(\theta_k). \tag{1}$$

Since the average of γ is clearly smaller than the peak of γ along θ_k (which is the growth rate in the static case), sheared toroidal flow has a stabilizing effect on ballooning modes in tokamaks; the critical plasma β for destabilization rises in the presence of sheared toroidal flows.

There are important differences in the theory of ballooning modes in three-dimensions configurations [13] relative to axisymmetric devices [14]. As in two-dimensional configurations, the ballooning transform for three-dimensional systems yields a lowest order solution from an ordinary differential equation along the field line [15]. In systems without a continuous symmetry, the ballooning mode eigenvalues depend upon three coordinates, Ψ, θ_k and α where α labels the field line. The local eigenvalues are required to satisfy periodicity conditions in both the toroidal and poloidal directions. As such, the ballooning mode growth rate is given by the general formula [13]

$$\gamma = \sum_{m,n} \gamma_{mn}(\Psi) e^{im\theta_k + in(\alpha + q\theta_k)}, \tag{2}$$

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where $q = q(\Psi)$ is the safety value and α is written $\alpha = \zeta - q\theta$ for straight field line poloidal and toroidal angles θ and ζ, respectively. In addition to the ballooning mode eigenvalue, the corresponding eigenfunction is also a spatial function of Ψ and α. Regions of ideal MHD ballooning instability tend be more localized on a magnetic surface relative to the equivalent tokamak; the most unstable ideal MHD ballooning eigenvalue occurs at a particular value of θ_k and field line label α on each magnetic surface.

Using the Cooper form for the ballooning eikonal allows one to reduce the order of the partial differential equation governing instability from three to two dimensions in axisymmetric geometry (time and distance along the field line). The equivalent procedure in three-dimensional configurations with sheared flow reduces the partial differential equation from four to three dimensions. Despite the extra dimension, one can solve the resulting equation in the limit of small rotation and rotation shear when the equivalent static plasma is near marginal stability. The result of this calculation yields a long time averaged ideal MHD ballooning mode growth rate given by the formula

$$\gamma =<\gamma >= \oint \frac{d\theta_k}{2\pi} \oint \frac{d\alpha}{2\pi}\gamma = \gamma_{0,0}, \tag{3}$$

where the subscript $(0,0)$ refers to the $m = n = 0$ term of Eq. (2). In addition to the physics leading to the tokamak expression for the average growth rate, the effect of the flow in stellarators is to convect the mode across field lines in the magnetic surface, leading to an additional averaging over α. As in the tokamak case, the effect of sheared flow in a stellarator is

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stabilizing.

In the following section, we introduce the MHD equilibrium quantities and equilibrium plasma flow used for quasisymmetric stellarator plasmas. In Section III, the linear MHD equations are presented using a ballooning formalism. In Section IV, a calculation of the growth rate is made using a procedure similar to that used in Ref. [8]. A summary of the results in given in Section V.

II. MHD EQUILIBRIUM AND FLOW

There is no complete theory of MHD equilibrium in the presence of flows for three-dimensional geometry. However, flow and flow shear effects can have an important effect on stability even with small values of velocity. The approach taken here is to treat the flow effects as small corrections to a magnetostatic equilibrium ($|\rho\mathbf{v}{\cdot}\nabla\mathbf{v}|/|\nabla p| \sim |\rho\mathbf{v}{\cdot}\nabla\mathbf{v}|/|\mathbf{J}{\times}\mathbf{B}| \sim \mathcal{O}(M_s^2) \ll 1$). In the following we assume the existence of magnetic surfaces and the lowest order force balance equation is given by

$$\mathbf{J} \times \mathbf{B} = \nabla p. \tag{4}$$

To describe the equilibrium magnetic field, we use Boozer coordinates [16] which describe the vector quantities in both a covariant and contravariant basis set, with Ψ as the poloidal flux function, θ as the poloidal angle and ζ as the toroidal angle,

$$\mathbf{B} = q(\Psi)\nabla\Psi \times \nabla\theta + \nabla\zeta \times \nabla\Psi = F(\Psi)\nabla\zeta + I(\Psi)\nabla\theta + h\nabla\Psi. \tag{5}$$

The Jacobian of this coordinate system is given by

$$\sqrt{g} = \frac{1}{\nabla\Psi \times \nabla\theta \cdot \nabla\zeta} = \frac{qF + I}{B^2}, \tag{6}$$

where the quantities q, F and I are all flux functions and the quantity $h = h(\Psi, \theta, \zeta)$ is related to the Pfirsch-Schlüter current.

