Derivation of paleoclassical key hypothesis

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The paleoclassical model of radial electron heat transport in resistive, current-carrying toroidal plasmas is based on a key hypothesis — that electron guiding centers move and diffuse with radially localized annuli of poloidal magnetic flux. This hypothesis is shown to result from transforming the drift-kinetic-equation to poloidal flux coordinates in situations where this flux is governed by a diffusion equation and analyzing the mathematical characteristic curves (guiding center trajectories) of the resultant drift-kinetic equation on the magnetic field diffusion time scale $\tau_\eta \equiv a^2/6D_\eta$. These effects add a τ_η time-scale Fokker-Planck-type spatial diffusion operator to the drift-kinetic equation.

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Recently, a new "paleoclassical" model for radial electron heat transport in current-carrying, resistive toroidal plasmas has been developed and published: basic concepts in a sheared slab model [1], comprehensive axisymmetric toroidal magnetic field model [2], and preliminary interpretations of various experimental results with it [3]. More extensive comparisons with experimental data [4] have shown that the minimum level of radial electron heat transport is often set by the paleoclassical model.

The key hypothesis of the paleoclassical model is that [1–3] in axisymmetric current-carrying toroidal plasmas, electron guiding centers move and diffuse along with and just as thin radially localized annuli of poloidal magnetic flux do on the resistivity-induced magnetic ("skin") diffusion time scale. (A similar hypothesis based on Landau damping was explored by Kadomtsev and Pogutse [5]). These advection and diffusion effects are represented by adding a Fokker-Planck-type spatial diffusion operator [2] to the right of the usual drift-kinetic equation [6]. This key hypothesis was originally motivated phenomenologically [2]. This note shows the hypothesis results naturally from a multiple-time-scale analysis of the mathematical characteristic curves (effective particle guiding center trajectories) of the drift-kinetic equation after transforming it from laboratory to poloidal magnetic flux coordinates when the poloidal flux satisfies a diffusion equation.

For an axisymmetric magnetic field described by $\mathbf{B} = I(\psi_p)\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi_p = \boldsymbol{\nabla}\psi_p\times\boldsymbol{\nabla}(q\,\theta - \zeta) \equiv \mathbf{B}_t + \mathbf{B}_p$ and a flux-surface-averaged $[\langle A(\mathbf{x})\rangle]$ equilibrium parallel Ohm's law, evolution equations for the toroidal magnetic flux ψ_t and the poloidal magnetic flux ψ_p are [2, 7, 8]

$$\frac{d\psi_t}{dt} \equiv \left.\frac{\partial\psi_t}{\partial t}\right|_{\mathbf{x}} + \bar{u}_G\frac{\partial\psi_t}{\partial\rho} = 0, \tag{1}$$

$$\left.\frac{\partial\psi_p}{\partial t}\right|_{\mathbf{x}} + \langle\mathbf{u}_G\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_p\rangle = D_\eta\,\Delta^+\psi_p - S_\psi, \quad D_\eta \equiv \frac{\eta_\parallel^{\rm nc}}{\mu_0}. \tag{2}$$

Here, $\bar{u}_G\,\partial\psi_t/\partial\rho \equiv \langle\mathbf{u}_G\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_t\rangle = q\,\langle\mathbf{E}\boldsymbol{\cdot}\mathbf{B}_p\rangle/I\langle R^{-2}\rangle$ is the advective "Grid velocity" (subscript G, relative to fixed laboratory coordinates) of ψ_t, D_η is the magnetic field diffusivity induced by neoclassical parallel resistivity $\eta_\parallel^{\rm nc}$ [2], and Δ^+ is a cylindrical-like second order differential operator in the radial direction [2, 7, 8]. The $D_\eta\,\Delta^+\psi_p = \eta_\parallel^{\rm nc}\langle\mathbf{J}\boldsymbol{\cdot}\mathbf{B}\rangle/I\langle R^{-2}\rangle$ term in (2) represents diffusion of poloidal flux ψ_p relative to the toroidal flux ψ_t, which is induced by resistivity and parallel current. The source of poloidal flux in (2) is [2] $S_\psi \equiv \partial\Psi/\partial t - (\mu_e/\nu_e)(\eta_0/\langle R^{-2}\rangle)\,dP/d\psi_p + \eta_\parallel^{\rm nc}\langle\mathbf{J}_S\boldsymbol{\cdot}\mathbf{B}\rangle/I\langle R^{-2}\rangle$, which includes sources of poloidal flux due to flux changes in a central solenoid (ohmic transformer "external boundary condition"), and parallel currents due to the bootstrap current and a non-inductive current source $\mathbf{J}_S$.

