Toroidal flow and radial particle flux in tokamak plasmas

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Many effects influence toroidal flow evolution in tokamak plasmas. Momentum sources and radial transport due to collisional processes and microturbulence-induced anomalous transport are usually considered. In addition, toroidal flow can be affected by non-axisymmetric magnetic fields; resonant components cause localized electromagnetic toroidal torques near rational surfaces in flowing plasmas and non-resonant components induce "global" toroidal flow damping torque throughout the plasma. Also, poloidal magnetic field transients on the magnetic field diffusion time scale can influence plasma transport. Many of these processes can also produce momentum pinch and intrinsic flow effects. This paper presents a comprehensive and self-consistent description of all these effects within a fluid moment context. Plasma processes on successive time scales (and constraints they impose) are considered sequentially: compressional Alfvén waves (Grad-Shafranov equilibrium, ion radial force balance); sound waves (pressure constant along a field line, incompressible flows within a flux surface); and ion collisions (damping of poloidal flow). Finally, plasma transport across magnetic flux surfaces is induced by the many second order (in the small gyroradius expansion) toroidal torque effects indicated above. Non-ambipolar components of the induced particle transport fluxes produce radial plasma currents. Setting the flux-surface-average of the net radial current induced by all these effects to zero yields the transport-time-scale equation for evolution of the plasma toroidal flow. It includes a combination of global toroidal flow damping and resonant torques induced by non-axisymmetric magnetic field components, poloidal magnetic field transients and momentum source effects, as well as the usual collision- and microturbulence-induced transport. On the transport time scale the plasma toroidal rotation determines the radial electric field for net ambipolar particle transport. The ultimate radial particle transport is composed of intrinsically ambipolar fluxes plus non-ambipolar fluxes evaluated at this toroidal-rotation-determined radial electric field.

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I. INTRODUCTION

Determining the magnitude, radial profile and evolution of toroidal flow in tokamak plasmas is an important issue for both present tokamak plasmas and ITER [1] — for $\mathbf{E}\times\mathbf{B}$ flow shear control of anomalous transport, prevention of locked modes, control of Edge Localized Modes (ELMs) via Resonant Magnetic Perturbations (RMPs) etc. Many effects [2] influence the evolution of toroidal flow in tokamak plasmas. Radial transport of toroidal flow due to collision-induced [3–7] and microturbulence-induced "anomalous" processes obtained using "mean-field" theory [8–13] are usually considered within the context of an equilibrium axisymmetric magnetic field model. In addition, the toroidal flow is constrained by faster time scale processes and can be affected by three-dimensional (3D) non-axisymmetric magnetic field components [14–21], magnetic field transients [22–25] and externally-supplied toroidal momentum sources (e.g., from neutral beam injection [26]).

Heretofore these diverse effects on the plasma toroidal flow have been mostly considered separately and in an *ad hoc* fashion. In this paper a detailed framework is developed for describing toroidal flow evolution in tokamaks that accounts for all these physical effects simultaneously and self-consistently. A "neoclassical equilibrium" is obtained for time scales longer than the collision time scales. It is briefly revisited to add self-consistently possible effects due to microturbulence and momentum sources (e.g., non-inductive current drives) on the lowest order plasma flows and parallel Ohm's law in tokamak experiments. The overall toroidal flow equation is developed using an approach originally developed [27, 28] for a neoclassical description of flows in non-axisymmetric (e.g., stellarator) toroidal magnetic confinement systems, but here extended to include microturbulence, magnetic field transients, and momentum source and sink effects.

Non-axisymmetric (NA) magnetic field components in tokamaks arise from coil irregularities, active control coils and plasma collective magnetic distortions [e.g., neoclassical tearing modes (NTMs) and resistive wall modes (RWMs)]. The NA components are typically very small compared to the equilibrium magnetic field: $\tilde{B}/B_0 \lesssim 10^{-3}$. Thus, they will be considered to be of first order or smaller in the small gyroradius expansion, which will greatly facilitate analysis of the 3D effects.

This paper uses a fluid moment approach to develop a comprehensive description of the various effects on the particle fluxes, toroidal flow, radial electric field and resultant net radial particle flux in a tokamak plasma. An earlier more limited and abridged version of this work has been presented elsewhere [2]. The magnetic field $\mathbf{B}$, electric potential ϕ and plasma parameters will be expanded in terms of their axisymmetric equilibrium plus

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gyroradius-small non-axisymmetric perturbations. Both externally induced perturbations and collective plasma fluctuations are taken into account. Effects on successively longer time scales will be considered sequentially: on the compressional Alfvén wave time scale the radial ion force balance yields a relation between the poloidal and toroidal flows within a flux surface and the radial electric field; on Coulomb collision time scales, the density, temperature and pressure of each plasma species become constant along a magnetic field line and flows within a magnetic flux surface become incompressible; and then, for times longer than the ion collision time, the poloidal ion flow is usually damped to a diamagnetic-type flow dependent on the ion temperature gradient.

Finally, there are many second order (in a small gyroradius expansion and the 3D NA magnetic field magnitudes) radial particle transport fluxes induced by toroidal torques on a plasma species, both collision-induced intrinsically ambipolar ones and possibly non-ambipolar ones. The collision-induced ones include classical [3], Pfirsch-Schlüter, banana-plateau [5, 6] and paleoclassical [29] particle fluxes, plus ones induced by parallel non-inductive and dynamo current-drives, and the $\mathbf{E}^A \times \mathbf{B}/B^2$ pinch. Possibly non-ambipolar fluxes are caused by: non-axisymmetric neoclassical toroidal viscosity (NTV) flow damping effects [14–17, 27, 28], perpendicular viscosities [3, 4, 7], toroidal torques on resonant surfaces due to non-ideal effects (e.g, due to resistivity [18] and two-fluid diamagnetic flows [19]) on NA resonant magnetic field components combined with NTV effects [20], polarization flows, microturbulence [8–13], magnetic field transients [22–25], and momentum sources. Non-ambipolar components of the particle fluxes cause radial plasma currents. Setting the flux-surface-average of the total radial current to zero (so from charge conservation and the time derivative of Gauss's law the radial electric field does not increase monotonically in time) yields [2, 15, 27, 28] a transport-time-scale toroidal flow evolution equation. The radial electric field is determined from the toroidal flow. The net radial particle flux is the sum of the collision-induced intrinsically ambipolar particle fluxes and the non-ambipolar particle fluxes evaluated at the toroidal-flow-determined radial electric field.

We make a number of *assumptions* to facilitate determining comprehensive equations for the particle fluxes and evolution of toroidal rotation in tokamak plasmas:

1) *Small gyroradius expansion,* which to zeroth order yields magnetohydrodynamic (MHD) force balance equilibrium, flows within flux surfaces at first order, and second order "radial" transport fluxes.

2) *Axisymmetric lowest order magnetic field structure,* with nested toroidal flux surfaces (i.e., no magnetic islands in region of interest).

3) *Gyroradius-small magnetic field non-axisymmetries,* such that toroidal non-axisymmetries (NA) in the magnetic field $\mathbf{B}$ are first order or smaller in the gyroradius expansion.

4) *Banana-plateau collisionalty regime* where electron and ion collision lengths are long compared to the poloidal periodicity length $2\pi Rq$ and hence plasma properties are constant on magnetic flux surfaces.

5) *Gyroradius small plasma fluctuations* which lead to second order microturbulence-induced "anomalous" radial plasma transport.

6) *Slow poloidal magnetic field transients* which at their fastest occur on the transport time scale.

7) *Poloidal damping of electron heat flow is neglected* in obtaining the parallel neoclassical Ohm's law.

No explicit inverse aspect ratio ($\epsilon \sim r/R_0$) expansion will be made, except in estimating the magnitude of some terms and in neglect of some minor $\mathcal{O}\{\epsilon^2\}$ terms. While the analysis will be valid over most of a tokamak plasma, because of assumptions 4) and 5) it might not apply to its very edge. In particular, very close to the magnetic separatrix in diverted tokamak plasmas assumption 4) can be violated, in which case a two-dimensional (radial, poloidal) description of plasma transport is needed. Also, in the edge of L-mode tokamak plasmas fluctuations can become so large that assumption 5) is violated, in which case a fluid-type turbulent plasma description may be needed there. Because of assumption 7), while scalings of all the collisional radial particle fluxes will be correct, some of their numerical coefficients will only be approximate. Finally, for simplicity we restrict our analysis to a two species plasma — electrons plus hydrogenic ions.

This paper is organized as follows. The next section (II) specifies the plasma and magnetic field models, and perturbation procedure. Section III discusses the successive time scales and the constraints they impose. The radial particle fluxes induced by the various effects indicated above and the plasma toroidal flow evolution equation that results from setting to zero the net radial currents from their non-ambipolar components are developed in Section IV. (Appendix A describes the relation of fluid flow velocities to guiding center velocities and Appendix B describes the gyroviscosity and its effects.) Some interesting properties of the various effects in the evolution equation for the plasma toroidal rotation are discussed in Section V. Finally, Section VI summarizes the main results obtained in this paper.

II. PLASMA AND MAGNETIC FIELD MODELS

We use a plasma description based on fluid moments of a general plasma kinetic equation for each species which includes the Vlasov operator, the Coulomb collision operator and sources. Some of the possible plasma particle and momentum sources and sinks are those due to: collisions with neutrals, fast ions injected via energetic neutral beams [26], current sources and sinks, and radio-frequency waves interacting with the plasma.

Such a description is sometimes called a two-fluid description. But because we are considering the banana-plateau collisionality regime, we will use the kinetically-derived closure relations for the (typically first order in the small gyroradius expansion) neoclassical parallel viscous forces $\langle \mathbf{B}_0 \cdot \nabla \cdot \boldsymbol{\pi} \rangle$ [6]. The magnetic field description will be based on a lowest order axisymmetric magnetic field which is composed of nested magnetic flux surfaces. That is, any non-axisymmetric magnetic field perturbations will be assumed to be small enough that they do not cause magnetic islands in the radial regions being considered. The dimensionless radial coordinate ρ will be based on the toroidal magnetic flux, as is customary for tokamak plasma transport modeling codes. The first order perturbations will include both the "average" (axisymmetric) poloidal variations in the equilibrium (Pfirsch-Schlüter effects) and effects due to "zero-toroidal-average," gyroradius-small perturbations.

The conservative forms of the density, momentum (force balance) equations for each plasma species obtained from the $\int d^3v\,(1, m\mathbf{v})$ fluid moments of a full plasma kinetic equation including sources are (in laboratory coordinates $\mathbf{x}$)

$$\partial n/\partial t|_{\mathbf{x}} + \nabla \cdot n\mathbf{V} = S_n, \qquad (1)$$

$$m \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} (n\mathbf{V}) + \nabla \cdot (mn\mathbf{V}\mathbf{V}) = nq\,(\mathbf{E} + \mathbf{V}\times\mathbf{B}) - \nabla p - \nabla \cdot \boldsymbol{\pi} + \mathbf{R} + \mathbf{S}_m. \qquad (2)$$

Here, $\mathbf{V}$ is the species flow velocity, $-\nabla \cdot \boldsymbol{\pi}$ is the viscous force density, $\mathbf{R}$ ($\mathbf{R}_e \simeq n_e e\,\eta \mathbf{J}$) is the Coulomb collision dynamical friction force density, and the other notation is standard [3]. The sources of density S_n and momentum $\mathbf{S}_m$ will be assumed to be small and to mostly contribute at the transport level (i.e., at second order in the gyroradius expansion). Also, the species label s has been suppressed for simplicity; it will be added as needed.

The lowest order axisymmetric equilibrium magnetic field $\mathbf{B}_0$ is composed of toroidal ($\mathbf{B}_t$) and poloidal ($\mathbf{B}_p$) components. It will be represented by

$$\begin{aligned} \mathbf{B}_0 &= \mathbf{B}_t + \mathbf{B}_p \equiv I\nabla\zeta + \nabla\zeta\times\nabla\psi_p \\ &= \nabla\psi_p\times\nabla(q\,\theta - \zeta) = \nabla\times(\psi_t\nabla\theta - \psi_p\nabla\zeta). \end{aligned} \qquad (3)$$

Here, $I(\psi_p) \equiv RB_t$ is the poloidal current function, $2\pi\,\psi_p(\rho)$ is the "equilibrium" poloidal magnetic flux which will be allowed to evolve on the slow transport time scale, ζ is the toroidal (axisymmetry) angle, $q(\psi_p) \equiv \mathbf{B}_0 \cdot \nabla\zeta / \mathbf{B}_0 \cdot \nabla\theta = d\psi_t/d\psi_p$ is the inverse of the rotational transform of the magnetic field $\mathbf{B}_0$, and θ is the "straight-field-line" (on a magnetic flux surface) poloidal angle. The Jacobian $\sqrt{g}$ of the transformation from laboratory ($\mathbf{x}$) to the (non-orthogonal) toroidal magnetic flux coordinates ρ, θ, ζ is

$$\sqrt{g} \equiv \frac{1}{\nabla\rho \cdot \nabla\theta\times\nabla\zeta} = \frac{d\psi_p/d\rho}{\mathbf{B}_0 \cdot \nabla\theta} = \frac{\psi_p'}{I/qR^2}, \qquad (4)$$

in which $\psi_p' \equiv d\psi_p/d\rho \simeq aRB_p$. Here, $\rho \equiv \sqrt{\psi_t/\psi_t(a)}$ is the dimensionless radial (flux surface label) coordinate based on the toroidal flux $2\pi\,\psi_t \equiv \iint d\mathbf{S}(\zeta) \cdot \mathbf{B}_0 = \int d^3x\,\mathbf{B}_0 \cdot \nabla\zeta$. It ranges from zero (at the magnetic axis) to unity (at the plasma edge or divertor magnetic separatrix of average minor radius a). Also, $R(\mathbf{x}) = R(\rho, \theta)$ is the major radius to the laboratory position $\mathbf{x}$.

The flux-surface-average (FSA) of a scalar function $f(\rho, \theta)$ is defined by

$$\langle f \rangle \equiv \frac{\int_0^{2\pi} d\theta\,\sqrt{g}\,f(\rho,\theta)}{\int_0^{2\pi} d\theta\,\sqrt{g}} = \frac{\int_0^{2\pi} d\theta\,f(\rho,\theta)/\mathbf{B}_0 \cdot \nabla\theta}{\int_0^{2\pi} d\theta/\mathbf{B}_0 \cdot \nabla\theta}. \qquad (5)$$

Flux-surface-averaging is an annihilation operator for the parallel derivative along a magnetic field line:

$$\langle \mathbf{B}_0 \cdot \nabla f \rangle = 0. \qquad (6)$$

In addition, the flux-surface-average of the divergence of a vector function $\mathbf{C}(\rho, \theta)$ is

$$\langle \nabla \cdot \mathbf{C} \rangle = \frac{d}{dV} \langle \mathbf{C} \cdot \nabla V \rangle = \frac{1}{V'} \frac{d}{d\rho} \left(V' \langle \mathbf{C} \cdot \nabla\rho \rangle \right), \qquad (7)$$

in which $V(\rho) \equiv \int_0^\rho d^3x = 2\pi \int_0^\rho d\rho \int_0^{2\pi} \sqrt{g}\,d\theta$ is the volume of the ρ flux surface and $V' \equiv dV/d\rho = 2\pi \int_0^{2\pi} \sqrt{g}\,d\theta$. Thus, the flux surface average is $\langle f \rangle = 2\pi \oint d\theta \sqrt{g}\,f/V'$.

The contravariant base vectors ($\mathbf{e}^i \equiv \nabla u^i$) for the $u^i = \rho, \theta, \zeta$ coordinate system are $\mathbf{e}^\rho \equiv \nabla\rho$, $\mathbf{e}^\theta \equiv \nabla\theta$, $\mathbf{e}^\zeta \equiv \nabla\zeta$. The covariant base vectors ($\mathbf{e}_i \equiv \partial\mathbf{x}/\partial u^i$) are $\mathbf{e}_\rho = \sqrt{g}\,\nabla\theta\times\nabla\zeta$, $\mathbf{e}_\theta = \sqrt{g}\,\nabla\zeta\times\nabla\rho$, $\mathbf{e}_\zeta = \sqrt{g}\,\nabla\rho\times\nabla\theta$. These non-orthogonal base vectors satisfy Kronecker delta relations: $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$. Because of toroidal axisymmetry, $\mathbf{e}^\zeta \equiv \nabla\zeta = \hat{\mathbf{e}}_\zeta/R$ and $\mathbf{e}_\zeta = R^2\nabla\zeta = R\,\hat{\mathbf{e}}_\zeta$ in which $\hat{\mathbf{e}}_\zeta \equiv \nabla\zeta/|\nabla\zeta|$ is the unit vector in the ζ direction.