Consistent with the equilibrium condition $\nabla \times (\mathbf{v}_o \times \mathbf{B}) = 0$, small $\mathcal{O}(M_s)$ neoclassical, $\mathbf{E} \times \mathbf{B}$ and parallel flows can be described by

$$\mathbf{v}_0 = \frac{\nabla\Psi \times \mathbf{B}}{B^2}\Omega(\Psi) + \frac{v_{||}}{B}\mathbf{B}, \tag{7}$$

with $v_{||}$ consistent with the equilibrium equation $\nabla \cdot \rho_o \mathbf{v}_o = 0$,

$$\mathbf{B} \cdot \nabla(\frac{\rho_o v_{||}}{B}) = -\nabla \cdot [\frac{\nabla\Psi \times \mathbf{B}}{B^2}\rho_o\Omega(\Psi)]. \tag{8}$$

As is the case with Pfirsch-Schlüter currents, self-consistent solutions for the parallel flows rely on the capability of inverting the operator $\mathbf{B} \cdot \nabla$. This is singular at rational surfaces and lead to large parallel flows near rational surfaces if $\oint dl/B$ is not a flux function. In the following, we assume that well-defined magnetic surfaces are established and that no singular currents or flows are present.

While for much of the following calculation, a more general formulation can be made, it is instructive to specialize to the case of a symmetric stellarator, which approximately describes the flow profile of a quasisymmetric stellarator. In this case, the magnetic field strength is independent of one variable, $B = B(\psi, M\theta - N\zeta)$, where M and N are integers denoting the

dominant field harmonic of the field strength. It is convenient to transform to the coordinate system

$$\alpha = \zeta - q\theta, \tag{9}$$

$$\eta = M\theta - N\zeta, \tag{10}$$

Here, α is a label of the field line and η denotes points along the field line. In the symmetric limit, $\partial B/\partial\alpha = 0$, Eq. (8) with $\rho_o = \rho_o(\Psi)$ yields

$$\mathbf{B} \cdot \nabla \frac{v_{\parallel}}{B} = \mathbf{B} \cdot \nabla [\Omega \frac{B_\alpha}{B^2}], \tag{11}$$

where $\mathbf{e}_\alpha = \sqrt{g}\nabla\Psi \times \nabla\eta/(M - Nq), B_\alpha = \mathbf{B} \cdot \mathbf{e}_\alpha = (MF + NI)/(M - Nq)$. This has the solution

$$\frac{v_{\parallel}}{B} = \Omega\frac{B_\alpha}{B^2} + f(\Psi), \tag{12}$$

where $f(\Psi)$ is an undetermined flux function. With this, the total flow profile is given by $\mathbf{v}_0 = \Omega\mathbf{e}_\alpha + f\mathbf{B}$. In neoclassical theory, flows within the flux surface tend to be damped in the direction of ∇B. As such, for a symmetric system, the dominant flow is anticipated to satisfy $\mathbf{v} \cdot \nabla B = 0$ [17]. Imposing this condition yields $f = 0$ and the equilibrium flow to be used in the calculation is given by

$$\mathbf{v}_0 = \Omega(\Psi)\mathbf{e}_\alpha. \tag{13}$$

In this formulation of the problem, we will treat the equilibrium flow as a small correction to the magnetostatic equilibrium. Formally, we will keep terms of order M_s in the following and neglect terms of higher order. This has the consequence that centrifugal effects that cause the pressure to

deviate from being a flux function are ignored. In practice, this is not anticipated to play an important in stellarator experiments. However, coriolis and convective acceleration terms will be included in the derivation of the linear stability properties.

III. LINEARIZED BALLOONING EQUATIONS

The single-fluid MHD equations are used in describe the perturbation

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) = \mathbf{J} \times \mathbf{B} - \nabla p, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{15}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\Gamma p \nabla \cdot \mathbf{v}, \tag{16}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{17}$$

where Γ depends upon the equation of state used.