Equation (1) shows that the toroidal flux ψ_t is stationary, except for the radial grid velocity $\bar{u}_G$, which, in steady-state, can be shown [2, 9] to be $\mathcal{O}(\epsilon^2, B_p^2/B_t^2)$ smaller than terms on the right of (2) for a torus of large aspect ratio ($\epsilon \equiv r/R_0 << 1$) with $B_p << B_t$. Thus, the toroidal flux ψ_t will be an essentially stationary frame of reference and it is customary to define a dimensionless, cylindrical-type radial variable ρ based on it:

$$\rho = \sqrt{\psi_t\,/\psi_t(a)} \simeq r/a, \quad \text{radial coordinate.} \tag{3}$$

Neoclassical transport fluxes are calculated using the ψ_p poloidal magnetic flux as the radial coordinate, but then specified relative to the ψ_t toroidal flux surfaces [2, 7–9].

The fundamental laboratory-coordinate-based drift-kinetic equation (DKE), correct through second order in the small gyroradius expansion ($\delta \sim v_\perp/\omega_c L_\perp << 1$, where $\omega_{ce} \equiv q_e B/m_e$) neglecting an $\mathcal{O}\{\beta\,\delta^2\}$ term due to $d\mu/dt \sim \beta\,\delta$ ($\beta \equiv 2\mu_0 P/B^2 << 1$), can be written as [6]

$$\left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} + (\mathbf{v}_\parallel + \mathbf{v}_D)\boldsymbol{\cdot}\boldsymbol{\nabla} f + \dot{\varepsilon}_g\frac{\partial f}{\partial\varepsilon_g} = \mathcal{C}\{f\}, \quad \text{lab DKE,} \tag{4}$$

in which $f = f(\mathbf{x}_g, \varepsilon_g, \mu, t)$ is the gyro-averaged, guiding center distribution function, $\mathbf{v}_\parallel \equiv \mathbf{B}(\mathbf{B}\boldsymbol{\cdot}\mathbf{v})/B^2$ is the parallel velocity, $\mathbf{v}_D = -(m_e/q_e)\mathbf{v}_\parallel\times\boldsymbol{\nabla}(v_\parallel/B)$ is the (first order) drift velocity, $\varepsilon_g \equiv mv_\parallel^2/2 + \mu B$ is the particle guiding center kinetic energy with magnetic moment $\mu \equiv mv_\perp^2/2B(\mathbf{x}_g, t)$, $\dot{\varepsilon}_g = \mu\,\partial B/\partial t + q_e(\mathbf{v}_\parallel + \mathbf{v}_D)\boldsymbol{\cdot}\mathbf{E}$, and $\mathcal{C}\{f\}$ is the Fokker-Planck Coulomb collision operator.