To lowest order in the gyroradius expansion the density n, temperature T and pressure $p \equiv nT$ of both plasma species will be constant [5] on the poloidal magnetic flux surfaces $\psi_p(\rho)$. In first order we allow for "zero-toroidal-average" non-axisymmetric (3D) perturbations (ζ-dependent, denoted by tilde, due to instability-induced fluctuations plus those induced by NA externally-imposed magnetic field components) and poloidal variations in the ζ-average (denoted by an overbar) plasma parameters. Thus, we expand spatial dependences of n, T and p as

$$p(\mathbf{x}) = p_0(\rho) + \delta\,[\bar{p}_1(\rho,\theta) + \tilde{p}_1(\rho,\theta,\zeta)] + \mathcal{O}\{\delta^2\}, \qquad (8)$$

in which $\delta \sim \varrho_s/a \ll 1$ is the small gyroradius expansion parameter. Here $\varrho_s = v_{Ts}/\omega_{cs}$ is the most probable gyroradius for a species s with thermal speed $v_{Ts} \equiv \sqrt{2T_s/m_s}$ and gyrofrequency $\omega_{cs} \equiv q_s B_0/m_s$.

The electric potential ϕ will be expanded similarly. In addition, we define $\mathbf{E} = -\nabla\phi + \mathbf{E}^A$ in which $\mathbf{E}^A \equiv -\partial\mathbf{A}/\partial t$ with $\mathbf{A} \equiv \psi_t\nabla\theta - \psi_p\nabla\zeta$ for the magnetic field in (3). The ζ-average inductive electric field is

$$\bar{\mathbf{E}}^A = (\partial\Psi/\partial t + \dot{\psi}_p)\nabla\zeta - \dot{\psi}_t\nabla\theta \sim \mathcal{O}\{\delta^2\}. \qquad (9)$$

Here, $2\pi\,\partial\Psi/\partial t$ is the (vacuum) loop voltage $V^{\zeta}_{\rm loop}(t)$ induced by the rate of change of magnetic flux in the ohmic transformer (central solenoid). As indicated, it is spatially constant but a function of time t; it represents a gauge transformation choice. Further, $2\pi\,\dot{\psi}_p$ represents the inductive (Lenz's law) voltage induced within the plasma by poloidal magnetic flux transients (e.g., from turning on a non-inductive current drive). The $\dot{\psi}_t$ term represents toroidal magnetic flux transients induced by radial motion and shaping of the plasma. All these slow, transport-time-scale transient effects are discussed in Section IV.A below.

Because of the toroidal axisymmetry in the equilibrium, the zero-toroidal-average perturbations can be expanded in a Fourier series. For example,

$$\tilde{p}_1 = \sum_n \hat{p}_n e^{-in\zeta}, \quad \hat{p}_n \equiv \frac{1}{2\pi}\int_0^{2\pi} d\zeta\, e^{in\zeta}\,\tilde{p}_1. \tag{10}$$

The toroidal average of a function $F(\mathbf{x})$ is defined by

$$\overline{F(\mathbf{x})} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\zeta\, F(\mathbf{x}), \tag{11}$$

which is a function of ρ, θ. Note that since $\overline{\tilde{p}_1} = 0$,

$$\overline{p(\mathbf{x})} = p_0(\rho) + \delta\,\bar{p}_1(\rho,\theta) + \mathcal{O}\{\delta^2\}. \tag{12}$$

The toroidal average of non-axisymmetric perturbations times functions that do not depend on ζ vanishes. However, the toroidal average of products of perturbations does not in general vanish; for example, $\overline{\tilde{p}_1\tilde{\phi}} = \sum_n \hat{p}_{-n}\,\hat{\phi}_n$ usually does not vanish.

The total (equilibrium plus perturbed) magnetic field will be expanded as

$$\mathbf{B} = \mathbf{B}_0(\rho,\theta) + \delta\,[\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\parallel] + \mathcal{O}\{\delta^2\}. \tag{13}$$

The parallel $\parallel$ and perpendicular $\perp$ components are defined relative to the equilibrium magnetic field $\mathbf{B}_0$:

$$\mathbf{A}_\parallel \equiv \frac{\mathbf{B}_0\cdot\mathbf{A}}{B_0^2}\,\mathbf{B}_0, \quad \mathbf{A}_\perp \equiv \frac{-\,\mathbf{B}_0\times(\mathbf{B}_0\times\mathbf{A})}{B_0^2}. \tag{14}$$

The magnetic field magnitude can be approximated by

$$\begin{aligned} B \equiv |\mathbf{B}| \;&=\; \left(B_0^2 + 2\,\delta\,B_0\tilde{B}_\parallel + \delta^2\tilde{B}_\perp^2 + \delta^2\tilde{B}_\parallel^2\right)^{1/2} \\ &\simeq\; B_0(\rho,\theta) + \delta\,\tilde{B}_\parallel(\rho,\theta,\zeta) + \mathcal{O}\{\delta^2\}. \end{aligned} \tag{15}$$

The magnetic field perturbations $\tilde{B}_\parallel$ and $\tilde{\mathbf{B}}_\perp$ are expanded in Fourier series analogous to that for $\tilde{p}_1$ in (10).

Perpendicular gradients of instability-induced fluctuations will be assumed to scale as $1/\delta$ to reflect the short radial scale length of drift-wave-type perturbations. Thus, for example, $\nabla_\perp\tilde{p}_1 \sim (1/\delta)\,\delta \sim \delta^0 \sim 1$. In contrast, parallel gradients of fluctuations will be assumed to scale with the overall tokamak plasma dimensions, and hence as δ^0; thus, $\nabla_\parallel\,\tilde{p}_1 \sim \delta^0\delta \sim \delta \ll 1$. Gradients of average plasma properties will also be assumed to scale as δ^0. Hence, while $\boldsymbol{\nabla}_\perp\tilde{p}_1 \sim \delta^0 \sim 1$, we scale $\boldsymbol{\nabla}_\perp\bar{p}_1 \sim \delta \ll 1$.

III. SUCCESSIVE TIME SCALES, PROCESSES

The plasma toroidal flow evolves on the long, transport time scale ($\sim$ seconds). Its evolution arises from effects that are formally second order or smaller in the gyroradius expansion. To obtain an equation for evolution of the plasma toroidal flow on this long time scale we need to take account of faster processes and constraints they impose on plasma behavior. To analyze the various time scale processes, we will analyze sequentially three independent (but not orthogonal) components of the momentum (force balance) equation: radial ($\mathcal{O}\{\delta^0\}$, enforces radial force balance on fast, compressional Alfvén time scale), parallel ($\mathcal{O}\{\delta\}$, determines first order flows on a flux surface on intermediate ion collision time scale), and toroidal ($\mathcal{O}\{\delta^2\}$, determines radial particle transport fluxes and the desired toroidal plasma flow equation).

Summing the density and momentum equations in (1) and (2) over species, we readily obtain to zeroth order the ideal magnetohydrodynamic (MHD) plasma equations $\partial\rho_m/\partial t + \boldsymbol{\nabla}\cdot\rho_m\mathbf{V} = 0$ and $\rho_m d\mathbf{V}/dt = \mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla}P$. In obtaining these equations we have neglected viscous force effects which are first order in the gyroradius expansion and source effects which are assumed to be of first or higher order. Neglecting electron inertia, the lowest order electron momentum equation yields the collisionless two-fluid Ohm's law $\mathbf{E}+\mathbf{V}\times\mathbf{B} = (\mathbf{J}\times\mathbf{B}-\boldsymbol{\nabla}p_e)/n_e e$. In these equations ρ_m is the mass density, $\mathbf{V} \simeq \mathbf{V}_i$ is the plasma ($\simeq$ ion) flow velocity, $\mathbf{J} \equiv n_i q_i\mathbf{V}_i - n_e e\mathbf{V}_e$ is the current density and $P = p_e + p_i$ is the total plasma pressure. Adding the non-relativistic Maxwell equations $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$ and $\partial\mathbf{B}/\partial t = -\boldsymbol{\nabla}\times\mathbf{E}$, and an isentropic equation of state $(d/dt)\,(\ln P/\rho_m^\Gamma) = \mathcal{O}\{\delta^2\} \to 0$, which is derivable from energy balance equations, completes the zeroth order plasma description.

The fastest time scale MHD processes are compressional Alfvén waves, which propagate perpendicular to magnetic field lines. On time scales longer than their natural wave periods ($\tau_A \sim a/c_A \lesssim \mu$sec), together with the condition $\mathbf{B}_0\cdot\boldsymbol{\nabla}P_0 = 0 \implies P_0 = P_0(\psi_p)$, they cause tokamak plasmas to come into an MHD force balance equilibrium with $\mathbf{J}_0\times\mathbf{B}_0 = \boldsymbol{\nabla}P_0 = (dP_0/d\psi_p)\,\boldsymbol{\nabla}\psi_p$. This relation together with the equilibrium Maxwell equations yields the Grad-Sharanov equation for determining the equilibrium poloidal flux function $\psi_p(\rho)$.

Replacing $\mathbf{J}_0\times\mathbf{B}_0$ by $\boldsymbol{\nabla}P_0 = \boldsymbol{\nabla}p_{e0} + \boldsymbol{\nabla}p_{i0}$ in the ζ-average ideal (collisionless) two-fluid Ohm's law yields

$$0 = n_{i0}q_i(\bar{\mathbf{E}}_0 + \bar{\mathbf{V}}_i\times\mathbf{B}_0) - \boldsymbol{\nabla}\bar{p}_{i0}, \tag{16}$$

in which the lowest order electric field is electrostatic: $\bar{\mathbf{E}}_0 \equiv -\boldsymbol{\nabla}\Phi_0(\psi_p)$. This MHD equilibrium ion force balance equation can also be obtained directly from the equilibium limit ($d/dt \longrightarrow 0$) of the ion force balance equation in (2) — by neglecting the frictional ($\mathbf{R} \sim \delta$) and viscous ($\boldsymbol{\nabla}\cdot\boldsymbol{\pi} \sim \delta$) forces, and momentum sources ($\mathbf{S}_m \sim \delta, \delta^2$), which are all higher order in the gyroradius expansion.

Taking the "radial" ($\mathbf{e}_\rho$) projection of (16) yields

$$\Omega_t \equiv \bar{\mathbf{V}}_i \cdot \boldsymbol{\nabla}\zeta = -\left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\psi_p}\right) + q\bar{\mathbf{V}}_i \cdot \boldsymbol{\nabla}\theta. \quad (17)$$

This equation provides a relation between the average (designated by overbar) toroidal flow $\bar{\mathbf{V}}_i \cdot \boldsymbol{\nabla}\zeta \sim \bar{V}_t/R$, lowest order radial electric field $\bar{E}_\rho \equiv -\hat{\mathbf{e}}_\rho \cdot \boldsymbol{\nabla}\Phi_0(\rho) = -|\boldsymbol{\nabla}\rho|\,(d\phi_0/d\rho) = -\,|\boldsymbol{\nabla}\rho|\,(d\Phi_0/d\psi_p)\,\psi_p'$, toroidal ion diamagnetic flow $-(1/n_{i0}q_i)(dp_{i0}/d\psi_p)$ and poloidal ion flow $\bar{\mathbf{V}}_i \cdot \boldsymbol{\nabla}\theta \sim \bar{V}_p/r$. However, it does not specify any of these quantities. Since in obtaining (17) we divided through by $q_i B_0$, the poloidal and toroidal ion flows are first order in the gyroradius ($\varrho_s \propto 1/q_s B_0$). Thus, these flows within tokamak flux surfaces are first order in the small gyroradius expansion; however, for simplicity of notation we will not always indicate this by adding a subscript 1 to these flows. Transport flows in the radial direction (i.e., $\bar{\mathbf{V}} \cdot \boldsymbol{\nabla}\rho$) will be of second or higher order in the gyroradius expansion.

Coulomb collisions cause electrons to thermalize (become Maxwellian) on the electron collision time scale ($\sim 1/\nu_e \gtrsim 10\,\mu$sec for $T_e \gtrsim 1$ keV, $n_e \lesssim 3{\times}10^{19}$ m^{-3} and $Z_{\rm eff} \lesssim 3$). Similarly, ions thermalize on the ion collision time scale ($\sim 1/\nu_i \gtrsim$ msec for $T_i \gtrsim 1$ keV). In doing so they cause their corresponding species temperatures to equilibrate along magnetic field lines over distances of order the collision length $\lambda \sim v_T/\nu > 10^2$ m. From assumption 4) above this causes [5] the species density and temperature to become constant on poloidal flux surfaces on the collision time scale of the species: $n_0 = n_0(\psi_p)$, $T_0 = T_0(\psi_p)$ for $t > 1/\nu$. On this same time scale the species flow velocity becomes a defineable and physically meaningful quantity.

Next, consider the form of the density equation (1) on time scales longer than the species collision time. Since the lowest (first) order equilibrium flows lie within a flux surface, they can be written in terms of their poloidal ($\bar{\mathbf{V}} \cdot \boldsymbol{\nabla}\theta$) and toroidal ($\bar{\mathbf{V}} \cdot \boldsymbol{\nabla}\zeta$) components as

$$\bar{\mathbf{V}}_1 \equiv \mathbf{e}_\theta \bar{\mathbf{V}} \cdot \boldsymbol{\nabla}\theta + \mathbf{e}_\zeta \bar{\mathbf{V}} \cdot \boldsymbol{\nabla}\zeta. \quad (18)$$

This representation applies to either electrons or ions. Alternatively, the equilibrium flows within a flux surface can be represented in terms of their components parallel to ($\|$) and cross ($\wedge$, perpendicular to $\mathbf{B}_0$ but within a flux surface) the equilibrium magnetic field $\mathbf{B}_0$:

$$\bar{\mathbf{V}}_1 = \bar{V}_\| \, \mathbf{B}_0/B_0 + \bar{\mathbf{V}}_\wedge. \quad (19)$$

The equilibrium cross flow $\bar{\mathbf{V}}_\wedge$ in each species is obtained by taking the cross product of $\mathbf{B}_0$ with the lowest order equilibrium ($\partial/\partial t \to 0$) momentum equation (2):

$$\bar{\mathbf{V}}_{s\wedge} = \frac{\mathbf{B}_0 \times \boldsymbol{\nabla}\psi_p}{B_0^2}\left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_p}\right). \quad (20)$$

The two terms in parentheses represent the usual equilibrium $\bar{\mathbf{E}}_0 \times \mathbf{B}_0$ ($\bar{\mathbf{V}}_E$) and diamagnetic ($\bar{\mathbf{V}}_*$) flows.

The comparable first order perturbed cross flow for each species is obtained by taking the cross product of the lowest order perturbed momentum equation from (2) with $\mathbf{B}_0$:

$$\tilde{\mathbf{V}}_{s\wedge} = \frac{1}{B_0^2}\,\mathbf{B}_0 \times \left(\boldsymbol{\nabla}\tilde{\phi}_1 + \frac{1}{n_{s0}q_s}\boldsymbol{\nabla}\tilde{p}_{s1}\right) \equiv \tilde{\mathbf{V}}_E + \tilde{\mathbf{V}}_*. \quad (21)$$

The terms in parentheses represent the perturbed $\tilde{\mathbf{E}} \times \mathbf{B}_0$ and diamagnetic flows. Since the first order perturbed flows have a nonzero radial component, they lead to the anomalous radial particle flux $\langle \overline{\tilde{n}_1 \tilde{\mathbf{V}}_\wedge \cdot \boldsymbol{\nabla}\rho} \rangle$. The relation of the fluctuating fluid flow velocity $\tilde{\mathbf{V}}_\wedge$ to the guiding center flow velocity usually calculated in gyrokinetics (see for example [12]) is discussed in Appendix A.