To examine stability, the above equations are linearized using the ballooning representation [13]

$$f_1 = \tilde{f} e^{i\epsilon^{-1}S}, \tag{18}$$

where $\epsilon << 1$ and S is the eikonal that satisfies

$$\mathbf{B} \cdot \nabla S = 0, \tag{19}$$

$$\frac{\partial S}{\partial t} + \mathbf{v}_o \cdot \nabla S = 0, \tag{20}$$

where the second condition follows the formulation introduced by Cooper [5]. In the coordinate system and flow profile used here, the eikonal is given by

$$S = \zeta - q(\theta - \theta_o) - \Omega t. \tag{21}$$

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As in the equivalent studies in axisymmetric geometry, the eikonal S is explicitly time-dependent with the associated issue that the eikonal solution are not eigenmode solutions. Rather, an initial value problem needs to be solved.

Using the ballooning transformation, the condition $\nabla \cdot \mathbf{B}_1 = 0$ allows one to write the lowest order magnetic perturbation

$$\mathbf{B}_1 = [\tilde{\psi}\frac{\mathbf{B} \times \nabla S}{B^2} + \tilde{B}_{||}\frac{\mathbf{B}}{B}]e^{i\epsilon^{-1}S}. \tag{22}$$

The perturbed parallel magnetic field is related to the perturbed pressure through the leading order momentum balance

$$\mu_o\tilde{p} + \tilde{B}_{||}B = 0. \tag{23}$$

The perpendicular part of the perturbed velocity is divergence-free to lowest order. This allows one to write the largest contribution to the perturbed velocity

$$\mathbf{v}_1 = [\tilde{\chi}\frac{\mathbf{B} \times \nabla S}{B^2} + \tilde{u}_{||}\mathbf{B}]e^{i\epsilon^{-1}S}, \tag{24}$$

Using the above forms in the perturbed MHD equations yields temporal evolution equations for five variables, $\tilde{\psi}, \tilde{p}, \tilde{\chi}, \tilde{u}_{||}, \tilde{\rho}$. However, since a low Mach number approximation is used where centrifugal forces are ignored, the mass density fluctuation decouples from the other four variables. The linearized MHD equations are given by

$$(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \alpha})\tilde{\psi} = \mathbf{B} \cdot \nabla\tilde{\chi}, \tag{25}$$

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$$(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \alpha})\tilde{p} + \tilde{\chi}\frac{dP_o}{d\Psi} = \frac{\Gamma p_0 B^2}{B^2 + \Gamma p_0 \mu_o}[-\mathbf{B} \cdot \nabla \tilde{u}_\parallel$$

$$+2\tilde{\chi}\frac{\mathbf{B} \times \nabla S}{B^2} \cdot \vec{\kappa} - \tilde{\psi}\frac{B_\alpha}{B^2}(\frac{d\Omega}{d\Psi} + \Omega\frac{Ndq/d\Psi}{M - Nq})], \qquad (26)$$

$$\rho_o B_0^2(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \alpha})\tilde{u}_\parallel + \rho_o\tilde{\chi}\nabla \cdot [\Omega \mathbf{e}_\alpha \times \nabla S] = -\tilde{\psi}\frac{dp_0}{d\Psi} - \mathbf{B}.\nabla\tilde{p}, \qquad (27)$$

$$(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \alpha})(\tilde{\chi}\frac{\rho_o\mu_o|\nabla S|^2}{B_0^2}) = \mathbf{B} \cdot \nabla(\frac{|\nabla S|^2}{B^2}\tilde{\psi}) - 2\tilde{p}\mu_o\frac{\mathbf{B} \times \nabla S}{B^2} \cdot \vec{\kappa}$$

$$+\rho_o\mu_o\tilde{u}_\parallel\nabla \cdot [\Omega \mathbf{e}_\alpha \times \nabla S] - \rho_o\mu_o\tilde{u}_\parallel B_\alpha(\frac{d\Omega}{d\Psi} + \Omega\frac{Ndq/d\Psi}{M - Nq}), \qquad (28)$$

where the magnetic field line curvature vector is given by $\vec{\kappa} = (\hat{\mathbf{b}} \cdot \nabla)\hat{\mathbf{b}}$. Other than the centrifugal terms that have been dropped, these equations are consistent with those used in Ref. [10] in the axisymmetric limit. As in the tokamak case, the effect of the flow impacts stability through the presence of coriolis effects (terms proportional to Ω) and convective acceleration terms proportional to the gradient of the equilibrium velocity. However, in this three-dimensional equilibrium case, there are additional terms proportional to Ω that convect the mode across different field lines within the magnetic surface. These terms are not present in ballooning stability properties of axisymmetric equilibrium as the ballooning eigenvalues are not dependent upon field line label.