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The drift-kinetic equation in (4) is usually transformed to the magnetic flux coordinates ψ_p, θ, ζ since charged particles gyrate about and move along $\mathbf{B}$. To see why the poloidal flux ψ_p is used as the radial coordinate, consider the $\mathbf{e}_\zeta \equiv R^2 \boldsymbol{\nabla}\zeta$ (covariant base vector) projection of Newton's second law with a Lorentz force $\mathbf{F} = q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$, and use $\mathbf{e}_\zeta\cdot\mathbf{E} = -\,\mathbf{e}_\zeta\cdot\partial\mathbf{A}/\partial t = \partial\psi_p/\partial t$ (i.e., neglect the scalar potential ϕ), to obtain [2] $d/dt(m_e\mathbf{e}_\zeta\cdot\mathbf{v}) = q_e[\partial\psi_p/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla}\rho)\partial\psi_p/\partial\rho] = q_e d\psi_p/dt$, which is valid on the gyromotion and longer time scales. Integrating this equation over t for an isolated conservative system yields the constancy of the canonical toroidal angular momentum: $p_\zeta = m_e\mathbf{e}_\zeta\cdot\mathbf{v} - q_e\psi_p =$ constant. Drift-kinetic theory [6] uses this constant of the motion, or its lowest order in small gyroradius form $p_\zeta \simeq -q_e\psi_p$, for its radial coordinate; hence ψ_p represents the average radial guiding center position about which the radial gyro and bounce motions are rapidly oscillating and tied to in axisymmetric toroidal systems. However, as will be shown below, in driven-dissipative systems such as resistive, current-carrying toroidal plasmas where the poloidal flux is injected from external sources and diffusing radially, ψ_p and p_ζ diffuse radially (due to $\partial\psi_p/\partial t \sim D_\eta\,\Delta^+\psi_p$) on the longer magnetic diffusion time scale $\tau_\eta \equiv a^2/6D_\eta$.

To transform the laboratory coordinate DKE to the magnetic flux coordinates ψ_p, θ, ζ, note that since ζ is the toroidal (axisymmetric) angle, it is invariant, i.e., $\partial\zeta/\partial t|_{\mathbf{x}} = 0$. Similarly, since θ motion is order β smaller than radial motion for the straight-field-line poloidal angle θ [9], it is also approximately invariant and its variation will be neglected, i.e., $\partial\theta/\partial t|_{\mathbf{x}} \simeq \mathcal{O}\{\beta, \epsilon^2, B_p^2/B_t^2\} \to 0$. However, $\partial\psi_p/\partial t|_{\mathbf{x}} \neq 0$, as given by (2). Thus, transforming (4) to the magnetic flux coordinates ψ_p, θ, ζ using $\partial f/\partial t|_{\mathbf{x}} = \partial f/\partial t|_{\psi_p,\theta,\zeta} + \partial\psi_p/\partial t|_{\mathbf{x}}\,\partial f/\partial\psi_p|_{\theta,\zeta}$ yields

$$\left.\frac{\partial f}{\partial t}\right|_{\psi_p} + [-\langle\mathbf{u}_G\cdot\boldsymbol{\nabla}\psi_p\rangle + D_\eta\,\Delta^+\psi_p - S_\psi]\frac{\partial f}{\partial\psi_p} + (\mathbf{v}_\parallel + \mathbf{v}_D)\cdot\boldsymbol{\nabla}f + \dot{\varepsilon}_g\frac{\partial f}{\partial\varepsilon_g} = \mathcal{C}\{f\}, \quad \psi_p \text{ DKE.} \quad (5)$$

This result is equivalent to Eq. (10) in [10] using the specification of $\partial\psi_p/\partial t|_{\mathbf{x}}$ from (2) so the drift-kinetics are self-consistent with Faraday's law and the resistive parallel Ohm's law. Since $D_\eta \sim \nu_e(c/\omega_p)^2$, ordering the electromagnetic skin depth $c/\omega_p \sim \delta\,L_\perp$, all the $\partial\psi_p/\partial t$ terms in the coefficient of $\partial f/\partial\psi_p$ in (5) are second order in the small gyroradius expansion $\delta << 1$. These terms are dropped in the first order neoclassical kinetic analysis in [10], but retained in obtaining fluid moment equations there. However, the second order terms from $\partial\psi_p/\partial t|_{\mathbf{x}}$ in (2) are critical for a self-consistent drift-kinetic analysis of ψ_p motion, advection and diffusion effects that lead to the paleoclassical processes.