Because the "equilibrium" density n_0 only changes on the transport time scale ($\partial/\partial t \sim \delta^2$) and the radial transport fluxes and density sources are second order, the lowest order average density equation (1) reduces to

$$\boldsymbol{\nabla} \cdot \bar{\mathbf{V}}_1 = 0 + \mathcal{O}\{\delta^2\}. \quad (22)$$

The divergence of a vector $\mathbf{C}$ is defined by $\boldsymbol{\nabla} \cdot \mathbf{C} = (1/\sqrt{g}\,)\sum_i (\partial/\partial u^i)(\sqrt{g}\,\mathbf{C} \cdot \mathbf{e}^i)$. Thus, using the facts that $\bar{\mathbf{V}}_1$ has no radial component and the equilibrium $\mathbf{B}_0$ is axisymmetric ($\partial/\partial\zeta \to 0$), this reduces to

$$(\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta)\,\frac{\partial}{\partial\theta}\left(\frac{\bar{\mathbf{V}}_1 \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta}\right) = 0. \quad (23)$$

The solution [6] of this partial differential equation is that a poloidal flow function U_θ is constant on a flux surface:

$$U_\theta(\psi_p) \equiv \frac{\bar{\mathbf{V}}_1 \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{\bar{V}_\|}{B_0} + \frac{\bar{\mathbf{V}}_\wedge \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta}. \quad (24)$$

Using the poloidal component of first order $\bar{\mathbf{E}}_0 \times \mathbf{B}_0$ and diamagnetic flows within a flux surface from (20) yields

$$\frac{\bar{\mathbf{V}}_\wedge \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{I}{B_0^2}\left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_p}\right) \equiv -\,\frac{I}{B_0^2}\,\Omega_\wedge, \quad (25)$$

in which $\Omega_\wedge(\psi_p) = \Omega_\wedge(\rho)$ is constant on a flux surface.

The currents flowing within the flux surface can be similarly determined. Using (19) and (20), we obtain

$$\mathbf{J} \equiv \sum_s n_{s0}q_s\bar{\mathbf{V}}_{s1} \equiv \mathbf{J}_\| + \mathbf{J}_\wedge = J_\| \frac{\mathbf{B}_0}{B_0} + \frac{\mathbf{B}_0 \times \boldsymbol{\nabla}P_0}{B_0^2}, \quad (26)$$

which is the usual sum of the parallel and diamagnetic current densities. Summing nq times the poloidal flow components in (24) and (25), or using the fact that for a quasi-neutral plasma the current density is also incompressible ($\boldsymbol{\nabla} \cdot \mathbf{J} = 0$), yields

$$K_J(\psi_p) \equiv \frac{\mathbf{J} \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{J_\|}{B_0} + \frac{I}{B_0^2}\frac{dP_0}{d\psi_p}. \quad (27)$$

The constant K_J is obtained from the flux surface average (FSA) of B_0^2 times this equation:

$$K_J = \frac{\langle B_0 J_\| \rangle}{\langle B_0^2 \rangle} + \frac{I}{\langle B_0^2 \rangle}\frac{dP_0}{d\psi_p}. \quad (28)$$

Using this form in (27) yields

$$B_0 J_\| = \frac{\langle B_0 J_\| \rangle B_0^2}{\langle B_0^2 \rangle} - I\,\frac{dP_0}{d\psi_p}\left(1 - \frac{B_0^2}{\langle B_0^2\rangle}\right). \qquad (29)$$

Here, $\langle B_0 J_\| \rangle$ is the flux-surface-average parallel current which will be determined from a parallel Ohm's law below and the last term represents the Pfirsch-Schlüter current whose flux-surface-average vanishes.

Similar considerations and orderings of the species energy balance equation and the heat flux fluid moment equation yield analogous formulas [6] for the first order equilibrium heat flows within a tokamak flux surface:

$$Q_\theta(\psi_p) \equiv \frac{\bar{\mathbf{q}}_1 \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{\bar{q}_\|}{B_0} + \frac{\bar{\mathbf{q}}_\wedge \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta}, \qquad (30)$$

$$\frac{\bar{\mathbf{q}}_{s\wedge} \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{5}{2}\,\frac{I}{q_s B_0^2}\,\frac{dT_{s0}}{d\psi_p}. \qquad (31)$$

Finally, the flux-surface-average of the parallel ($\mathbf{B}_0 \cdot$) component of the lowest order parallel ion heat flux equation [6] is $\langle \mathbf{B}_0 \cdot \bar{\mathbf{R}}_{\mathbf{q}_i} \rangle = 0 + \mathcal{O}\{\delta^2\}$ in which $\bar{\mathbf{R}}_{\mathbf{q}_i} \propto -\nu_i \bar{\mathbf{q}}_i$ is the ion heat friction force. Solving (30) for $\bar{q}_{i\|}$ and substituting it in this lowest order relation yields [6]

$$Q_{i\theta} = \frac{1}{\langle B_0^2 \rangle}\left\langle B_0^2\,\frac{\bar{\mathbf{q}}_{i\wedge} \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta}\right\rangle = \frac{5}{2}\,\frac{n_{i0} T_{i0}\, I}{q_i \langle B_0^2\rangle}\,\frac{dT_{i0}}{d\psi_p}. \qquad (32)$$

The (first order) average parallel force balance for each species is obtained from the parallel ($\mathbf{B}_0 \cdot$) component of the momentum equation. Its most convenient form here is obtained by using the density equation (1) to write

$$\begin{aligned} & m(\partial/\partial t)(n\mathbf{V}) + \boldsymbol{\nabla}\cdot(mn\mathbf{V}\mathbf{V} + \boldsymbol{\pi}) \\ &= mn[\partial \mathbf{V}/\partial t + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V}] + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} + m\mathbf{V}S_n. \end{aligned} \qquad (33)$$

The viscous stress tensor $\boldsymbol{\pi}$ has components along $\mathbf{B}$ ($\boldsymbol{\pi}_\|$), across magnetic field lines but within flux surfaces ($\boldsymbol{\pi}_\wedge$) and perpendicular to flux surfaces ($\boldsymbol{\pi}_\perp$) [3]:

$$\boldsymbol{\pi} = \delta^0 \boldsymbol{\pi}_\| + \delta\,\boldsymbol{\pi}_\wedge + \delta^2 \boldsymbol{\pi}_\perp. \qquad (34)$$

For the parallel force balance the most important viscous force will be the parallel component of $-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\|$ which we will discuss below. The gyroviscous stress $\boldsymbol{\pi}_\wedge$ indicates a diamagnetic-type $\mathbf{B}\times\boldsymbol{\nabla}\mathbf{V}$ effect. The gyroviscous force $-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\wedge$ it induces is higher order for equilibrium flows. However, it is significant for fluctuating flows — see Appendix B. The forces due to perpendicular viscous stresses $\boldsymbol{\pi}_\perp$ are higher order and will not contribute to the first order parallel momentum equation.

Thus, taking the FSA of the $\mathbf{B}_0$ (parallel) component of the ζ average of the momentum equation in (2) with all quantities expanded as in (8), employing (33), and eliminating $\bar{\phi}_1$ and $\bar{p}_1$ terms using (6) yields the $\mathcal{O}\{\delta\}$ parallel force balance equation for each species:

$$\begin{aligned} mn_0 \frac{\partial \langle B_0 \bar{V}_\| \rangle}{\partial t} \;=\; & n_0 q \langle \mathbf{B}_0 \cdot \bar{\mathbf{E}}^A \rangle - \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\pi}}_\| \rangle + \langle \mathbf{B}_0 \cdot \bar{\mathbf{R}} \rangle \\ & + \langle \mathbf{B}_0 \cdot (\bar{\mathbf{S}}_m - m\bar{\mathbf{V}}\bar{S}_n) \rangle \\ & - \langle \mathbf{B}_0 \cdot (mn_0 \overline{\tilde{\mathbf{V}}_1 \cdot \boldsymbol{\nabla}\tilde{\mathbf{V}}_1} + \overline{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\wedge} \rangle \\ & + n_0 q \langle \mathbf{B}_0 \cdot \overline{\tilde{\mathbf{V}}_\wedge \times \tilde{\mathbf{B}}_\perp} \rangle. \end{aligned} \qquad (35)$$

The $mn_0 \tilde{\mathbf{V}}_1 \cdot \boldsymbol{\nabla}\tilde{\mathbf{V}}_1$ term represents the force caused by the Reynolds stress introduced by the fluctuations. To lowest order the ζ-average gyroviscous force $\overline{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\wedge}$ produces (see for example [30]) the "gyroviscous cancellation" of the $\tilde{\mathbf{V}}_* \cdot \boldsymbol{\nabla}\tilde{\mathbf{V}}$ contribution to the Reynolds stress force — see Appendix B. The Maxwell-stress-type term $n_0 q \langle \mathbf{B}_0 \cdot \overline{\tilde{\mathbf{V}}_\wedge \times \tilde{\mathbf{B}}_\perp} \rangle = -n_0 q \langle \overline{\tilde{\mathbf{B}}_\perp \cdot [\boldsymbol{\nabla}\tilde{\phi}_1 + (1/n_0 q)\boldsymbol{\nabla}\tilde{p}_1]} \rangle$ will yield dynamo-type contributions to the parallel Ohm's law below. Both of these contributions are typically small for drift-wave-type fluctuations in the hot core of tokamak plasmas [12, 31].

The electron Coulomb collision frictional force density is given by [3] $\bar{\mathbf{R}}_e \simeq n_{e0} e(\mathbf{J}_\|/\sigma_\| + \mathbf{J}_\perp/\sigma_\perp)$. The perpendicular and parallel (Spitzer) electrical conductivities are $\sigma_\perp \equiv n_{e0} e^2/m_e \nu_e$ and $\sigma_\| \simeq 1.96\,\sigma_\perp$ for $Z = 1$. Thus, the equilibrium limit ($\partial/\partial t \ll \nu_e$) of the electron momentum equation (35) yields an equation for the flux-surface-average parallel current [5, 6] in the plasma:

$$\begin{aligned} \langle B_0 J_\| \rangle \;=\; & \sigma_\| \langle \mathbf{B}_0 \cdot \bar{\mathbf{E}}^A \rangle + (\sigma_\|/n_{e0} e)\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\pi}}_{e\|} \rangle \\ & + \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm CD} \rangle + \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm dyn} \rangle. \end{aligned} \qquad (36)$$

The flux-surface-average parallel electron viscous force can be written as [6] $\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\pi}}_{e\|} \rangle \simeq m_e n_{e0} \mu_{e00} \langle B_0^2 U_{e\theta} \rangle$ in which the neglected poloidal electron heat flow damping effects induced by $Q_{e\theta}$ only modify some numerical coefficients. Here, $\mu_{e00} \simeq 2.24\sqrt{\epsilon}\,\nu_e$ is the viscous damping frequency [6] (for $Z_i = 1$, $\sqrt{\epsilon} \ll 1$) of the poloidal electron flow $U_{e\theta}$ in which ϵ is the inverse aspect ratio which here represents the variation of $|\mathbf{B}_0| = B_0(\rho,\theta)$ on a flux surface: $\epsilon \equiv (B_{\max} - B_{\min})/(B_{\max} + B_{\min}) \simeq r/R_0$. Using the definition $J_\| = n_{e0} e\,(V_{i\|} - V_{e\|})$, and (24), (25) for both electrons and ions, the parallel viscous force contribution to the parallel Ohm's law in (36) can be written as

$$\frac{\sigma_\|}{n_{e0} e}\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\pi}}_{e\|} \rangle \simeq -\frac{\sigma_\|}{\sigma_\perp}\,\frac{\mu_{e00}}{\nu_e}\langle B_0 J_\| \rangle + \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle. \qquad (37)$$

Here, the bootstrap current is defined by

$$\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle \simeq -\frac{\sigma_\|}{\sigma_\perp}\,\frac{\mu_{e00}}{\nu_e}\left(\frac{I}{\psi_p'}\,\frac{dP_0}{d\rho} - n_{e0} e U_{i\theta} \langle B_0^2 \rangle\right). \qquad (38)$$

Thus, in (36) the parallel electron viscous force in (37) produces [6, 29] both the trapped particle modification ($\propto \mu_{e00} \sim \sqrt{\epsilon}\,\nu_e$) of the Spitzer parallel electrical conductivity $\sigma_\|$ and the bootstrap current.

The "non-inductive" current drive (CD) in (36) is induced by electron momentum and particle sources:

$$\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm CD} \rangle \equiv -(\sigma_\|/n_{e0} e)\langle \mathbf{B}_0 \cdot (\bar{\mathbf{S}}_{em} - m_e \bar{\mathbf{V}}_e \bar{S}_{en} \rangle. \qquad (39)$$

Most non-inductive current drives (e.g., due to electron cyclotron current drive) result from the average electron parallel momentum $\langle \mathbf{B}_0 \cdot \bar{\mathbf{S}}_{em} \rangle$ they impart to the plasma.

Fluctuation-induced electron Reynolds and Maxwell-type stresses cause a dynamo-type parallel current:

$$\begin{aligned} \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm dyn} \rangle \;=\; & \frac{\sigma_\|}{n_{e0} e}\langle \mathbf{B}_0 \cdot (m_e n_{e0} \overline{\tilde{\mathbf{V}}_e \cdot \boldsymbol{\nabla}\tilde{\mathbf{V}}_e} + \overline{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{e\wedge}}\,) \rangle \\ & + \sigma_\| \langle \mathbf{B}_0 \cdot \overline{\tilde{\mathbf{V}}_{e\wedge} \times \tilde{\mathbf{B}}_\perp} \rangle. \end{aligned} \qquad (40)$$

In tokamak plasmas the $\|$ electron Reynolds stress and dynamo-type terms are usually negligible [31, 32].

Taking account of the bootstrap current and trapped-particle effects from the parallel electron viscous force in (37), (36) becomes a neoclassical parallel Ohm's law that includes non-inductive and dynamo current-drives:

$$\begin{aligned}&\langle \mathbf{B}_0\boldsymbol{\cdot}\bar{\mathbf{E}}^A\rangle = \eta_\|^{\rm nc}\,\langle B_0 J_\|\rangle \\ &- (1/\sigma_\|)\,[\,\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm bs}\rangle + \langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm CD}\rangle + \langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm dyn}\rangle\,]. \quad (41)\end{aligned}$$

Here, the neoclassical parallel electrical resistivity is

$$\eta_\|^{\rm nc} \simeq \frac{1}{\sigma_\|}\left(1+\frac{\sigma_\|}{\sigma_\perp}\,\frac{\mu_{e00}}{\nu_e}\right). \quad (42)$$

Summing the electron and ion FSA parallel momentum equations (35) and using the quasineutrality constraint $\sum_s n_{s0}q_s = 0$ and the momentum conservation relation for Coulomb collisions between species of $\sum_s \bar{\mathbf{R}}_s = 0$ yields the overall plasma parallel force balance:

$$\begin{aligned}\rho_m\frac{\partial\langle B_0\bar{V}_\|\rangle}{\partial t} &\simeq -\textstyle\sum_s\langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{i\|}\rangle \\ &- \textstyle\sum_s\langle \mathbf{B}_0\boldsymbol{\cdot}(\,m_s n_{s0}\overline{\tilde{\mathbf{V}}_s\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_s} + \overline{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_{s\wedge}}\,)\,\rangle \\ &+ \langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\tilde{\mathbf{J}}_\wedge\times\tilde{\mathbf{B}}_\perp}\,\rangle, \\ &+ \langle \mathbf{B}_0\boldsymbol{\cdot}\textstyle\sum_s(\bar{\mathbf{S}}_{sm} - m_s\bar{\mathbf{V}}_s\bar{S}_{sn})\rangle. \quad (43)\end{aligned}$$

The sums over species are dominated by the ion contributions since the corresponding electron momentum contributions are typically a factor of $\sqrt{m_e/m_i}\sim 1/60 \ll 1$ smaller. The FSA of the equilibrium parallel ion viscous force, which is $\mathcal{O}\{\delta\}$, is given by [6]

$$\langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{i\|}\rangle \simeq m_i n_{i0}\langle B_0^2\rangle\left[\mu_{i00}U_{i\theta}+\mu_{i01}\frac{-2}{5n_iT_{i0}}Q_{i\theta}\right]. \quad (44)$$

Here, $\mu_{i00}\simeq 1.1\sqrt{\epsilon}\,\nu_i$ and $\mu_{i01}\simeq 1.17\,\mu_{i00}$ for $Z=1$ and $\sqrt{\epsilon}\ll 1$, in the asymptotic banana collisionality regime [6] [i.e., for $\nu_{*i}\equiv \nu_i Rq/\epsilon^{3/2}v_{Ti}\ll 1$]. They are damping frequencies for the poloidal ion flow and heat flow.