IV. STABILITY ANALYSIS

In this section, a stability condition of ideal MHD ballooning modes in the presence of sheared flow is derived. We approach this problem by using

the asymptotic technique developed by Waelbroeck and Chen for tokamak plasmas [8].

Before preceding with the problem with flow, we review the threory for static plasmas. In this limit, the eikonal is no longer time dependent; the eikonal solution is the eigenmode solution and the growth rate is identified with a local eigenvalue. In the case of modes that are near marginality, an asymptotic technique can be developed to describe the local eigenvalue ω^2 where $\tilde{\xi} \sim e^{-i\omega t}$. The conventional definition of the radial displacement vector with the identity $\tilde{\chi} = -i\omega\xi$ is used in the following. When the mode is near marginality, one can employ an asymptotic expansion recognizing two spatial scales along the field line, a short scale with $\eta \sim \mathcal{O}(1)$ and a long scale with $\eta \sim \mathcal{O}(1/\omega\tau_a)$. On the short scale, the parallel momentum balance is dominated by $\tilde{\psi}(dp_0/d\Psi) + \mathbf{B} \cdot \nabla\tilde{p} = 0$. This decouples the perturbed parallel velocity and the associated sound wave physics. Additionally, the radial induction equation is given by $\tilde{\psi} = \mathbf{B} \cdot \nabla\xi$. On the short scale, the eigenmode is described by the marginal ideal ballooning equation given by

$$\mathbf{B} \cdot \nabla(\frac{|\nabla S|^2}{B^2}\mathbf{B} \cdot \nabla\xi) + 2\mu_o\frac{\mathbf{B} \times \nabla S}{B^2} \cdot \vec{\kappa}\frac{dp_o}{d\Psi}\xi = 0. \tag{29}$$

Conventional techniques [18] demonstrate that the large η limit of this solution is given by

$$\xi \sim A_o(|\eta|^{-1/2-\nu} + \Delta(\theta_k, \alpha)|\eta|^{-1/2+\nu}), \tag{30}$$

where $\nu = \sqrt{1/4 - D_I}$ and D_I is the Mercier coefficient. In three-dimensional configurations, the quantity Δ is different for every field line and every value

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of θ_k. This property also holds for quasisymmetric stellarators as well. For an explicit example, see [19, 20].

For the solution valid on the long $\eta \sim 1/\omega\tau_a \gg 1$ scale, a two-scale analysis is applicable. Details of this calculation are given in the Appendix and results in an equation for the $\overline{\xi} = \oint d\tilde{\eta}/2\pi\xi$ given by

$$\frac{\partial}{\partial\eta}\eta^2\frac{\partial\overline{\xi}}{\partial\eta} + D_I\overline{\xi} = -\eta^2\omega^2\tau_a^2\overline{\xi}, \tag{31}$$

where τ_a^2 scales inversely with the Alfven speed and is defined by Eq. (A-8). The solution to Eq. (31) is given by $\xi \sim |\eta|^{-1/2}K_\nu(-i\omega\tau_a)$. Expanding this expression for small η and asymptotically matching this solution with that given by Eq. (30) yields the matching condition

$$(\frac{\gamma\tau}{2})^{2\nu} = (\frac{-i\omega\tau_a}{2})^{2\nu} = -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\Delta(\theta_k, \alpha), \tag{32}$$

which simplifies in the $D_I \to 0$ limit to

$$\gamma = -\frac{\Delta(\theta_k, \alpha)}{\tau_a}. \tag{33}$$

For equilibria near marginal stability, the asymptotic matching parameter Δ is simply identified as proportional to the linear growth rate.