The mathematical characteristic curves (effective particle guiding center trajectories) of the partial differential equation in (5) are determined from $d\varepsilon_g/dt = \dot{\varepsilon}_g$ and

$$\frac{d\mathbf{x}_g}{dt} = \mathbf{v}_\parallel + \mathbf{v}_D + [-\langle\mathbf{u}_G\cdot\boldsymbol{\nabla}\psi_p\rangle + D_\eta\,\Delta^+\psi_p - S_\psi]\,\mathbf{e}_{\psi_p}, \quad (6)$$

in which $\mathbf{e}_{\psi_p} \equiv \partial\mathbf{x}/\partial\psi_p = (\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta)/(\boldsymbol{\nabla}\psi_p\cdot\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta)$ is the covariant ψ_p base vector constructed such that $\mathbf{e}_{\psi_p}\cdot\boldsymbol{\nabla}\psi_p = 1$. Without the $\mathbf{e}_{\psi_p}$ terms, the mathematical characteristic curves of (6) are the usual (hyperbolic) guiding center particle trajectories governed by Hamilton's equations. However, with the $D_\eta\,\Delta^+\psi_p$ term the characteristic curves become parabolic (diffusive) on the magnetic diffusion time scale τ_η — because the tokamak poloidal magnetic field system is not an isolated, conservative system; rather, it is a driven-dissipative system in which poloidal magnetic flux is externally supplied and plasma resistivity causes a magnetic field diffusivity $D_\eta = \eta/\mu_0$ that diffusively dissipates ψ_p.

The guiding center motion equations within a flux surface obtained from (6) are normal [6], e.g., $d\theta/dt \simeq (v_\parallel/B)\mathbf{B}\cdot\boldsymbol{\nabla}\theta$. But the "radial" guiding center equation obtained from the $\boldsymbol{\nabla}\psi_p(\rho, t)$ projection of (6) is different:

$$d\psi_p/dt = \mathbf{v}_D\cdot\boldsymbol{\nabla}\psi_p + [-\langle\mathbf{u}_G\cdot\boldsymbol{\nabla}\psi_p\rangle + D_\eta\,\Delta^+\psi_p - S_\psi]. \quad (7)$$

This equation for the ψ_p guiding center motion will be solved via a multiple-time-scale analysis. The first term represents the usual first order (in δ) radial drift of the guiding center off the ψ_p flux surface, $\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi_p = (m_e/q_e)(v_\parallel/B)(\mathbf{B}\cdot\boldsymbol{\nabla}\theta)\partial/\partial\theta(Iv_\parallel/B)$; it is oscillatory and bounce averages to zero [6]. Neglecting the other terms in (7) and integrating it over time using $d\theta/dt$ from above yields the constancy of the guiding center canonical toroidal angular momentum $p_{\zeta g} = m_e I v_\parallel/B - q_e\psi_p$.

The other terms in (7) are of order δ^2 in the small gyroradius expansion; they are relevant on the longer magnetic field diffusion time scale τ_η. Averaging (7) over the fast bounce motion time scale, the solubility condition for this equation (to prevent secular growth of the oscillatory term on the bounce and longer time scales) becomes

$$d\psi_p(\bar{\rho}_g)/dt + \langle\mathbf{u}_G\cdot\boldsymbol{\nabla}\psi_p(\bar{\rho}_g)\rangle = D_\eta\,\Delta^+\psi_p(\bar{\rho}_g) - S_\psi, \quad (8)$$

in which $\bar{\rho}_g \equiv \bar{\mathbf{x}}_g\cdot\boldsymbol{\nabla}\rho$ is the average radial position of the guiding center — banana center for trapped particles, an average poloidal flux surface for untrapped particles [6].