Physically, the ion parallel viscous force $\langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{i\|}\rangle$ is caused by ion collisional drag by trapped particles ($f_t\sim 1.46\sqrt{\epsilon}$) on the untrapped (circulating, $f_c\equiv 1-f_t$) ions that carry the parallel (poloidal) flow. This viscous force damps the poloidal (ion) flow to an ion-temperature-gradient diamagnetic-type flow because hotter ions are more collisionless than bulk ions and hotter regions at smaller radii damp less than colder ones at larger radii.

Thus, (43) is an evolution equation for the average parallel ion flow $\bar{V}_{i\|}$ or the average poloidal ion flow function $U_{i\theta}$. The parallel and poloidal ion flows come into equilibrium (see [33] and references cited therein) on the ion collision time scale $t > 1/\nu_i\sim$ msec. In the absence of the usually small parallel momentum sinks and sources on the second through fourth lines of (43), the equilibrium poloidal ion flow can be obtained by setting the parallel ion viscous force in (44) to zero:

$$U_{i\theta}^0 \simeq -\frac{\mu_{i01}}{\mu_{i00}}\,\frac{-2}{5n_iT_{i0}}\,Q_{i\theta} = c_p\,\frac{I}{q_i\langle B_0^2\rangle}\,\frac{dT_{i0}}{d\psi_p}. \quad (45)$$

The poloidal flow coefficient $k=c_p\equiv \mu_{i01}/\mu_{i00}$ is 1.17 for $\nu_{*i}\ll 1$, $\sqrt{\epsilon}\ll 1$. However, c_p depends on the ion collisionality regime and ϵ [6]; with impurities it also depends on gradients of impurity densities and temperatures [34]. It is often evaluated using the NCLASS code [35].

When sinks and sources on the second and third lines of (43) are significant, they cause a poloidal ion flow of

$$\begin{aligned}U_{i\theta}(\psi_p) = U_{i\theta}^0 &- \frac{\langle \mathbf{B}_0\boldsymbol{\cdot}(\,m_i n_{i0}\overline{\tilde{\mathbf{V}}_i\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_i} + \overline{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_{i\wedge}}\,)\,\rangle}{m_i n_{i0}\mu_{i00}\langle B_0^2\rangle} \\ &+ \frac{\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\tilde{\mathbf{J}}_\wedge\times\tilde{\mathbf{B}}_\perp}\,\rangle + \langle \mathbf{B}_0\boldsymbol{\cdot}\sum_s(\bar{\mathbf{S}}_{sm} - m_s\bar{\mathbf{V}}_s\bar{S}_{sn})\rangle}{m_i n_{i0}\mu_{i00}\langle B_0^2\rangle}. \quad (46)\end{aligned}$$

The additions to the poloidal ion flow $U_{i\theta}^0$ in (45) are induced by Reynolds stresses, dynamo-type effects, and ion "flow drive" effects due to $\langle \mathbf{B}_0\boldsymbol{\cdot}\bar{\mathbf{S}}_{im}\rangle$, respectively. They are usually thought to be small in tokamak plasmas — see Eqs. (22), (25) and discussion in [12].

Having determined the average poloidal ion flow $\bar{\mathbf{V}}_i\boldsymbol{\cdot}\boldsymbol{\nabla}\theta = U_{i\theta}\,\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta$, we substitute it into the Ω_t expression obtained in (17) to obtain the more specific toroidal rotation frequency relation (for $t > 1/\nu_i \gtrsim$ msec)

$$\Omega_t = -\left(\frac{d\Phi_0}{d\psi_p}+\frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\psi_p}\right) + \Omega_{*p} = \Omega_{i\wedge}+\Omega_{*p}. \quad (47)$$

Here, the poloidal flow contribution to Ω_t is

$$\Omega_{*p}(\rho,\theta)\equiv \frac{I}{R^2}U_{i\theta}(\rho)\simeq \frac{c_p\,I^2}{q_iR^2\langle B_0^2\rangle}\,\frac{dT_{i0}}{d\psi_p}, \quad (48)$$

in which the last approximate form neglects the usually small parallel momentum sink and source terms. Since for rigid body rotation Ω_t must be a flux function, tokamak plasmas rotate toroidally as a rigid body only when the poloidal flow is negligible so $\Omega_t\simeq\Omega_{i\wedge}(\psi_p)$.

The neoclassical literature contains a number of forms for the average ion flow velocity $\bar{\mathbf{V}}_1$ in addition to those given in (18) and (19). In particular, using the definition of $\Omega_\wedge$ at the end of (25) in (47), one can write the first order average ion flow within a flux surface as [6]

$$\bar{\mathbf{V}}_i = \Omega_{i\wedge}\mathbf{e}_\zeta + U_{i\theta}\mathbf{B}_0 = \Omega_{i\wedge}(\psi_p)R^2\boldsymbol{\nabla}\zeta + U_{i\theta}(\psi_p)\mathbf{B}_0. \quad (49)$$

The $\boldsymbol{\nabla}\zeta$ projection of this representation of $\bar{\mathbf{V}}_i$ readily yields the toroidal rotation frequency given in (47) above. Also, the plasma parallel flow speed can be written as

$$\begin{aligned}\bar{V}_\| &\equiv \mathbf{B}_0\boldsymbol{\cdot}\bar{\mathbf{V}}_i/B_0 = \Omega_{i\wedge}(\psi_p)\,I(\psi_p)/B_0 + U_{i\theta}(\psi_p)B_0 \\ &\simeq -R\left(\frac{d\Phi_0}{d\psi_p}+\frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\psi_p}\right) + k\frac{R_0^2}{q_iR}\,\frac{dT_{i0}}{d\psi_p}. \quad (50)\end{aligned}$$

In (48)–(50), $R\equiv R(\rho,\theta)\simeq R_0(1+\epsilon\cos\theta)$.

For given radial ion pressure and temperature gradients, (47) provides a consistency relation between the plasma toroidal rotation $\Omega_t \equiv \bar{\mathbf{V}}\boldsymbol{\cdot}\boldsymbol{\nabla}\theta \simeq \bar{V}_t/R$ and the lowest order radial electric field $\bar{E}_\rho \equiv -|\boldsymbol{\nabla}\rho|\psi_p'(d\Phi_0/d\psi_p)$. However, it does not specify either of these quantities. To proceed further we need to determine either the radial electric field or the toroidal flow — from their own evolution equations. We proceed by calculating the radial particle transport fluxes and hence net radial current in the spirit of seeking to calculate the radial electric field.

IV. TRANSPORT TIME SCALE

The collision-induced radial particle fluxes of electrons and ions are mostly determined from the second order flux $n_0\bar{\mathbf{V}}_2\cdot\boldsymbol{\nabla}\psi_p$. To determine them we first note that

$$\mathbf{e}_\zeta\cdot\mathbf{V}\times\mathbf{B}_0 = -\mathbf{V}\cdot\mathbf{e}_\zeta\times\mathbf{B}_0 = \mathbf{V}\cdot\boldsymbol{\nabla}\psi_p = \psi_p'\mathbf{V}\cdot\boldsymbol{\nabla}\rho. \quad (51)$$

Thus, the radial particle flux can be determined from the toroidal angular component ($\mathbf{e}_\zeta\,\cdot$) of the force balance (2), i.e., from the toroidal torques on the plasma species.

Next, consider the $\mathbf{e}_\zeta$ projection of the $\boldsymbol{\nabla}\cdot mn\mathbf{V}\mathbf{V}$ term in (2). Employing a vector, tensor identity for a general tensor T, it can be written in the form:

$$\mathbf{e}_\zeta\cdot\boldsymbol{\nabla}\cdot\mathsf{T} = \boldsymbol{\nabla}\cdot\,(\mathbf{e}_\zeta\cdot\mathsf{T}) - \boldsymbol{\nabla}\mathbf{e}_\zeta\,\boldsymbol{:}\,\mathsf{T}. \quad (52)$$

Using a cylindrical coordinate system R,ζ,Z with $R=0$ being the ζ symmetry axis, since $\boldsymbol{\nabla}R=\hat{\mathbf{e}}_R$ and $\boldsymbol{\nabla}\hat{\mathbf{e}}_\zeta = \hat{\mathbf{e}}_\zeta\partial\hat{\mathbf{e}}_\zeta/\partial\zeta = -\,\hat{\mathbf{e}}_\zeta\,(1/R)\hat{\mathbf{e}}_R$, one can show that

$$\boldsymbol{\nabla}\mathbf{e}_\zeta = \boldsymbol{\nabla}(R\,\hat{\mathbf{e}}_\zeta) = \hat{\mathbf{e}}_R\hat{\mathbf{e}}_\zeta - \hat{\mathbf{e}}_\zeta\hat{\mathbf{e}}_R. \quad (53)$$

Since this is an anti-symmetric tensor, its contraction with any symmetric tensor T_S vanishes: $\boldsymbol{\nabla}\mathbf{e}_\zeta\,\boldsymbol{:}\,\mathsf{T}_S=0$. Hence, for any symmetric tensor T_S, (52) simplifies to

$$\mathbf{e}_\zeta\cdot\boldsymbol{\nabla}\cdot\mathsf{T}_S = \boldsymbol{\nabla}\cdot\,(\mathbf{e}_\zeta\cdot\mathsf{T}_S). \quad (54)$$

Therefore, since $mn\mathbf{V}\mathbf{V}$ is a symmetric tensor, the toroidal component of the terms on the left side of the momentum equation (2) can be written as

$$\begin{aligned}&\mathbf{e}_\zeta\cdot[\partial/\partial t|_\mathbf{x}\,(mn\mathbf{V}) + \boldsymbol{\nabla}\cdot(mn\,\mathbf{V}\mathbf{V})]\\ &= \partial/\partial t|_\mathbf{x}\,[mn(\mathbf{e}_\zeta\cdot\mathbf{V})] + \boldsymbol{\nabla}\cdot[mn\,(\mathbf{e}_\zeta\cdot\mathbf{V})\mathbf{V}]. \quad (55)\end{aligned}$$

Thus, the ζ average of the toroidal ($\mathbf{e}_\zeta\,\cdot$) component of the momentum equation (2) for each species with all quantities expanded as in (8) can be written (in laboratory coordinates $\mathbf{x}$) to lowest order as:

$$\begin{aligned}&\partial/\partial t|_\mathbf{x}\,[mn_0(\mathbf{e}_\zeta\cdot\bar{\mathbf{V}}_1)] + \boldsymbol{\nabla}\cdot\overline{mn(\mathbf{e}_\zeta\cdot\mathbf{V}_1)\mathbf{V}_1}\\ &= n_0q(\mathbf{e}_\zeta\cdot\bar{\mathbf{E}}^A + \bar{\mathbf{V}}_2\cdot\boldsymbol{\nabla}\psi_p) + q\,\overline{\tilde{n}_1\tilde{\mathbf{V}}_1}\cdot\boldsymbol{\nabla}\psi_p\\ &+n_0q\,\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{V}}_1\times\tilde{\mathbf{B}}} - \mathbf{e}_\zeta\cdot\overline{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}} + \mathbf{e}_\zeta\cdot\bar{\mathbf{R}} + \mathbf{e}_\zeta\cdot\bar{\mathbf{S}}_m. \quad (56)\end{aligned}$$

After taking into account the requirement for ambipolarity of net radial particle fluxes, the flux surface average of (56) will produce an equation for the toroidal angular momentum density $mn_0\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{V}}_1\rangle$ and hence Ω_t.

A. Transient magnetic field effects

The poloidal and toroidal magnetic fields in tokamaks evolve during plasma startup and the approach to steady-state on the slow magnetic "skin" diffusion time scale. Also, non-inductive current drive and other poloidal flux transient effects, particularly radially localized ones like electron cyclotron heating (ECH) and current drive (ECCD), can cause transient poloidal field effects on transport time scales of the plasma thermodynamic parameters n, T, $\mathbf{e}_\zeta\cdot\bar{\mathbf{V}}_1$. Thus, we need to allow for transient magnetic flux effects in analyzing (56).

To do so, we need to determine our density and toroidal flow equations on poloidal flux surfaces ψ_p, upon which drift kinetics and gyrokinetics are based — because Grad-Shafranov equilibria determine the poloidal flux surfaces, and neoclassical transport [22] and microturbulence-induced anomalous transport are determined relative to them. Magnetic field transient effects were not considered above because they are $\mathcal{O}\{\delta^2\}$; hence they were negligible in obtaining the $\mathcal{O}\{\delta^0\}$ radial force balance, and $\mathcal{O}\{\delta\}$ parallel force balance and flows within flux surfaces.

Equations for the slow, transport-time-scale evolution of the poloidal ψ_p and toroidal ψ_t magnetic fluxes can be obtained from Faraday's law. Representing the magnetic field by $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, Faraday's law can be written as $\boldsymbol{\nabla}\times(\partial\mathbf{A}/\partial t|_\mathbf{x} - \boldsymbol{\nabla}\phi + \mathbf{E}^A) = 0$. Using the vector potential $\mathbf{A}=\psi_t\boldsymbol{\nabla}\theta - \psi_p\boldsymbol{\nabla}\zeta$ for the $\mathbf{B}_0$ field given in (3), the temporal evolution of the toroidal and poloidal magnetic fluxes at a given laboratory position $\mathbf{x}$ are

$$\left.\frac{\partial\psi_t}{\partial t}\right|_\mathbf{x} = -\,\mathbf{e}_\theta\cdot\mathbf{E}^A = -\frac{qR^2\,\mathbf{B}_p\cdot\mathbf{E}^A}{I}, \quad (57)$$

$$\left.\frac{\partial\psi_p}{\partial t}\right|_\mathbf{x} = +\,\mathbf{e}_\zeta\cdot\mathbf{E}^A - \frac{\partial\Psi}{\partial t} = R^2\boldsymbol{\nabla}\zeta\cdot\mathbf{E}^A - \frac{\partial\Psi}{\partial t}. \quad (58)$$

Here, $\partial\Psi/\partial t \equiv V^\zeta_{\rm loop}(t)/2\pi$ is a spatial constant (gauge variable) that is usually positive [see (66)–(68) below] to drive current in the plasma. It represents the toroidal loop voltage induced in vacuum by the rate of change of magnetic flux in the central solenoid (ohmic transformer) of a tokamak. In tokamak modeling codes that include flux surface evolution [23–25] it is usually taken to be zero; then the tranformer-induced flux change is imposed as a boundary condition on the plasma domain and causes the poloidal flux to increase linearly with time for an ohmically heated tokomak plasma in steady state. The $\partial\Psi/\partial t \neq 0$ term is introduced here so that in near steady-state conditions the poloidal flux function $\psi_p(\rho,t)$ represents poloidal flux changes within the plasma and does not increase approximately linearly with time t.