Returning to the case with shear flow present, the flow velocity and characteristic growth rates are treated as small quantities, $\Omega\tau_a \sim \tau_a\partial/\partial t \ll 1$. As in the static case, a matched asymptotic procedure can be carried out. With sheared flow, the eikonal is time dependent and the wavenumber is given by

$$\nabla S = \nabla\alpha + \nabla q(\theta_{ko} - \frac{d\Omega}{dq}t), \tag{34}$$

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where $q = q(\Psi)$ can be used as a magnetic surface label. In the presence of a sheared flow relative to the q profile, the effective wave number is time-dependent. The radial displacement is identified with

$$(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \alpha})\xi = \tilde{\chi}.$$ (35)

As with the static case, asymptotic matching can be carried out using solutions on the two disparate scales. On the short scale, the eigenmode is described by the marginal ideal ballooning equation, Eq. (29), with the notable difference that the ballooning angle θ_k is replaced by the time-dependent wave-number $\theta_k(t) = \theta_{ko} - (d\Omega/dq)t$. Consequently, the large η limit of this solution contains a matching parameter that is time dependent

$$\xi \sim A_o\{|\eta|^{-1/2-\nu} + \Delta[\theta_k(t), \alpha]|\eta|^{-1/2+\nu}\}.$$ (36)

On the longer scale, the effect of inertia and flow enter. On this scale, the governing equation is a partial differential equation involving derivatives in time, derivatives along the field line and derivatives across field lines. The governing equation is given by

$$\frac{\partial}{\partial \eta}\eta^2\frac{\partial \bar{\xi}}{\partial \eta} + D_I\bar{\xi} = (\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \alpha})\eta^2\tau_a^2(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \alpha})\bar{\xi}.$$ (37)

The equivalent expression for tokamak equilibrium does not contain the terms proportional to derivatives along α. This is due to the α dependence of the local ballooning eigenvalues and eigenfunctions in three-dimensional equilibria. Treating τ_a as a parameter, one can solve Eq. (37) by integrating over

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the characteristic $d\alpha/dt = \Omega$ ($\alpha = \alpha_o + \Omega t$). Equation (37) then becomes

$$\frac{\partial}{\partial \eta} \eta^2 \frac{\partial \overline{\xi}}{\partial \eta} + D_I \overline{\xi} = \eta^2 \tau_a^2 \frac{d^2 \overline{\xi}}{dt^2}. \tag{38}$$

Fourier transforming in the time variable with

$$\hat{\xi} = \int_{-\infty}^{+\infty} dt e^{-i\omega t} \overline{\xi}, \tag{39}$$

recovers the equation

$$\frac{\partial}{\partial \eta} \eta^2 \frac{\partial \hat{\xi}}{\partial \eta} + D_I \hat{\xi} = -\omega^2 \tau_a^2 \eta^2 \hat{\xi}, \tag{40}$$

The solution of this equation is given by $\hat{\xi} \sim |\eta|^{-1/2} K_\nu(-i\omega\tau)$. In the special limit $D_I = 0$, the inverse Fourier transform yields an evolution equation for the amplitude given by

$$\frac{d\overline{\xi}}{dt} = -\frac{\Delta[\theta_k(t), \alpha(t)]}{\tau_a} \overline{\xi}, \tag{41}$$

with the solution

$$\overline{\xi} = \overline{\xi}_o e^{-\int^t \frac{\Delta(t')}{\tau_a} dt'}. \tag{42}$$

Note that the integrand is the linear growth rate for the static limit with the θ_k and α dependence replaced with terms that progress in time owing to the flow effects. Noting Eq. (2), we can identify a long time averaged growth rate as the average over θ_k and α

$$\gamma = < \gamma > = -\oint \frac{d\theta_k}{2\pi} \oint \frac{d\alpha}{2\pi} \frac{\Delta(\theta_k, \alpha)}{\tau_a} = \gamma_{0,0}. \tag{43}$$

This expression is the primary result of this paper. The effect of sheared flow is to convect the mode along θ_k and across field lines in the magnetic surface.