To analyze motion of the average guiding center position $\bar{\rho}_g$, expand $\psi_p(\bar{\rho}_g)$ in a Taylor series about an arbitrary initial toroidal-flux-based radial position ρ_0, which is close enough to $\bar{\rho}_g$ so that $|\bar{\rho}_g - \rho_0| << 1$:

$$\psi_p(\bar{\rho}_g, t) = \psi_0(\rho_0, t) + (\bar{\rho}_g - \rho_0)\psi_p' + \cdots, \quad \psi_p' \equiv \partial\psi_p/\partial\rho|_{\rho_0}. \quad (9)$$

The first term represents the "global" variation of the poloidal flux; the second indicates possible radial motion of the average position of the guiding center $\bar{\rho}_g(x, t)$, in which $x \equiv \rho - \bar{\rho}_{g0}$ is the (dimensionless) radial distance from the initial toroidal flux surface at $\bar{\rho}_{g0}$. Note that $\delta\psi_g \equiv \psi_p(\bar{\rho}_g) - \psi_0(\rho_0) \simeq -[p_{\zeta g}(\bar{\rho}_g) - p_{\zeta g}(\rho_0)]/q_e$ and that $|\delta\psi_g| << \psi_p(\rho_0)$. Thus, $p_{\zeta g}(\bar{\rho}_g)$ is tied to $\delta\psi_g$ and since $D_\eta\,\Delta^+\psi_p(\bar{\rho}_g) = D_\eta\,\Delta^+(\delta\psi_g + \psi_0)$, diffusion of $\delta\psi_g$ [2] will cause diffusion of $\psi_p(\bar{\rho}_g)$ and hence $p_{\zeta g}(\bar{\rho}_g)$, $\bar{\rho}_g$.

Substituting (9) into (8) yields

$$\frac{\partial\bar{\rho}_g}{\partial t} + \bar{u}_G\frac{\partial\bar{\rho}_g}{\partial x} = -\,\bar{u}_{\dot{\psi}} + \bar{D}_\eta\,\frac{\partial^2\bar{\rho}_g}{\partial x^2}, \quad (10)$$

in which the following quantities have been defined:

$$\bar{u}_{\dot\psi} \equiv -\,\dot\psi/\psi_p', \quad \dot\psi \equiv D_\eta\, \Delta_0^+ \psi_0 - S_\psi, \tag{11}$$

where the subscript 0 on Δ_0^+ means radial derivatives are taken with respect to ρ_0 and [2]

$$\bar{D}_\eta \equiv D_\eta/\bar{a}^2, \quad 1/\bar{a}^2 \equiv \langle|\boldsymbol{\nabla}\rho|^2/R^2\rangle/\langle R^{-2}\rangle \sim 1/a^2. \tag{12}$$

The diffusive term on the right side of (10) results from $\Delta^+\bar\rho_g(x,t) \simeq (1/\bar{a}^2)\,\partial^2\bar\rho_g(x,t)/\partial x^2$ [2] because $\bar\rho_g$ is initially localized radially, i.e., $\bar\rho_g(x,t\!=\!0) = \bar\rho_{g0}\,\delta(x)$.

Equation (10) is a diffusion equation for the average radial position of the guiding center $\bar\rho_g(x,t)$. For an initial radial position at $\bar\rho_{g0}$ and short times ($t << \tau_\eta$), the radially localized solution of (10) is (for $t \geq 0$):

$$\bar\rho_g(\rho,t) = -\bar{u}_{\dot\psi} t + \bar\rho_{g0}\, \frac{e^{-[\rho-\bar\rho_{g0}-\bar{u}_G t]^2/4\bar{D}_\eta t}}{(4\pi\bar{D}_\eta t)^{1/2}}. \tag{13}$$

Thus, the intuitive physical explanation of the key hypothesis of the paleoclassical model is that the average radial guiding center position $\bar\rho_g$ diffuses with $\delta\psi_g$ because, as indicated in (9) and the sentences after it, on the bounce time scale $p_{\zeta g}$ and $\bar\rho_g$ become "tied to" this thin annulus of poloidal flux; then, on the longer magnetic diffusion time scale τ_η, $\delta\psi_g$ diffuses radially [2] which causes $p_{\zeta g}$ to diffuse and carry the average electron guiding center $\bar\rho_g$ with it, as indicated in (10),(13).