Multiplying these equations by $1/R^2$, taking their ζ averages and flux-surface-averaging them yields

$$\left.\frac{\partial\psi_t}{\partial t}\right|_\mathbf{x} = -\frac{q\langle\mathbf{B}_p\cdot\bar{\mathbf{E}}^A\rangle}{I\langle R^{-2}\rangle} \equiv -\,\bar{u}_G\frac{\partial\psi_t}{\partial\rho} \equiv \dot{\psi}_t, \quad (59)$$

$$\left.\frac{\partial\psi_p}{\partial t}\right|_\mathbf{x} = \frac{\langle\boldsymbol{\nabla}\zeta\cdot\bar{\mathbf{E}}^A\rangle}{\langle R^{-2}\rangle} - \frac{\partial\Psi}{\partial t}. \quad (60)$$

The small radial "grid" speed $\bar{u}_G$ of toroidal flux surfaces relative to laboratory coordinates $(\mathbf{x})$ defined in (59) is [23]

$$\bar{u}_G \equiv \langle \mathbf{u}_G \cdot \boldsymbol{\nabla}\rho \rangle = \frac{\langle \mathbf{B}_p \cdot \bar{\mathbf{E}}^A \rangle}{\psi_p' I \langle R^{-2} \rangle}. \tag{61}$$

Since our radial coordinate ρ is based on the toroidal flux surfaces, this grid speed represents the radial speed of our coordinate system relative to fixed laboratory coordinates $(\mathbf{x})$. It is caused by inductive poloidal electric fields produced by the poloidal magnetic field system as it moves the tokamak plasma radially or changes its cross-section (e.g., ellipticity, triangularity) in the $\zeta =$ constant plane. Note also that on a flux surface the time rate of change of the toroidal flux surface differential volume $V' \equiv dV/d\rho$ is given by [23, 25] the compressibility of the toroidal flux surfaces:

$$\frac{1}{V'} \frac{\partial V'}{\partial t}\bigg|_{\rho} = \langle \boldsymbol{\nabla} \cdot \mathbf{u}_G \rangle = \frac{1}{V'} \frac{\partial}{\partial \rho} (V' \langle \mathbf{u}_G \cdot \boldsymbol{\nabla}\rho \rangle). \tag{62}$$

Using the relation $\mathbf{B}_p \equiv \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\psi = \mathbf{B}_0 - I \boldsymbol{\nabla}\zeta$, (61) can be rearranged to yield for the toroidal electric field

$$\langle \boldsymbol{\nabla}\zeta \cdot \bar{\mathbf{E}}^A \rangle / \langle R^{-2} \rangle = \langle \mathbf{B}_0 \cdot \bar{\mathbf{E}}^A \rangle / (I \langle R^{-2} \rangle) - \bar{u}_G \psi_p'. \tag{63}$$

The parallel electric field $\langle \mathbf{B}_0 \cdot \bar{\mathbf{E}}^A \rangle$ in the plasma is obtained from the parallel Ohm's law in (41). The parallel plasma current can be written in terms of ψ_p as [5, 29]

$$\mu_0 \langle B_0 J_\parallel \rangle = I \langle R^{-2} \rangle \, \Delta^+ \psi_p, \tag{64}$$

in which the second order cylindrical-type operator is

$$\Delta^+ \psi_p \equiv \frac{I}{\langle R^{-2} \rangle V'} \frac{\partial}{\partial \rho} \left[\left\langle \frac{|\boldsymbol{\nabla}\rho|^2}{R^2} \right\rangle \frac{V'}{I} \frac{\partial \psi_p}{\partial \rho} \right] \simeq \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial \psi_p}{\partial \rho}. \tag{65}$$

Substituting (63) into (60), and using the neoclassical parallel Ohm's law in (41) and the definition of $\langle B_0 J_\parallel \rangle$ in (64) yields a diffusion equation for the poloidal flux [29]:

$$\frac{\partial \psi_p}{\partial t}\bigg|_{\mathbf{x}} = D_\eta \, \Delta^+ \psi_p - S_\psi - \bar{u}_G \psi_p'. \tag{66}$$

Here, the magnetic field diffusivity D_η and parallel "current-drive sources" S_ψ of poloidal flux are [29, 39]

$$D_\eta \equiv \frac{\eta_\parallel^{\rm nc}}{\mu_0}, \tag{67}$$

$$S_\psi = \frac{\partial \Psi}{\partial t} + \frac{1/\sigma_\parallel}{I \langle R^{-2} \rangle} [\langle \mathbf{B}_0 \cdot (\mathbf{J}_{\rm bs} + \mathbf{J}_{\rm CD} + \mathbf{J}_{\rm dyn}) \rangle]. \tag{68}$$

As indicated in (59), the toroidal flux moves only slightly relative to laboratory coordinates — due to the usually small grid speed $\bar{u}_G$. In contrast, the poloidal flux is much more mobile; it moves radially in response to non-ideal MHD effects from the parallel electric field $\langle \mathbf{B}_0 \cdot \bar{\mathbf{E}}^A \rangle$ caused by changes in $\eta_\parallel^{\rm nc}$ and $\langle B_0 J_\parallel \rangle$, the bootstrap current, and non-inductive and dynamo current drives. Thus, tokamak modeling codes [23, 25] usually use a magnetic flux coordinate system based on the toroidal flux in the plasma region, as we have in this paper. Taking account of (59), the temporal change of the poloidal flux on a toroidal flux surface is given by

$$\dot{\psi}_p \equiv \frac{\partial \psi_p}{\partial t}\bigg|_{\psi_t} = D_\eta \, \Delta^+ \psi_p - S_\psi. \tag{69}$$

The density and momentum equations in (1) and (2) are written in terms of laboratory coordinates $(\mathbf{x})$. However, drift-kinetic and gyrokinetic analyses used for obtaining collision- and fluctuation-induced transport fluxes are performed [22] in terms of poloidal magnetic flux coordinates ψ_p, θ, ζ. In particular, they are performed on poloidal magnetic flux surfaces — so the guiding center canonical toroidal angular momentum $p_{\zeta g} \equiv m v_\parallel I/B - q\psi_p$ naturally emerges as a constant of motion.

Transforming the drift-kinetic equation and hence its fluid moments in (1), (2) from laboratory to ψ_p, θ, ζ coordinates requires taking account of the fact that the equation for the poloidal magnetic flux ψ_p in (66) includes possible transient and grid motion effects, and poloidal flux diffusion. A transformation procedure based on the mathematical characteristics of the drift-kinetic equation on poloidal flux surfaces has been developed [36]; also, a simplified model illustrating its key physics has been presented [37]. In response to a Comment [38] on the derivation in [36], a more systematic derivation based on an analysis of the second order radial guiding center motion was published [39]. The net result [36, 37, 39] of this coordinate transformation is that a "paleoclassical" diffusion-type operator $\mathcal{D}\{f\}$ is added to the right side of the drift-kinetic equation, which then becomes

$$\frac{\partial f}{\partial t}\bigg|_{\psi_p} + (\mathbf{v}_\parallel + \mathbf{v}_d) \cdot \boldsymbol{\nabla} f + \dot{\mathcal{E}} \frac{\partial f}{\partial \mathcal{E}} = \mathcal{C}\{f\} + \mathcal{D}\{f\}. \tag{70}$$

Here, in general we have $f = f(\psi_p, \theta, \zeta, \mu, \mathcal{E}, t)$. For a lowest order distribution function f that is constant on a flux surface, [i.e., $\bar{f}(\rho) \equiv f(\psi_p, \mu, \mathcal{E}, t)$], the FSA of the paleoclassical operator can be written using (7) as [39]

$$\langle \mathcal{D}\{\bar{f}(\rho)\} \rangle \equiv -\dot{\rho}_{\psi_p} \frac{\partial \bar{f}}{\partial \rho} + \langle \boldsymbol{\nabla} \cdot \bar{f} \mathbf{u}_G \rangle + \frac{1}{V'} \frac{\partial^2}{\partial \rho^2} (V' \bar{D}_\eta \bar{f}). \tag{71}$$

Here, $\dot{\rho}_{\psi_p}$ indicates motion in the toroidal-flux-based ρ coordinate as the radial position of a given ψ_p surface changes during poloidal flux transients within the tokamak plasma. Since remaining on a given ψ_p surface during transients requires $d\psi_p = 0 = d\rho \, \partial \psi_p / \partial \rho + dt \, \partial \psi_p / \partial t$, the ρ motion to remain on the same ψ_t surface is [36, 37, 39]

$$\dot{\rho}_{\psi_p} \equiv \dot{\psi}_p / \psi_p'. \tag{72}$$

The second term in (71) indicates grid (coordinate) speed effects as indicated in (62). The final term represents paleoclassical transport effects due to the guiding centers of charged particles being carried along with the diffusing poloidal magnetic flux [36, 37, 39]. The normalized magnetic diffusivity in (71) has units of s^{-1} [29]:

$$\bar{D}_\eta \equiv \frac{D_\eta}{\bar{a}^2}, \quad \frac{1}{\bar{a}^2} \equiv \frac{1}{\langle R^{-2}\rangle}\left\langle \frac{|\boldsymbol{\nabla}\rho|^2}{R^2}\right\rangle. \tag{73}$$

Here, $\bar{a}$ is an effective minor radius of the ρ flux surface.

The scaling of the operator $\mathcal{D}\{f\}$ in (71) is determined from the scaling of its diffusion coefficient, which is the magnetic field diffusivity from (67): $D_\eta \equiv \eta_\parallel^{\rm nc}/\mu_0 \sim \nu_e \delta_e^2$ in which $\delta_e \equiv c/\omega_p$ is the electromagnetic skin depth. Since δ_e is of order the ion gyroradius, the terms $\mathcal{D}\{f\}$ introduces in the drift-kinetic equation are $\mathcal{O}\{\delta^2\}$; they have been negligible up to now because they were not needed in the $\mathcal{O}\{\delta^0, \delta^1\}$ analyses in Section III.

B. Density conservation equation

Before solving the ζ-average toroidal momentum equation (56) for the collisional radial particle flux $n_0\bar{\mathbf{V}}_2\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_p$, we discuss the net particle flux that arises in the density conservation equation. The relevant density conservation (continuity) equation can be obtained from the flux-surface-average (FSA) of the $\int d^3v$ moment of the drift-kinetic equation in (70). First, we note that [25] for a lowest order Maxwellian distribution function $\bar{f}_M(\psi_p)$ that is constant on a flux surface, $\langle \int d^3v\, \partial \bar{f}_M/\partial t|_{\psi_p}\rangle = \langle \partial n/\partial t|_{\psi_p}\rangle = (1/V')(\partial/\partial t|_{\psi_p}(V' n_0)$. Also, the FSA of the $\mathcal{D}\{f\}$ contribution to the right side of the density equation is $\langle \int d^3v\,\mathcal{D}\{\bar{f}_M\}\rangle = \langle \mathcal{D}\{n_0\}\rangle$. Thus, taking the $\int d^3v$ moment of the drift kinetic equation in (70), averaging over toroidal angle ζ and taking its flux surface average using (7), we find that in terms of the toroidal-flux-surface-based coordinate ρ the $\mathcal{O}\{\delta^2\}$ density conservation equation on a poloidal flux surface ψ_p is

$$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_p}(V' n_0) + \dot{\rho}_\psi \frac{\partial n_0}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\,\Gamma) = \langle \bar{S}_n\rangle. \tag{74}$$

This density equation can also be obtained by taking the FSA of the ζ average of (1) with all quantities expanded as in (8) and adding $\langle \mathcal{D}\{n_0\}\rangle$ to its right side to take account of its transformation from laboratory ($\mathbf{x}$) to poloidal flux coordinates. The quantity $dN \equiv (V' n_0)\, d\rho$ is the number of particles contained between the ρ and $\rho + d\rho$ flux surfaces. The quantity $V' n_0$ is an "adiabatic" plasma property [5, 6, 23–25] that is conserved in the absence of particle fluxes and density sources, including those induced by poloidal flux changes induced by $\dot{\rho}_{\psi_p} \propto \dot{\psi}_p \neq 0$.

For each plasma species the particle flux in (74) is

$$\Gamma \equiv \Gamma_\nu^a + \Gamma_{\rm an}^{na} + \Gamma_{\rm pc}^a. \tag{75}$$

It includes: the "direct," second order ambipolar (superscript a) collision-induced flux,

$$\Gamma_\nu^a \equiv n_0\,\langle (\bar{\mathbf{V}}_2 - \mathbf{u}_G)\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle; \tag{76}$$

possibly non-ambipolar (superscript na) "anomalous" (subscript an) fluxes induced by correlations of first order density fluctuations and fluid flows,

$$\Gamma_{\rm an}^{na} \equiv \langle\, \overline{\tilde{n}_1 \tilde{\mathbf{V}}_\wedge} \boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle; \tag{77}$$

and the ambipolar paleoclassical particle flux given by

$$\Gamma_{\rm pc}^a = -\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\bar{D}_\eta n_0) = -\,\bar{D}_\eta \frac{\partial n_0}{\partial\rho} - n_0 V_{\rm pc}. \tag{78}$$

Here, the paleoclassical particle pinch speed (s^{-1}) is

$$V_{\rm pc} = \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\bar{D}_\eta). \tag{79}$$

The grid velocity $\mathbf{u}_G$ has been incorporated into the definition of the collisional part of Γ_ν^a in (76) via $\langle \bar{\mathbf{V}}_2\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle \to \langle (\bar{\mathbf{V}}_2 - \mathbf{u}_G)\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle$ because while [5, 6, 22, 23] collisional (classical and neoclassical) transport theory determines particle fluxes across poloidal flux surfaces it uses a toroidal flux coordinate system ($\rho \sim \sqrt{\psi_t}$) that moves with grid velocity $\mathbf{u}_G$.

All the contributions to the particle flux in (75) are $\mathrm{O}\{\delta^2\}$ or smaller in the gyroradius expansion. The units of Γ are $\mathrm{m}^{-3}\ \mathrm{s}^{-1}$ (because ρ is dimensionless and $|\boldsymbol{\nabla}\rho| \sim 1/a \sim \mathrm{m}^{-1}$). Note also that the particle fluxes are defined relative to $\boldsymbol{\nabla}\rho$ here rather than to $\boldsymbol{\nabla}\psi_p$ as was used in [2]; thus, the $\Gamma^\psi \equiv \langle \boldsymbol{\Gamma}\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_p\rangle \equiv \langle \boldsymbol{\Gamma}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle \psi_p' = \Gamma\psi_p'$ in [2] is ψ_p' times the Γ defined here.

The toroidal force balance equation (56) also needs to be transformed from laboratory to poloidal flux coordinates. Analogous to the effects in the density equation [see discussion between Eqs. (69) and (74) above], the transformation adds a term $\mathbf{e}_\zeta\boldsymbol{\cdot}\mathcal{D}\{\int d^3v\, m\mathbf{v}f\} = \mathbf{e}_\zeta\boldsymbol{\cdot}\mathcal{D}\{mn_0\bar{\mathbf{V}}_1\}$ to the right side of (56). Then, since the unit vector $\hat{\mathbf{e}}_\zeta \equiv \boldsymbol{\nabla}\zeta/|\boldsymbol{\nabla}\zeta| = R\boldsymbol{\nabla}\zeta$ does not change with ρ and hence $\partial\hat{\mathbf{e}}_\zeta/\partial\rho = 0$, we find that the flux surface average of this contribution becomes

$$\begin{aligned}\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\mathcal{D}\{mn_0\bar{\mathbf{V}}_1\}\rangle \;\simeq\; & -\dot{\rho}_{\psi_p}\frac{\partial}{\partial\rho}[mn_0\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\bar{\mathbf{V}}_1\rangle] \\ & + \langle \boldsymbol{\nabla}\boldsymbol{\cdot}[mn_0(\mathbf{e}_\zeta\boldsymbol{\cdot}\bar{\mathbf{V}}_1)\mathbf{u}_G]\rangle \\ & + \frac{1}{V'}\frac{\partial^2}{\partial\rho^2}[V'\bar{D}_\eta mn_0\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\bar{\mathbf{V}}_1\rangle].\end{aligned} \tag{80}$$

Here, the approximate equality indicates that $\mathcal{O}\{\epsilon^2\}$ inverse aspect ratio terms have been neglected in bringing the R factor in $\mathbf{e}_\zeta \equiv R\,\hat{\mathbf{e}}_\zeta$ through the ρ derivatives.

Thus, transforming (56) to poloidal flux coordinates, taking its FSA, and solving it for the second order radial particle fluxes for each species yields

$$\begin{aligned}\psi_p'(\Gamma_\nu^a+\Gamma_{\rm an}^{na}) \;&\equiv\; \langle n_0(\bar{\mathbf{V}}_2-\mathbf{u}_G)\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_p\rangle + \langle\, \overline{\tilde{n}_1\tilde{\mathbf{V}}_\wedge}\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_p\rangle \\ &=\; -\,\frac{1}{q}\,\left[\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{R}}\rangle + n_0 q\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{E}}^A\rangle - \langle\mathbf{e}_\zeta\boldsymbol{\cdot}\overline{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}}\rangle\right] \\ &\quad-\;\frac{1}{q}\,\langle\mathbf{e}_\zeta\cdot\mathcal{D}\{mn_0\bar{\mathbf{V}}_1\}\rangle \\ &\quad+\;\frac{1}{qV'}\frac{\partial}{\partial t}\bigg|_{\psi_p}\big(V'mn_0\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\bar{\mathbf{V}}_1\rangle\big) \\ &\quad+\;\frac{1}{qV'}\,\frac{\partial}{\partial\rho}\left(V'mn_0\langle\,\overline{(\mathbf{e}_\zeta\boldsymbol{\cdot}\tilde{\mathbf{V}}_1)(\tilde{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\rho)}\,\rangle\right) \\ &\quad-\;\langle\mathbf{e}_\zeta\cdot n_0\overline{\tilde{\mathbf{V}}_1\times\tilde{\mathbf{B}}}\rangle \\ &\quad-\;\frac{1}{q}\,\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{S}}_m\rangle. \end{aligned} \tag{81}$$

Here, (7) has been used to obtain the fluctuation-induced contribution to the particle flux on the fourth line. Successive lines in (81) indicate that the radial particle flux has many $\mathcal{O}\{\delta^2\}$ components: collision-induced fluxes, collision-induced paleoclassical processes, polarization flux due to time-dependent toroidal flows, microturbulence-induced Reynolds stresses, effects due to the Maxwell stress $\overline{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}$ induced by fluctuations and non-axisymmetric magnetic field components, and toroidal momentum sources.