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For stellarators with small amount of net current, ballooning stability calculations are often used to predict the onset of plasma β limits. Generally, the onset of ballooning instability in stellarators occurs at highly localized regions of $\alpha - \theta_k$ space. Flow and flow shear cause the resulting eigenmode to be averaged over all values in this space. Hence, the flow effects tend to be highly stabilizing for ideal MHD ballooning modes in three-dimensional equilibria.

It is worth noting that this result is derived assuming the dominant flow profile of a quasisymmetric stellarator. For more general plasmas, the flow profile can have significant structure within the flux surface. This effect would produce additional convecting terms in the linear equations that complicate the analysis. However, it seems unlikely that additional flow terms would fundamentally affect the averaging procedure that yields the above result; we speculate that the result given in Eq. (43) would hold in the more general case.

V. SUMMARY

In this work, the ideal MHD ballooning stability properties of three-dimensional equilibrium in the presence of sheared flow is examined. This case may be particularly relevant for the class of quasisymmetric stellarators where a near symmetry leads to a low viscous damping rate in the direction of symmetry and hence significant plasma flows can exist.

In the small centrifugal force limit, flow and flow shear are known to have

a stabilizing effect on a number of MHD instabilities. In particular, flow shear stabilization of ideal MHD ballooning modes in axisymmetric tokamak equilibrium have been addressed by a number of authors [5, 6, 7, 8, 9, 10, 11, 12]. The result of the present work generalizes these results to three-dimensional equilibrium. Application of the conventional ballooning formalism to stellarator equilibrium with small flow reduces the problem to a partial differential equation in three dimensions. In the limit of small flow, this equation can be solved and an analytic estimate of the effect of sheared flow on ballooning instability can be performed using matched asymptotic solutions. The calculation produces the result given in Eq. (43), which demonstrates the highly stabilizing effect of sheared flow on ballooning modes in stellarators.

This work can be extended or applied in a number of directions. Applications of these results to actual stellarator plasmas, 3-D equilibrium solutions with realistic flows or analytically derived local equilibria [19, 20] can be carried out. Previous authors have noted the issue of modeling flow effects with ideal MHD since the effects of Landau damping on the sound wave are not property accounted for [21, 22]. While this effect is not addressed here, a correct treatment of this effect would presumably enhance stabilization. Finally, we note that a similar averaging process may also apply to the question of drift wave stability in three-dimensions. In the axisymmetric tokamak context, Connor et al considered a model drift wave problem with a sheared diamagnetic rotation and derived a similar formula to the Waelbroeck and Chen formula [23]. The equivalent question for stellarator plasmas remains

to be addressed.

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APPENDIX: MATCHING EQUATIONS FOR THE LONG SCALE

A conventional multiple scale analysis is used with the small parameter ϵ associated with small inertial effects $\tau_a \partial/\partial t \sim \Omega \tau_a \sim \epsilon$. Largely, this analysis parallels the calculation outlines in Ref. [8]. To determine the relevant equation on the long scale, all perturbations are written as functions of the slowly varying part with $\partial/\partial \eta \sim \mathcal{O}(\epsilon) \ll 1$ and a part that varies on the rapid scale $\partial/\partial \eta \sim \mathcal{O}(1)$. The slowly varying part of the displacement ξ is written $\bar{\xi}$ and the rapidly varying part is written $\delta \xi = \xi - \bar{\xi}$. It is useful to replace the pressure perturbation with the variable Q defined by

$$Q = \tilde{p} + \frac{dp_o}{d\Psi}\xi. \tag{A-1}$$

Perusal of the equations finds the dominant balances are give by the orderings $\tilde{u}_{\parallel} \sim \bar{\xi}$, $Q \sim \delta\xi \sim \tilde{\psi} \sim \epsilon\bar{\xi}$.

Before proceeding to the equations, an investigation of terms proportional to the magnetic field line curvature are required. It is convenient to identify those parts that secularly grow with η. Namely, the curvature can be written

$$2\frac{\mathbf{B} \times \nabla S}{B^2}\vec{\kappa} = V_o + q'\frac{\eta'}{M - Nq}\mathbf{B} \cdot \nabla\lambda, \tag{A-2}$$

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where $V_o = 2\vec{\kappa} \cdot \mathbf{B} \times [\nabla\zeta - q\nabla\theta]/B^2$, $q' = dq/d\Psi$, λ is the coefficient to the Pfirsch-Schlüter current $\mathbf{B} \cdot \nabla(J_{||}/B) = -(dp_o/d\Psi)\mathbf{B} \cdot \nabla\lambda$ and $\eta' = \eta + N\alpha + d\Omega/dqt$. For the symmetric stellarator, $\lambda = B_\alpha/B^2$.