The $\bar\rho_g$ advection speed $\bar{u}_G$ is caused by the ψ_p surfaces advecting with the ψ_t toroidal flux surfaces which have a radial grid velocity $\bar{u}_G$ relative to laboratory coordinates. The $\bar{u}_{\dot\psi}$ term represents the effect of a time-varying poloidal flux ($\dot\psi \neq 0$) on $\bar\rho_g$. Using $\psi_p \simeq \psi_p(\rho_0)+x\psi_p'+\dot\psi t$ and setting $d\psi_p = 0$, one finds that an electron on a given ψ_p flux surface moves a distance $\delta x = -\dot\psi\tau_c/\psi_p'$ in an electron gyroperiod $\tau_c \equiv 2\pi/\omega_{ce}$; hence, the $-\bar{u}_{\dot\psi} t = \dot\psi\, t/\psi_p'$ term in (13) represents the change in ψ_p radial position for the guiding center to remain at the initial radial x, ρ position after one gyroperiod, as it should from physical considerations. (Note that these ψ_p coordinate transformation effects are physically and mathematically different from the Ware pinch effect [7], which is a second order radial $\mathbf{E}\times\mathbf{B}$ flux $\propto \epsilon^{1/2}\mathbf{e}_\zeta\cdot\mathbf{E}^A/B_p$ caused by the effect of the first order inductive toroidal electric field force $q_e\mathbf{e}_\zeta\cdot\mathbf{E}^A = q_e\partial\Psi/\partial t$ on trapped particles.)

Thus, in the ψ_p coordinate system the average radial position of the electron guiding center moves (due to $\bar{u}_{\dot\psi} = -\dot\psi/\psi_p'$), advects (due to $\bar{u}_G$), and diffuses (due to the diffusion coefficient $\bar{D}_\eta$) on the magnetic field diffusion time scale. The radial speed $d\bar\rho_g/dt$ [11] and Fokker-Planck coefficients that represent the motion and advection [11], diffusion effects on $\bar{x}_g \equiv \bar\rho_g - \bar\rho_{g0}$ are [2]

$$\frac{d\bar\rho_g}{dt} = \frac{\dot\psi}{\psi_p'}, \quad \frac{\langle\Delta\bar{x}_g\rangle}{\Delta t} = -\,\bar{u}_G, \quad \frac{\langle(\Delta\bar{x}_g)^2\rangle}{\Delta t} = 2\,\bar{D}_\eta. \tag{14}$$

Using the covariant radial base vector $\mathbf{e}_\rho \equiv \partial\mathbf{x}/\partial\rho = \sqrt{g}\,\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta$ with $\sqrt{g} \equiv 1/\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta$ and $\mathbf{e}_\rho\cdot\boldsymbol{\nabla}\rho = 1$, the general geometry Fokker-Planck coefficients are [2]:

$$\frac{\langle\Delta\mathbf{x}_g\rangle}{\Delta t} \equiv \frac{\langle\Delta\bar{x}_g\rangle}{\Delta t}\mathbf{e}_\rho, \quad \frac{\langle\Delta\mathbf{x}_g\,\Delta\mathbf{x}_g\rangle}{\Delta t} \equiv \frac{\langle(\Delta\bar{x}_g)^2\rangle}{\Delta t}\mathbf{e}_\rho\mathbf{e}_\rho. \tag{15}$$

Since these Fokker-Planck coefficients are the same as those for the radially localized poloidal magnetic flux $\delta\psi_g \equiv \psi_p(\bar\rho_g) - \psi_0(\rho_0) << \psi(\rho_0)$ [2], these results explicitly justify the key paleoclassical hypothesis that electrons move, advect and diffuse with $\delta\psi_g$. In principle, the results are valid for time scales longer than the transit time $\tau_t \equiv \bar{R}q/v_{Te}$ [2] (or gyroperiod for $1/\nu_e < \tau_t$ where a similar averaging over the oscillatory gyromotion yields the same $\mathcal{D}\{f\}$), but short compared to the magnetic diffusion time scale τ_η. However, since the equilibrium parallel Ohm's law and hence (2) are only valid [2] on time scales longer than the electron collision time $1/\nu_e$, for these Fokker-Planck coefficients one has

$$1/\nu_e < \Delta t << \tau_\eta \equiv a^2/6D_\eta, \quad \text{validity time scale.} \tag{16}$$