C. Ambipolar collisional particle fluxes

Consider first particle fluxes induced by the collisional friction force $\bar{\mathbf{R}}_e \simeq n_e e(\mathbf{J}_\parallel/\sigma_\parallel + \mathbf{J}_\wedge/\sigma_\perp) = -\bar{\mathbf{R}}_i$ (neglect of poloidal heat flows changes some numerical coefficients but does not affect the qualitative results or scalings):

$$\psi_p'\Gamma_\nu^a \equiv -\,\frac{1}{q_s}\,\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{R}}_s\rangle - n_0\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{E}}^A\rangle. \tag{82}$$

A vector identity relating the toroidal, parallel and poloidal directions is useful for analyzing $\langle\mathbf{e}_\zeta\cdot\bar{\mathbf{R}}_s\rangle$:

$$\mathbf{e}_\zeta \equiv R^2\boldsymbol{\nabla}\zeta = \frac{I\,\mathbf{B}_0}{B_0^2} - \frac{\mathbf{B}_0\times\boldsymbol{\nabla}\psi_p}{B_0^2}. \tag{83}$$

The collisional-friction-induced toroidal torque is thus

$$\begin{aligned}\frac{1}{q_s}\,\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\bar{\mathbf{R}}\rangle \;&=\; \frac{I}{q_s}\left\langle\frac{\mathbf{B}_0\boldsymbol{\cdot}\bar{\mathbf{R}}}{B_0^2}\right\rangle - \frac{1}{q_s}\left\langle\frac{\mathbf{B}_0\times\boldsymbol{\nabla}\psi_p}{B_0^2}\cdot\bar{\mathbf{R}}\right\rangle \\ &=\; -\,\frac{n_{e0}I}{\sigma_\parallel}\left\langle\frac{B_0J_\parallel}{B_0^2}\right\rangle + \frac{n_{e0}}{\sigma_\perp}\left\langle\frac{|\boldsymbol{\nabla}\psi_p|^2}{B_0^2}\right\rangle\frac{dP_0}{d\psi_p}.\end{aligned} \tag{84}$$

The FSA $J_\parallel$ term can be worked out using (29) and (36):

$$\begin{aligned}\left\langle\frac{B_0J_\parallel}{B_0^2}\right\rangle &= \frac{\langle B_0J_\parallel\rangle}{\langle B_0^2\rangle} + \left\langle J_\parallel B_0\left(\frac{1}{B_0^2} - \frac{1}{\langle B_0^2\rangle}\right)\right\rangle \\ &= \frac{\sigma_\parallel\langle\mathbf{B}_0\boldsymbol{\cdot}\bar{\mathbf{E}}^A\rangle + (\sigma_\parallel/n_{e0}e)\langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{e\parallel}\rangle}{\langle B_0^2\rangle} \\ &\quad+\frac{\langle\mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm CD}\rangle + \langle\mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm dyn}\rangle}{\langle B_0^2\rangle} \\ &\quad- I\,\frac{dP_0}{d\psi_p}\left\langle\frac{1}{B_0^2}\left(1-\frac{B_0^2}{\langle B_0^2\rangle}\right)^2\right\rangle.\end{aligned} \tag{85}$$

Next, we use this result in (84) to obtain the total collision-induced particle flux from (75), which has 7 contributions: classical (cl), Pfirsch-Schlüter (PS), banana-plateau [6] (bp) collisionality regime and paleoclassical (pc) fluxes plus the radial particle fluxes due to the non-inductive and dynamo parallel currents $\langle\mathbf{B}_0\cdot\mathbf{J}_{\rm CD}\rangle$ and $\langle\mathbf{B}_0\cdot\mathbf{J}_{\rm dyn}\rangle$, and the $\bar{\mathbf{E}}_A\times\mathbf{B}_0/B_0^2$ pinch effect [5],

$$\Gamma_\nu^a+\Gamma_{\rm pc}^a = \Gamma_{\rm cl}+\Gamma_{\rm PS}+\Gamma_{\rm bp}+\Gamma_{\rm pc}+\Gamma_{\rm CD}+\Gamma_{\rm dyn}+\Gamma_{E^A}. \tag{86}$$

Here, for each species we have defined

$$\Gamma_{\rm cl} \;\equiv\; -\,\frac{n_{e0}}{\sigma_\perp}\left\langle\frac{|\boldsymbol{\nabla}\rho|^2}{B_0^2}\right\rangle\frac{dP_0}{d\rho}, \tag{87}$$

$$\Gamma_{\rm PS} \;\equiv\; -\,\frac{n_{e0}I^2}{\sigma_\parallel\psi_p'^2}\left\langle\frac{1}{B_0^2}\left(1-\frac{B_0^2}{\langle B_0^2\rangle}\right)^2\right\rangle\frac{dP_0}{d\rho}, \tag{88}$$

$$\Gamma_{\rm bp} \;\equiv\; \frac{I}{e\langle B_0^2\rangle\psi_p'}\,\langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{e\parallel}\rangle, \tag{89}$$

$$\Gamma_{\rm pc} \;\equiv\; -\,\bar{D}_\eta\frac{dn_{e0}}{d\rho} - n_{e0}V_{\rm pc}, \tag{90}$$

$$\Gamma_{\rm CD} \;\equiv\; \frac{n_{e0}I}{\sigma_\parallel\psi_p'\langle B_0^2\rangle}\,\langle\mathbf{B}_0\cdot\mathbf{J}_{\rm CD}\rangle, \tag{91}$$

$$\Gamma_{\rm dyn} \;\equiv\; \frac{n_{e0}I}{\sigma_\parallel\psi_p'\langle B_0^2\rangle}\,\langle\mathbf{B}_0\cdot\mathbf{J}_{\rm dyn}\rangle, \tag{92}$$

$$\Gamma_{\rm E^A} \;\equiv -\frac{n_{e0}}{\psi_p'}\left[\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\bar{\mathbf{E}}^A\rangle\left(1-\frac{I^2\langle R^{-2}\rangle}{\langle B_0^2\rangle}\right) - \frac{I\langle\mathbf{B}_p\boldsymbol{\cdot}\bar{\mathbf{E}}^A\rangle}{\langle B_0^2\rangle}\right]. \tag{93}$$

Note that all 7 of these particle fluxes are the same for electrons and ions [5, 6, 40]. Thus, they are ambipolar (superscript a) and cause no FSA radial plasma current:

$$\langle\mathbf{J}^a\boldsymbol{\cdot}\,\boldsymbol{\nabla}\rho\rangle \equiv \sum_s q_s\,(\Gamma_{s\nu}^a + \Gamma_{s\rm pc}^a) = 0. \tag{94}$$

For a Fick's law particle flux form $\Gamma = -\langle D\boldsymbol{\nabla}n\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle = -D\langle|\boldsymbol{\nabla}\rho|^2\rangle(dn_0/d\rho)$, the collision-induced diffusion coefficients are of order $D_{\rm cl}\sim\nu_e\varrho_e^2$, $D_{\rm PS}\sim(q^2\sigma_\perp/\sigma_\parallel)D_{\rm cl}$, $D_{\rm bp}\sim\mu_{e00}\varrho_e^2B_t^2/B_p^2\sim(q^2/\epsilon^{3/2})D_{\rm cl}$, and $D_{pc}\equiv\eta_\parallel^{\rm nc}/\mu_0\sim D_{\rm cl}/\beta_e$, which is the largest D for the usual situation where $\beta_e\equiv p_e/(B_0^2/2\mu_0)\ll\epsilon^{3/2}/q^2\ll 1$. In addition there are non-diffusive radial fluxes due to the Ware pinch [5, 6] [$\propto\langle\mathbf{B}_0\cdot\bar{\mathbf{E}}^A\rangle$, from $\Gamma_{\rm bp}$ via the parallel viscous force in (37) using (41)], paleoclassical pinch flow (79), outward flows due to non-inductive co-current-drive and dynamo effects, and $\bar{\mathbf{E}}^A\times\mathbf{B}_0/B_0^2$ inward pinch [5].

D. Possibly non-ambipolar particle fluxes

Next, consider the particle flux induced by the toroidal ($\mathbf{e}_\zeta\,\boldsymbol{\cdot}$) component of the viscous force density $-\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}$:

$$\psi_p'\Gamma_\pi \equiv (1/q_s)\,\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}\rangle. \qquad (95)$$

The viscous stress tensor $\bar{\boldsymbol{\pi}}$ was described in (34) above. The parallel stresses, which are in principle the largest, can be written in the Chew-Goldberger-Low (CGL) form:

$$\boldsymbol{\pi}_\parallel = (p_\parallel - p_\perp)(\mathbf{BB}/B^2 - \mathbf{I}/3). \qquad (96)$$

They can be split into their axisymmetric (superscript A) and non-axisymmetric (superscript NA) parts, which when averaged over the fluctuations can be written as

$$\bar{\boldsymbol{\pi}}_\parallel = \bar{\boldsymbol{\pi}}_\parallel^{\rm A} + \bar{\boldsymbol{\pi}}_\parallel^{\rm NA}. \qquad (97)$$

Using equilibrium, average quantities in the definition of $\boldsymbol{\pi}_\parallel$ in (96) and $\mathbf{J}_0\times\mathbf{B}_0 = (\boldsymbol{\nabla}\times\mathbf{B}_0)\times\mathbf{B}_0/\mu_0 = \boldsymbol{\nabla}P_0$ yields

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_\parallel^{\rm A} = \mathbf{B}_0\,(\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\left(\frac{\bar{p}_\parallel - \bar{p}_\perp}{B_0^2}\right) + \frac{\bar{p}_\parallel - \bar{p}_\perp}{B_0^2}\,\boldsymbol{\nabla}\left(\frac{B_0^2}{2} + \mu_0 P_0\right) - \frac{1}{3}\boldsymbol{\nabla}(\bar{p}_\parallel - \bar{p}_\perp). \qquad (98)$$

Taking the toroidal angular ($\mathbf{e}_\zeta\,\boldsymbol{\cdot}$) component of this viscous force, the terms on the second line vanish because of axisymmetry ($\partial/\partial\zeta \to 0$) in the equilibrium. Thus, using (6), the flux-surface-average of the toroidal viscous torque in an axisymmetric magnetic field vanishes [5, 6]:

$$\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_\parallel^{\rm A}\rangle = I\left\langle (\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\left(\frac{\bar{p}_\parallel - \bar{p}_\perp}{B_0^2}\right)\right\rangle = 0. \qquad (99)$$

Alternatively, since $\bar{\boldsymbol{\pi}}_\parallel^{\rm A}$ is a symmetric tensor, one can use the vector, tensor identity in (54) and the FSA property in (7) to obtain the same result. Physically, this toroidal viscous force vanishes because there is no toroidal inhomogeneity in the equilibrium magnetic field strength to impede flow in the toroidal (axisymmetry) direction.

However, non-axisymmetric (NA, 3D) magnetic field components in (15) cause toroidal torques on toroidally flowing plasmas that are typically second order in $\tilde{B}_\parallel$ and hence in the gyroradius expansion. They are usually calculated [15–17] in the absence of fluctuation and momentum source effects from the relation $\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{s\parallel}^{\rm NA}\rangle = q_s\psi_p'\Gamma_{s\pi\parallel}^{\rm NA}$. The net ion torques, which are typically a factor of order $(m_i/m_e)^{1/2} \sim 60$ larger than the electron ones, can be written in the generic form [2, 14–17, 41]

$$\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{i\parallel}^{\rm NA}\rangle = m_i n_{i0}\mu_{i\,t}\frac{\tilde{B}_{\rm eff}^2}{B_0^2}[\langle R^2\Omega_t\rangle - \langle R^2\Omega_*\rangle]. \qquad (100)$$

Here, $\mu_{i\,t}$ (s^{-1}) is an effective ion viscous toroidal damping frequency due to TTMP [14, 15], banana-drift [16, 41] and ripple-trapping effects (see [17] and references cited therein). And $(\tilde{B}_{\rm eff}/B_0)^2$ is a weighted sum over all m, n components of $\tilde{B}_\parallel$ which represents the effective magnetic field non-axisymmetry (NA) through which the tokamak plasma flows toroidally.

The intrinsic (or "offset" [42, 43]) rotation frequency

$$\langle\Omega_*\rangle \equiv \langle R^2\Omega_*\rangle/\langle R^2\rangle \simeq (c_p + c_t)(1/q_i)(dT_{i0}/d\psi_p) \quad (101)$$

is a diamagnetic-type toroidal rotation frequency proportional to the radial ion temperature gradient. It is caused by ions of different energy drifting radially at different speeds. In most NA transport processes, ions on the tail of the ion distribution drift radially more rapidly than thermal ions; this causes $\Omega_* < 0$ for $dT_{i0}/d\psi_p < 0$. The poloidal coefficient c_p in (101) was discussed after (45); the toroidal coefficient c_t ranges from -0.67 to 2.4, depending on which non-axisymmetric process is involved [2, 41].

Next, we consider the cross and perpendicular stresses and their effects. The cross stress $\boldsymbol{\pi}_\wedge \sim \mathbf{B}_0\times\boldsymbol{\nabla}\mathbf{V}$ is a symmetric, diamagnetic-type stress tensor (see [30] and references cited therein) for which an approximate form is given in (B1). The gyroviscous force it induces is given in (B2). The flux-surface-average of its toroidal angular component vanishes for the equilibrium first order neoclassical flow velocity $\bar{\mathbf{V}}_1$ for up-down symmetric plasmas [7], which we will assume. (For up-down asymmetric plasmas it is very small unless there is "extreme asymmetry," perhaps near a divertor separatrix in the Pfirsch-Schlüter collisionality regime [7]). Also, for collisionless plasmas it vanishes [44]. Thus, for equilibrium flows we will assume

$$\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_\wedge\rangle = 0. \qquad (102)$$

However, for fluctuations it does not vanish. That is, $\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\,\overline{\boldsymbol{\pi}_\wedge}\rangle \neq 0$ — see Appendix B.

The ion perpendicular stresses cause both classical [3] and neoclassical [4, 7] radial transport of the plasma toroidal angular rotation Ω_t through their respective perpendicular viscous forces $\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{s\perp}^{\rm cl}\rangle$ and $\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{\pi}}_{s\perp}^{\rm nc}\rangle$. They cause diffusive radial transport of Ω_t with diffusion coefficients (classical [3]) $\chi_t^{\rm cl} \simeq 0.3\,\nu_i\varrho_i^2$ and (neoclassical [4]) $\chi_t^{\rm nc} \simeq 0.1\,q^2\nu_i\varrho_i^2$. It is important to note that the neoclassical toroidal angular momentum diffusivity $\chi_t^{\rm nc}$ is a factor of order $0.1\,\epsilon^{3/2}$ smaller than the ion neoclassical radial ion heat diffusivity [5] $\chi_i \sim (q^2/\epsilon^{3/2})\,\nu_i\varrho_i^2$. In addition, neoclassical radial transport of toroidal momentum includes a momentum pinch type effect [4, 7] that is comparable in magnitude to the diffusive radial transport of toroidal momentum. In most tokamak plasmas of fusion interest the classical and neoclassical toroidal angular momentum diffusivities χ_t are less than 0.1 m^2/s; hence the radial transport of Ω_t they induce is usually negligible. The combination of all these cross and perpendicular classical and neoclassical viscous stress effects are discussed in detail in [7].