A number of quantities vary on the short scale owing to Pfirsch-Schlüter-like effects that modify parallel flows and currents due to compressibility and quasineutrality, respectively. The dominant contributions to the pressure evolution equation describe a parallel flow responding to the compression of the perturbed perpendicular flow. Solution of this flow yields the relation

$$\tilde{u}_{||} = q'\frac{\eta}{M - Nq}(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\alpha})\overline{\xi}(\lambda - \frac{<\lambda B^2>}{<B^2>}) + \frac{<u_{||}B^2>}{<B^2>}, \qquad \text{(A-3)}$$

where the flux surface averaging operator on the short scale is defined

$$<f> = \frac{\oint d\eta\sqrt{g}f}{\oint d\eta\sqrt{g}}. \qquad \text{(A-4)}$$

Inserting this solution into the parallel momentum balance equation, we find $<u_{||}B^2>$ is negligible and the variation of Q on the short scale is determined by the equation

$$\mathbf{B} \cdot \nabla Q = -q'\frac{\eta}{M - Nq}(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\alpha})(\lambda B^2 - B^2\frac{<\lambda B^2>}{<B^2>})(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\alpha})\overline{\xi}, \quad \text{(A-5)}$$

The quantity $\delta\xi$ can be found from the largest contribution to the quasineutrality equation. This balance describes the perturbed Pfirsch-Schlüter currents flowing. Standard techniques finds the solution

$$\mathbf{B} \cdot \nabla\delta\xi = (<\frac{1}{\sqrt{g}}> \frac{B^2}{|\nabla\Psi|^2}\frac{1}{<B^2/|\nabla\Psi|^2>} - \frac{1}{\sqrt{g}})(M - Nq)\frac{\partial\overline{\xi}}{\partial\eta}$$
$$+ \frac{\mu_o p'_o(M - Nq)}{q'\eta}\overline{\xi}(\frac{B^2}{|\nabla\Psi|^2}\frac{<B^2\lambda/|\nabla\Psi|^2>}{<B^2/|\nabla\Psi|^2>} - \frac{B^2\lambda}{|\nabla\Psi|^2}). \qquad \text{(A-6)}$$

Applying the flux surface averaging operator, Eq. (A-4), to the quasineutrality equation gives the relation

$$\frac{\partial}{\partial \eta} q' \eta^2 < \frac{|\nabla \Psi|^2}{B^2 (\sqrt{g^2})} > \frac{\partial \bar{\xi}}{\partial \eta} + \frac{\partial}{\partial \eta} \frac{q'^2 \eta^2}{M - Nq} < \frac{|\nabla \Psi|^2}{B^2 \sqrt{g}} \mathbf{B} \cdot \nabla \delta \xi >$$

$$+ 2 p_o' \mu_o \bar{\xi} < V_o > + \frac{q' \eta}{M - Nq} < (p_o' \mu_o \delta \xi - Q) \mathbf{B} \cdot \nabla \lambda >$$

$$= (\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \alpha}) \frac{q'^2 \eta^2}{(M - Nq)^2} < \frac{\rho_o \mu_o |\nabla \Psi|^2}{B^2} > (\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \alpha}) \bar{\xi}. \qquad (A-7)$$

Inserting the expression derived above for $\delta \xi$ and Q, Eq. (37) is derived with identification of the standard Mericer index D_I and the Alfven time τ_a defined by

$$\tau_a^2 = \frac{\rho_o \mu_o}{(M - Nq)^2} \frac{< \frac{B^2}{|\nabla \Psi|^2} >}{< \frac{1}{\sqrt{g}} >^2} [< \frac{|\nabla \Psi|^2}{B^2} > + < \lambda^2 B^2 > - \frac{< \lambda B^2 >^2}{< B^2 >}]. \qquad (A-8)$$

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