Thus, transforming the drift-kinetic equation from laboratory coordinates to flux coordinates with ψ_p as the radial coordinate and taking account of the $\dot\psi \neq 0$, grid motion and radial diffusion effects on the average position of electron guiding centers by adding a Fokker-Planck spatial diffusion operator to the usual DKE yields the magnetic-field-diffusion-modified drift-kinetic equation (MDKE) [2] in magnetic flux coordinates ψ_p, θ, ζ:

$$\left.\frac{\partial f}{\partial t}\right|_{\psi_p} + (\mathbf{v}_\parallel + \mathbf{v}_D)\cdot\boldsymbol{\nabla} f + \dot\varepsilon_g\frac{\partial f}{\partial\varepsilon_g} = \mathcal{C}\{f\} + \mathcal{D}\{f\}, \quad \text{MDKE}, \tag{17}$$

in which $f = f(\psi_p, \theta, \zeta, \varepsilon_g, \mu, t)$. Here, motion, advection and diffusion of electron guiding centers is taken into account via a Fokker-Planck-type spatial diffusion operator

$$\mathcal{D}\{f\} \equiv -\dot\psi\frac{\partial f}{\partial\psi_p} - \boldsymbol{\nabla}\cdot\frac{\langle\Delta\mathbf{x}_g\rangle}{\Delta t}f + \frac{1}{2}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\cdot\frac{\langle\Delta\mathbf{x}_g\,\Delta\mathbf{x}_g\rangle}{\Delta t}f\right). \tag{18}$$

For spatial distributions $f(\rho)$, the flux-surface-average of this Fokker-Planck spatial diffusion operator is [2]

$$\langle\mathcal{D}\{f(\rho)\}\rangle \simeq \bar{u}_{\dot\psi}\frac{\partial f}{\partial\rho} + \frac{1}{V'}\frac{\partial}{\partial\rho}\left(V'\bar{u}_G f + \frac{\partial}{\partial\rho}(V'\bar{D}_\eta f)\right), \tag{19}$$

in which $\mathcal{O}\{\epsilon^2, B_p^2/B_t^2\}$ terms have been neglected and $V' \equiv dV/d\rho$ where $V(\rho)$ is the ρ flux surface volume.

All three terms in $\mathcal{D}\{f\}$ are $\mathcal{O}\{\delta^2\}$ in the small gyroradius expansion. The $-\dot\psi\,\partial f/\partial\psi_p$ term in (18) accounts for ψ_p transient effects; the $\bar{u}_G$ advection term causes transport fluxes to be determined relative to ψ_t surfaces [7–9]. Both terms are needed to obtain the proper neoclassical fluid moment equations [9, 10]. The $\bar{D}_\eta$ diffusion contribution is the new, paleoclassical feature that results from analyzing the second order diffusive mathematical characteristic curves (effective guiding center trajectories for non-Hamiltonian systems) on the magnetic diffusion time scale τ_η. The present analysis shows the $\bar{u}_{\dot\psi}$, $\bar{u}_G$, and

$\bar{D}_\eta$ effects are all part of a deductive, complete (through order δ^2) transformation of the DKE from laboratory to ψ_p-based magnetic flux coordinates.