Paleoclassical processes due to the $\langle \mathbf{e}_\zeta\boldsymbol{\cdot}\mathcal{D}\{mn_0\bar{\mathbf{V}}_1\}\rangle$ defined in (80) also produce radial transport of toroidal angular momentum. The D_η-induced part of electron and ion paleoclassical effects can be characterized as perpendicular viscous torques on the toroidal angular momentum density of each species. The ion paleoclassical viscous torque is larger than that for electrons by $m_i/m_e \sim 3672 \gg 1$. It is caused by the ion D_η-induced term in (80):

$$\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}^{\rm pc}_{i\perp} \rangle \simeq -\frac{1}{V'} \frac{\partial^2}{\partial \rho^2} \left[V' \bar{D}_\eta L_t \right] = -\frac{1}{V'} \frac{\partial}{\partial \rho} \left[V' \left(\bar{D}_\eta \frac{\partial L_t}{\partial \rho} + L_t V_{\rm pc} \right) \right], \quad (103)$$

in which the plasma toroidal angular momentum (torque) density is

$$L_t \equiv m_i n_{i0} \langle \mathbf{e}_\zeta \cdot \bar{\mathbf{V}}_i \rangle = m_i n_{i0} \langle R^2 \Omega_t \rangle. \quad (104)$$

The ion term $-\dot{\rho}_{\psi_p} (\partial/\partial\rho)\, m_i n_{i0} \langle \mathbf{e}_\zeta \cdot \bar{\mathbf{V}}_i \rangle$ in (80) leads to an additional toroidal torque density on the right side of (56) of $-\dot{\rho}_{\psi_p} (\partial L_t/\partial \rho)$. It is caused by poloidal magnetic field transients in the plasma that produce $\dot{\psi}_p \neq 0$ and is analogous to the $\dot{\rho}_{\psi_p} \partial n_0/\partial \rho$ term in (74). The $\langle \boldsymbol{\nabla} \cdot m_i n_{i0} (\mathbf{e}_\zeta \cdot \bar{\mathbf{V}}_i) \mathbf{u}_G \rangle$ ion term in (80) contributes, as in particle fluxes, to properly determining the neoclassical perpendicular viscosity discussed above when there is grid motion of the toroidal flux surfaces.

Paleoclassical radial momentum transport in (103) causes radial diffusion of L_ζ with a diffusivity $\chi_t^{\rm pc} \equiv D_\eta$, plus a momentum pinch $V_{\rm pc}$ that is the same as that for the density, i.e., (79). The paleoclassical momentum diffusivity is the same as the magnetic field diffusivity: $D_\eta \equiv \eta_\parallel^{\rm nc}/\mu_0 \sim 1400\, Z_{\rm eff}/T_e({\rm eV})^{3/2}$, which becomes less than about 0.1 m^2/s for $T_e \gtrsim 1$ keV. While it can be significant in ohmic-level tokamak plasmas and towards the plasma edge, it is usually negligible in the hot plasma core unless fluctuation-induced transport is suppressed.

Taking into account all of the classical, neoclassical and paleoclassical processes of radial transport of Ω_t, the total collision-induced perpendicular viscous torque is

$$\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}_{s\perp} \rangle \equiv \langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot (\bar{\boldsymbol{\pi}}^{\rm cl}_{s\perp} + \bar{\boldsymbol{\pi}}^{\rm nc}_{s\perp} + \bar{\boldsymbol{\pi}}^{\rm pc}_{s\perp}) \rangle \simeq \langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}^{\rm pc}_{s\perp} \rangle. \quad (105)$$

Combining this with the parallel viscous force due to 3D non-axisymmetric magnetic field components, the radial particle fluxes induced by $\|$, $\perp$ viscous forces become

$$\psi_p' \Gamma_{\pi\|} \equiv (1/q_s) \langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}^{\rm NA}_{s\|} \rangle, \quad (106)$$

$$\psi_p' \Gamma_{\pi\perp} \equiv (1/q_s) \langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}_{s\perp} \rangle. \quad (107)$$

The other particle fluxes in (81) are induced by polarization flows, perturbation-induced Reynolds and Maxwell stresses, poloidal magnetic flux transients and momentum sources:

$$\psi_p' \Gamma_{\rm pol} \equiv \frac{1}{q_s V'} \left.\frac{\partial}{\partial t}\right|_{\psi_p} \left(V' m_s n_{s0} \langle \mathbf{e}_\zeta \cdot \bar{\mathbf{V}}_{s1} \rangle \right), \quad (108)$$

$$\psi_p' \Gamma_{\rm Rey} \equiv \frac{1}{q_s V'} \frac{\partial}{\partial \rho} (V' \Pi_{s\rho\zeta}), \quad (109)$$

$$\psi_p' \Gamma_{\rm Max} \equiv -\langle \mathbf{e}_\zeta \cdot n_{s0} \overline{\tilde{\mathbf{V}}_1 \times \tilde{\mathbf{B}}} \rangle, \quad (110)$$

$$\psi_p' \Gamma_{\rm JxB} \equiv -\langle \mathbf{e}_\zeta \cdot n_{s0} \overline{\tilde{\mathbf{V}}^{\rm NA}_{\|mn} \times \tilde{\mathbf{B}}^{\rm NA}_{\perp mn}} \rangle, \quad (111)$$

$$\psi_p' \Gamma_{\dot{\psi}_p} \equiv \frac{\dot{\rho}_{\psi_p}}{q_s} \frac{\partial}{\partial \rho} (m n_0 \langle \mathbf{e}_\zeta \cdot \bar{\mathbf{V}}_1 \rangle), \quad (112)$$

$$\psi_p' \Gamma_{\rm S} \equiv -(1/q_s) \langle \mathbf{e}_\zeta \cdot \bar{\mathbf{S}}_{sm} \rangle. \quad (113)$$

The toroidal Reynolds stress $\Pi_{s\rho\zeta}$ has been considered in fluid models [9, 10], but is now usually obtained from gyrokinetic-based theories [11–13]. The definition of it that emerges from this fluid-based approach is

$$\Pi_{s\rho\zeta} \equiv m_s n_{s0} \langle \overline{(\boldsymbol{\nabla}\rho \cdot \tilde{\mathbf{V}}_{s\wedge})(\tilde{\mathbf{V}}_s \cdot \mathbf{e}_\zeta)} \rangle + \langle \boldsymbol{\nabla}\rho \cdot \overline{\boldsymbol{\pi}_\wedge} \cdot \mathbf{e}_\zeta \rangle. \quad (114)$$

The gyroviscous stress $\overline{\boldsymbol{\pi}_\wedge}$ in the last term is described in Appendix B and an approximate form of it is specified in (B4); this term provides a partial gyroviscous cancellation [see discussion before (B2)] of the first term in (114). Also, we have split the $\tilde{\mathbf{V}} \times \tilde{\mathbf{B}}$ contributions into a fluctuation-induced Maxwell stress part $\Gamma_{\rm Max}$ and a part $\Gamma_{\rm JxB}$ induced by externally-imposed non-axisymmetric (NA) resonant (at $q = m/n$) magnetic perturbations, which will be discussed after (120) below.

The sum of the rest of the particle fluxes in (81), in addition to those included in (86), for each species is thus

$$\Gamma^{na}_{\rm an} = \Gamma_{\pi\|} + \Gamma_{\pi\perp} + \Gamma_{\rm pol} + \Gamma_{\rm Rey} + \Gamma_{\rm Max} + \Gamma_{\rm JxB} + \Gamma_{\dot{\psi}_p} + \Gamma_S. \quad (115)$$

These 8 particle flux components are all, in principle, different for electrons and ions, and hence non-ambipolar (superscript na).

E. Radial current, toroidal rotation equation

Next, we consider the flux surface average of the plasma current continuity equation obtained by multiplying the FSA density equation (74) by the charge q_s and summing over plasma species ($\langle \rho_q \rangle \equiv \sum_s n_{s0} q_s$):

$$\frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_{\psi_p} (V' \langle \rho_q \rangle) + \dot{\rho}_{\psi_p} \frac{\partial \langle \rho_q \rangle}{\partial \rho} + \frac{1}{V'} \frac{\partial}{\partial \rho} (V' \langle \mathbf{J} \cdot \boldsymbol{\nabla} \rho \rangle) = 0. \quad (116)$$

Here, $V' \langle \mathbf{J} \cdot \boldsymbol{\nabla} \rho \rangle \equiv \sum_s q_s V' \Gamma_s$ is the net FSA plasma current (in Amperes) flowing radially across a magnetic flux surface. For simplicity, we have assumed the particle sources are ambipolar and hence that $\sum_s q_s \langle \bar{S}_{sn} \rangle = 0$ — because inside tokamak plasmas the electron and ion density sources S_{sn} are usually equal and hence ambipolar. Thus, using the FSA of Gauss' law ($\langle \epsilon_0 \boldsymbol{\nabla} \cdot \bar{\mathbf{E}}_0 \rangle = \langle \rho_q \rangle$)

and (7), this last equation becomes

$$\frac{1}{V'}\frac{\partial}{\partial\rho}\left[V'\left(\epsilon_0\frac{\partial}{\partial t}\bigg|_{\psi_p}\langle\bar{\mathbf{E}}_0\cdot\boldsymbol{\nabla}\rho\rangle+\langle\mathbf{J}\cdot\boldsymbol{\nabla}\rho\rangle\right)\right] + \dot{\rho}_{\psi_p}\frac{\partial}{\partial\rho}\left[\frac{\epsilon_0}{V'}\frac{\partial}{\partial\rho}(V'\langle\bar{\mathbf{E}}_0\cdot\boldsymbol{\nabla}\rho\rangle)\right] = 0. \qquad (117)$$

The radial electric field contribution here is $\langle\bar{\mathbf{E}}_0\cdot\boldsymbol{\nabla}\rho\rangle = \langle|\boldsymbol{\nabla}\rho|^2\rangle\partial\Phi_0/\partial\rho$. It is proportional to one of the key terms in the toroidal rotation Ω_t given in (47). Hence, the radial ion current due to the ion polarization particle flow $\Gamma_{i\mathrm{pol}}$ in (108) includes a term that is of the form $(\partial/\partial t)(m_i n_{i0}\langle R^2\rangle\partial\Phi_0/\partial\psi_p)$ in (117). This term is larger than the vacuum field term $\partial\langle\epsilon_0\bar{\mathbf{E}}_0\cdot\boldsymbol{\nabla}\rho\rangle/\partial t$ in (117) by a factor of $m_i n_{i0}\langle R^2\rangle/(\epsilon_0\psi_p'^2\langle|\boldsymbol{\nabla}\rho|^2\rangle)\simeq c^2/c_{Ap}^2\gg 1$, in which $c_{Ap}\equiv B_p/\sqrt{\mu_0 m_i n_{i0}}$ is the Alfvén speed in the poloidal magnetic field. The electric field term proportional to $\dot{\rho}_{\psi_p}$ is of a similar magnitude relative to this inertial electric field term from $\Gamma_{i\mathrm{pol}}$. Thus, since typically $c^2/c_{Ap}^2\sim 10^5\gg 1$, both of the $\langle\bar{\mathbf{E}}_0\cdot\boldsymbol{\nabla}\rho\rangle$ electric field terms in (117) are negligible compared to the neoclassical MHD polarizability [45] arising from the $(\partial/\partial t)(m_i n_{i0}\langle R^2\rangle\partial\Phi_0/\partial\psi_p)$ term in $\partial L_t/\partial t$. This inertial term produces the neoclassical perpendicular dielectric $\hat{\epsilon}_\perp = c^2/c_{Ap}^2$ for this plasma when the poloidal flow is in equilibrium on the transport time scale, for which the equilibrium poloidal flow is given in (46).

Thus, as in stellarator neoclassical transport theory [15, 27, 28], since the $\langle\bar{\mathbf{E}}_0\cdot\boldsymbol{\nabla}\rho\rangle$ terms are negligible in (117), this equation requires that there should be no net radial current in the plasma on the transport time scale:

$$0=\langle\mathbf{J}\cdot\boldsymbol{\nabla}\rho\rangle=\sum_s q_s\Gamma_s=\sum_s q_s\Gamma_{\mathrm{san}}^{na}. \qquad (118)$$

As indicated, only non-ambipolar particle fluxes cause net radial currents. Setting to zero the net radial plasma current caused by contributions produced by all 8 non-ambipolar particle fluxes in (115) for both species yields

$$\underbrace{\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_p}(V'L_t)}_{\text{inertia}} \simeq -\underbrace{\langle\mathbf{e}_\zeta\cdot\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\pi}}_{i\|}^{\mathrm{NA}}\rangle}_{\text{non-res. NA }\tilde{B}_\|} - \underbrace{\langle\mathbf{e}_\zeta\cdot\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\pi}}_{i\perp}\rangle}_{\text{cl, neo, paleo}} - \underbrace{\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\Pi_{i\rho\zeta})}_{\text{ion Reynolds stress}} + \underbrace{\langle\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}\rangle}_{\text{Maxwell stress}}$$
$$+\underbrace{\langle\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{J}}_{\|mn}^{\mathrm{NA}}\times\tilde{\mathbf{B}}_{\perp mn}^{\mathrm{NA}}}\rangle}_{\text{resonant NA }\tilde{B}_{mn}} - \underbrace{\dot{\rho}_{\psi_p}\partial L_t/\partial\rho}_{\dot{\psi}_p\neq 0\text{ trans.}} + \underbrace{\langle\mathbf{e}_\zeta\cdot\textstyle\sum_s\bar{\mathbf{S}}_{sm}\rangle}_{\text{mom. sources}}. \qquad (119)$$

This comprehensive conservation equation for the toroidal angular momentum $L_t\equiv m_i n_{i0}\langle R^2\Omega_t\rangle$ of a tokamak plasma is the primary result of this paper. Here, the approximate equality indicates that we have neglected the electron $\|$, $\perp$ viscous and Reynolds stresses which are usually smaller by a factor of order $\sqrt{m_e/m_i}\sim 1/60$. Similar to the density, $V'L_t$ represents the plasma toroidal angular momentum between the ρ and $\rho+d\rho$ flux surfaces; it is also an adiabatic plasma property that vanishes in the absence of dissipative processes and momentum sources.

The Maxwell stress contributions in (119) are induced by microturbulence. They can be made more explicit by using Ampere's law for the fluctuations and the vector identity $\mathbf{C}\times(\boldsymbol{\nabla}\times\mathbf{C})=\boldsymbol{\nabla}(|\mathbf{C}|^2/2)-\mathbf{C}\cdot\boldsymbol{\nabla}\mathbf{C}$:

$$\langle\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}\rangle=\langle\mathbf{e}_\zeta\cdot\frac{1}{\mu_0}\overline{(\boldsymbol{\nabla}\times\tilde{\mathbf{B}})\times\tilde{\mathbf{B}}}\rangle=\langle\mathbf{e}_\zeta\cdot\frac{1}{\mu_0}\overline{\tilde{\mathbf{B}}\cdot\boldsymbol{\nabla}\tilde{\mathbf{B}}}\rangle. \qquad (120)$$

However, for radially local fluctuations that do not connect to "external" magnetic fields, the average of this contribution over a narrow radial region of the plasma vanishes — see Eq. (28) and discussion in [12].

The $\langle\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}\rangle$ toroidal torque vanishes for ideal MHD perturbations throughout the hot core of tokamak plasmas [18]. However, in the vicinity of a low order rational surface (e.g., $q=m/n$) non-ideal effects can allow an externally imposed non-axisymmetric (NA) magnetic field component to induce a finite parallel current $\tilde{\mathbf{J}}_{\|mn}$ and nonzero $\tilde{\mathbf{B}}_{\perp mn}$ within the thin non-ideal boundary layer [18, 19] around the low order rational surface. This process produces the toroidal torque $\langle\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{J}}_{\|mn}^{\mathrm{NA}}\times\tilde{\mathbf{B}}_{\perp mn}^{\mathrm{NA}}}\rangle$ in (119) in the vicinity of low order $q=m/n$ rational surfaces in a toroidally rotating plasma.