The $\mathcal{D}\{f\}$ operator should be included in the ψ_p-coordinate DKE for any axisymmetric resistive, current-carrying plasma for which, since $\eta\,\langle\mathbf{J}\boldsymbol{\cdot}\mathbf{B}\rangle \propto D_\eta\,\Delta^+\psi_p$, the poloidal flux and hence electron guiding centers are diffusing radially, irrespective of how current is driven and even in steady-state ($\dot{\psi} = 0$) with $\partial\psi_p/\partial t|_{\mathbf{x}} = 0$ — because, mathematically, ψ_p obeys a diffusion equation and is the coordinate being transformed to, and physically, in driven-dissipative $\mathbf{B}_p$ field systems the thin poloidal flux annulus $\delta\psi_g$ (and hence $p_{\zeta g}$) is always locally diffusing radially [2]. Since the electron guiding centers are tied to $\delta\psi_g$, in transforming to the ψ_p (and hence $p_{\zeta g}$) coordinate one is effectively transforming to a locally diffusing radial coordinate system. [For quasi-symmetric stellarators that have both vacuum (ψ_V) and current-driven (ψ_J) "poloidal" magnetic fluxes, D_η should be multiplied by [3] $\psi'_J/(\psi'_V+\psi'_J)$ since then in transforming (8) into (10) $\boldsymbol{\nabla}\psi_p \to (\psi'_V+\psi'_J)\boldsymbol{\nabla}\rho$ whereas $\Delta^+\psi_p \propto \langle\mathbf{J}\boldsymbol{\cdot}\mathbf{B}\rangle \to \Delta^+\psi_J$.]

While the MDKE was derived here for electrons, it applies equally well to ions for the time scales indicated in (16), as long as the usual finite gyroradius expansion is applicable. The $\mathcal{D}\{f\}$ Fokker-Planck spatial diffusion operator should also be added to the right of the usual gyrokinetic equations [6]. Since $D_\eta \sim \nu_e(c/\omega_p)^2$ is second order in the electromagnetic skin depth c/ω_p ($\simeq 0.1$ cm for $n_e \simeq 3\times10^{19}$ m^{-3}), addition of $\mathcal{D}\{f\}$ to the electron and ion drift-kinetic and gyrokinetic equations does not significantly modify plasma properties at zeroth order (density, temperature, pressure) or first order (flows, currents, heat flows along and within flux surfaces) in the small gyroradius, c/ω_p expansion. However, it will affect second order processes; in particular, it leads to paleoclassical radial electron heat transport [1–4]. Also, for perturbations that are radially localized, the $\mathcal{D}\{f\}$ operator introduces a new dissipative process with a rate of order $\nu_e(k_x c/\omega_p)^2 \lesssim \nu_e$ [2]. Such paleoclassical processes could be more important than second order finite ion gyroradius effects where $\varrho_i \equiv v_\perp/\omega_{ci} < c/\omega_p$, which occurs where the local $\beta_i \equiv n_e T_i/(B^2/2\mu_0) < m_e/m_i$ (e.g., for $T_i < 300$ eV where $B \sim 2$ T, $n_e \sim 10^{19}$ m^{-3}).

The lowest order $f \to f_0(\rho)$ is a Maxwellian [2]. The paleoclassical transport introduced via $\mathcal{D}\{f_0\}$ is a direct second order secular process — because it results from particle guiding centers being carried with another diffusing quantity, i.e., $\delta\psi_g$. Thus, plasma transport operators resulting from $\mathcal{D}\{f_0\}$ are not in the usual form of the divergence of radial flows. Rather, as can be seen from (19) with $\bar{u}_G = \bar{u}_{\dot{\psi}} = 0$, they become second derivatives of zeroth order plasma quantities (density, temperature); hence, when forced into the usual radial transport flux forms, they embody [2, 3] both diffusive and "pinch-type" (inward flow) density and temperature transport.

Because $\mathcal{D}\{f_0\}$ is the same in the electron and ion MDKEs, the axisymmetric component [2] of paleoclassical density transport will be the same for both species; hence, it will be automatically ambipolar. However, the helically-resonant effects [1–3], which result in the important multiplier $M \sim 10$ for paleoclassical electron heat transport, probably only apply to electrons. Thus, the helically-resonant component of paleoclassical transport is probably not ambipolar; rather, it is likely to induce an equilibrium radial electric field which, on time scales longer than the poloidal flow damping time ($\sim 1/\nu_i$, ion collision time), will lead to a toroidal flow. In addition, for transients ($\dot{\psi} \neq 0$) the helically-resonant responses probably move radially. Determination of such non-ambipolar and transient paleoclassical processes, and their effects on the radial electric field, toroidal flow and plasma transport, are left for future research.

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