F. Rotation, radial electric field, net particle flux

The flux surface average toroidal plasma rotation frequency $\langle\Omega_t\rangle$ is defined in terms of L_t by

$$\begin{aligned}\langle\Omega_t\rangle &\equiv \frac{L_t}{m_i n_{i0}\langle R^2\rangle}=\frac{\langle R^2\Omega_t\rangle}{\langle R^2\rangle}=\Omega_\wedge+\frac{\langle R^2\Omega_{*p}\rangle}{\langle R^2\rangle}\\ &= -\left(\frac{d\Phi_0}{d\psi_p}+\frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\psi_p}\right)+\frac{I}{\langle R^2\rangle}U_{i\theta}\\ &\simeq -\frac{1}{\psi_p'}\left(\frac{d\Phi_0}{d\rho}+\frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\rho}-\frac{c_p}{q_i}\frac{dT_{i0}}{d\rho}\right). \qquad (121)\end{aligned}$$

Thus, determination of L_t and hence $\langle R^2\Omega_t\rangle$ from a solution of (119) also determines the radial electric field:

$$\begin{aligned} E_\rho \;&\equiv\; \frac{\boldsymbol{\nabla}\rho}{|\boldsymbol{\nabla}\rho|}\boldsymbol{\cdot}\bar{\mathbf{E}}_0 = -\,|\boldsymbol{\nabla}\rho|\,\frac{d\Phi_0}{d\rho} \\ &=\; |\boldsymbol{\nabla}\rho|\left(\langle\Omega_t\rangle\psi_p' + \frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\rho} - \frac{I}{\langle R^2\rangle}U_{i\theta}\psi_p'\right) \\ &\simeq\; |\boldsymbol{\nabla}\rho|\left(\langle\Omega_t\rangle\psi_p' + \frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\rho} - \frac{c_p}{q_i}\frac{dT_{i0}}{d\rho}\right). \quad (122) \end{aligned}$$

The $|\boldsymbol{\nabla}\rho|$ factor accounts for the bunching together ($|\boldsymbol{\nabla}\rho| > 1/a$) of poloidal flux surfaces on the tokamak's outboard side but stretched separations ($|\boldsymbol{\nabla}\rho| < 1/a$) on the inboard side — due to the Shrafranov shift, etc.

The resultant E_ρ (or $\langle\Omega_t\rangle$) causes the electron and ion non-ambipolar radial particle fluxes to become equal (i.e., ambipolar) and produce no net radial current:

$$\Gamma^{na}_{\rm ean}(E_\rho) = Z_i\Gamma^{na}_{\rm ian}(E_\rho) \;\;\Longrightarrow\;\; \langle\mathbf{J}^{na}_{\rm an}(E_\rho)\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle = 0. \quad (123)$$

Note that obtaining no net radial current in the plasma is the same as determining $\langle\Omega_t\rangle$ (or E_ρ) from the evolution equation for L_t given in (119). Because (118) and hence (119) are determined predominantly by the sum of the non-ambipolar ion particle fluxes, the radial electric field E_ρ obtained here is what is referred to as the "ion root" in the stellarator literature [46].

Thus, from (75) the net ambipolar radial particle flux is the sum of the intrinsically ambipolar fluxes $\Gamma^a_\nu + \Gamma^a_{\rm pc}$ and the non-ambipolar fluxes $\Gamma^{na}_{\rm an}$ evaluated at E_ρ:

$$\Gamma^{\rm net}_e = \Gamma^a_{e\nu} + \Gamma^a_{\rm epc} + \Gamma^{na}_{\rm ean}(E_\rho) = \Gamma^{\rm net}_i. \quad (124)$$

As indicated, since the non-ambipolar ion particle fluxes mostly sum to zero to yield $\langle\mathbf{J}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle \simeq eZ_i\Gamma^{na}_{\rm ian}(E_\rho) \sim 0$, it is easiest to determine the net ambipolar particle flux using the non-ambipolar electron fluxes.

Since electron perpendicular-viscosity-driven classical and neoclassical particle fluxes are $m_e/m_i \lesssim 1/3672 \ll 1$ smaller than the corresponding already small ion ones, the lowest order electron particle flux is likely to be determined by the paleoclassical particle flux ($\Gamma_{\rm pc}$) plus the electron Reynolds and Maxwell stress particle fluxes:

$$\begin{aligned} \Gamma^{\rm net}_e \;&\simeq\; \Gamma^a_{\rm pc} + \Gamma^{na}_{e\rm Rey}(E_\rho) + \Gamma^{na}_{e\rm Max}(E_\rho) \\ &=\; -\,\bar{D}_\eta\frac{dn_{e0}}{d\rho} - n_{e0}V_{\rm pc} \\ &\quad - \frac{1}{e}\,\frac{1}{\psi_p' V'}\,\frac{\partial}{\partial\rho}\left[V'\Pi_{e\rho\zeta}\right] \\ &\quad - \frac{1}{\psi_p'}\langle\mathbf{e}_\zeta\boldsymbol{\cdot}n_{e0}\overline{\tilde{\mathbf{V}}_e\boldsymbol{\times}\tilde{\mathbf{B}}}\,\rangle. \quad (125) \end{aligned}$$

However, electron particle fluxes due to parallel viscous force effects from NA magnetic fields (via $\Gamma_{e\pi||}$) and momentum source effects (via Γ_{eS}) may be important for particular tokamak plasma situations. Also, during poloidal magnetic flux transients the FSA density equation (74) has the additional contribution $\dot{\rho}_{\psi_p}\,\partial n_{e0}/\partial\rho = (\dot{\psi}_p/\psi_p')(\partial n_{e0}/\partial\rho)$. This added ambipolar, radial motion effect could lead, for example, to the "density pump-out" effect (see for example [47]) as Electron Cyclotron Heating (ECH) is applied to a tokamak plasma, locally rapidly increases T_e, reduces $D_\eta \propto \eta^{\rm nc}_{\|} \sim 1/T_e^{3/2}$, and causes $\dot{\rho}_{\psi p} \propto \dot{\psi}_p < 0$ and hence $\partial n_{e0}/\partial t \sim -\,\dot{\rho}_{\psi_p}\,\partial n_e/\partial\rho < 0$ there.

V. DISCUSSION

The seven contributions on the right of the toroidal torque equation (119) have a wide variety of effects. In the following we describe briefly for each contribution, in order of its appearance in (119), the parameter regime in which it is likely to be significant and if it were dominant what effect it would have on the plasma toroidal rotation frequency $\langle\Omega_t\rangle$:

1) *Neoclassical toroidal viscosity due to NA fields:* When field errors are very large or large non-axisymmetric (NA) control fields are deliberately applied, the neoclassical toroidal viscous (NTV) torque in (100) acts as a drag throughout the plasma. It relaxes the plasma toroidal rotation toward an "intrinsic" or "offset" [42, 43] rotation frequency in the "counter" (to the current) direction given by (101): $\langle\Omega_t\rangle \simeq \langle\Omega_*\rangle \simeq (c_p + c_t)(1/q_i\psi_p')(dT_{i0}/d\rho) < 0$.

2) *Collision-induced perpendicular viscosities:* In plasmas with approximately ohmic-level heating and near the plasma edge where $T_e \lesssim 1$ keV, and when fluctuation-induced transport is suppressed, paleoclassical perpendicular viscosity in (103) may be dominant and cause $\langle\Omega_t\rangle \simeq 0$ through a combination of radial diffusion of L_t with diffusivity D_η from (67) and an inward pinch $V_{\rm pc}$ from (79).

3) *Fluctuation-induced ion Reynolds stress:* Various properties of microturbulence can lead [8–13] to radial diffusion, plus non-diffusive typically co-current "intrinsic" rotation (or "pinch velocity") [12, 48–53] and "residual stress" [13, 54] effects that do not depend on either $\langle\Omega_t\rangle$ or its radial derivative.

4) *Fluctuation-induced Maxwell stress:* This tends to be small for low β plasmas, but its electron component could be important in the radial particle flux.

5) *Resonant non-axisymmetric (NA) magnetic fields:* They produce a toroidal torque in the vicinity of low order resonant rational surfaces where $q = m/n$. Toroidal flow inhibits penetration of resonant field errors into the plasma by producing a shielding effect on the rational surfaces. Above a critical field error amplitude — termed the *penetration threshold* — the plasma rotation can no longer suppress the resonant torque, and plasma rotation at a m/n rational surface rapidly (in a few ms) locks to the wall (laboratory frame) [18, 21]. Then, the diffusive radial momentum transport due to perpendicular collision-iduced viscosity and the ion Reynolds stress slows toroidal rotation throughout the plasma on a fraction of the global momentum confinement time scale [55]. The net result is then a toroidal rotation profile that is locked to the wall on low order rational surfaces with some residual toroidal rotation between rational surfaces if the plasma toroidal torque source $\langle \mathbf{e}_\zeta \cdot \sum_s \bar{\mathbf{S}}_{sm} \rangle$ is large.

6) *Poloidal magnetic flux transients:* Poloidal flux transients that cause $\dot{\psi}_p \neq 0$ move $L_t \equiv m_i n_{i0} \langle R^2 \Omega_t \rangle$ radially at a speed $d\rho/dt = \dot{\rho}_{\psi_p} = \dot{\psi}_p / \psi_p'$. The rotation frequency $\langle \Omega_t \rangle$ typically decreases with minor radius; thus, from (72) one usually has $\partial L_t / \partial \rho < 0$. If, in addition, $\dot{\psi}_p < 0$ (e.g., due to EC heating or turn-on of a non-inductive co-current source), this effect dynamically reduces the plasma toroidal angular momentum on the magnetic field diffusion time scale: $(1/V')\partial(V' L_t)/\partial t|_{\psi_p} \simeq -\,\dot{\rho}_{\psi_p} \partial L_t / \partial \rho \sim \bar{D}_\eta \, \partial L_t / \partial \rho < 0$.

7) *Momentum sources:* A net toroidal momentum source $\langle \mathbf{e}_\zeta \cdot \sum_s \bar{S}_{sm} \rangle$ applies toroidal torque to the plasma. For example, uni-directional tangential energetic neutral beam injection (NBI) used to heat tokamak plasmas also causes a toroidal torque on the plasma via the injected fast ions collisionally transferring [26] their momentum to the background plasma — in the direction of the NBI. In steady state this torque on the plasma is usually balanced by the fluctuation-induced ion Reynolds stress discussed in 3) above. Similar considerations apply to the radial current induced by "energetic" electrons involved in lower hybrid current drive (LHCD) experiments [56]. In LHCD, the wave momentum input transferred to the energetic electrons provides a plasma momentum source via collisional transfer of the momentum in energetic electrons to the background plasma. This LHCD plasma momentum source is in the direction opposite to the wave momentum input — because the negatively charged energetic electrons cause a current in the opposite direction from the wave momentum input. Thus, co-current LHCD produces a plasma torque in the counter-current direction.

In the preceding discussion, particularly point 7), it was assumed that the only effects of the sources and sinks are through their particle and momentum inputs. However, for tangential NBI the mechanical momentum source from the collisional slowing down of the injected untrapped fast ions only comprises [26] a fraction $\sim 1-\epsilon$ of the total momentum input — because at birth the $v_\parallel$ of the NBI-injected fast ions is less than their toroidal velocity. The remaining $\sim \epsilon$ fraction results from the change in the fast ion guiding center canonical toroidal angular momentum $p_{\zeta g} \equiv m v_\parallel I/B - q_i \psi_p$ caused by the difference of their birth radius from the bounce average of their radial guiding center position: $\langle \Delta \psi_{\rm f} \rangle_\theta \sim \psi_p' \langle \Delta \rho_{\rm f} \rangle_\theta$. For a fast ion injection rate [26] $\dot{n}_{\rm f}$ this causes a radial fast ion current $\langle \mathbf{J}_{\rm f} \cdot \boldsymbol{\nabla} \rho \rangle \simeq \dot{n}_{\rm f} q_{\rm f} \langle \Delta \rho_{\rm f} \rangle_\theta$. At the plasma edge this radial current can also be induced by direct fast ion losses to the surrounding vacuum vessel wall.

Equation (118) did not include this fast-ion-induced current. However, it should be added there [57], in which case this equation becomes $\langle \mathbf{J} \cdot \boldsymbol{\nabla} \rho \rangle + \langle \mathbf{J}_{\rm f} \cdot \boldsymbol{\nabla} \rho \rangle = 0$. Multiplying (51) by nq we see that $\mathbf{J} \cdot \boldsymbol{\nabla} \psi_p = (\mathbf{J} \cdot \boldsymbol{\nabla} \rho)\, \psi_p' = \mathbf{e}_\zeta \cdot \mathbf{J} \times \mathbf{B} \simeq R\, J_r B_p$. Thus, the toroidal torque the fast ion current induces can be accommodated in the formalism of this paper by including an extra fast ion momentum source [57]:

$$\langle \mathbf{e}_\zeta \cdot \bar{\mathbf{S}}_{{\rm f}m} \rangle \equiv -\,\langle \mathbf{J}_{\rm f} \cdot \boldsymbol{\nabla} \rho \rangle\, \psi_p' \simeq -\,R B_p \dot{n}_{\rm f} q_{\rm f} \langle \Delta \rho_{\rm f} \rangle_\theta. \quad (126)$$

This is positive for tangentially co-injected ions that on average drift inward of their birth radius and cause $\langle \Delta \rho_{\rm f} \rangle_\theta < 0$. This extra toroidal angular momentum input due to a fast-ion-induced radial current causes a "return current" in the plasma [57]: $\langle \mathbf{J} \cdot \boldsymbol{\nabla} \rho \rangle = -\,\langle \mathbf{J}_{\rm f} \cdot \boldsymbol{\nabla} \rho \rangle$.

For tangential NBI this momentum input is typically a small fraction ($\sim \epsilon$) of the direct momentum input and hence can usually be neglected; however, for near-perpendicular NBI this fast-ion-induced radial current momentum source effect can be dominant. Similar considerations apply to the radial current induced by the energetic electrons involved in lower hybrid current drive (LHCD) [56] due to the average radial guiding center motion of the "extra species" of energetic electrons. In addition, a probe inserted into a plasma [58] that draws or emits a current $I_{\rm probe} \equiv V' \langle \mathbf{J}_{\rm probe} \cdot \boldsymbol{\nabla} \rho \rangle$ produces an analogous toroidal momentum source $\langle \mathbf{e}_\zeta \cdot \bar{\mathbf{S}}_{pm} \rangle = -\,(I_{\rm probe}/V')\, \psi_p'$ at the radial position of the probe.

VI. SUMMARY

Three independent components of the plasma force balance have been considered on sequential time scales: 1) plasma MHD force balance (16) on the fast Alfvén time scale which leads to a fundamental relation (17) between toroidal and poloidal flows, and $\mathbf{E} \times \mathbf{B}$ and ion diamagnetic flows; 2) plasma parallel force balance (43) which on the intermediate ion collision time scale leads to the specification (46) of the ion poloidal flow in terms mainly of the radial ion temperature gradient; and finally, 3) toroidal force balance (56) on the slow transport time scale which leads to radial particle fluxes.

The transport-time-scale density equation on a poloidal flux surface is specified in (74). The net particle flux $\Gamma \equiv \langle \boldsymbol{\Gamma} \cdot \boldsymbol{\nabla} \rho \rangle$ is composed of: 7 ambipolar collisional fluxes (86) due to classical, Pfirsch-Schlüter, banana-plateau neoclassical, paleoclassical, current-drive, dynamo current and $\bar{\mathbf{E}}^A \times \mathbf{B}/B^2$ pinch transport processes;

plus 8 non-ambipolar fluxes (115) due to parallel (non-axisymmetric, NA) and perpendicular viscosities, polarization flows, fluctuation-induced Reynolds and Maxwell stresses, externally-imposed resonant NA magnetic fields, poloidal flux transients, and momentum source effects.

Finally, a new comprehensive evolution equation for the plasma toroidal angular momentum density $L_t \equiv m_i n_{i0}\langle R^2\Omega_t\rangle$ in (119) has been obtained by requiring the net radial current to vanish on the transport time scale. The net radial current is determined by summing over species current contributions from all 8 non-ambipolar particle fluxes. The toroidal rotation frequency $\langle\Omega_t\rangle$ in (121) and hence radial electric field in (122) are obtained from L_t. For this radial electric field the net electron and ion particle fluxes become equal (ambipolar) (124) and are determined mainly by the electron particle fluxes indicated in (125).

This paper has also developed a general, self-consistent framework for taking account of all the anomalous transport effects of microturbulence on tokamak plasmas. These effects are explicitly included in equations for the parallel Ohm's law (36) and (40), poloidal ion flow (46), particle fluxes (109) and (110), momentum flux (114), and plasma toroidal angular momentum (119).

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Appendix A: Fluid and guiding center flows, fluxes

Since the first order perturbed flows have a nonzero radial component, they lead to the anomalous radial particle flux $\langle\overline{\tilde n_1\tilde{\mathbf V}_\wedge}\cdot\boldsymbol\nabla\rho\rangle$. The species fluid flow velocity $\mathbf V_\wedge \equiv \mathbf V_E + \mathbf V_*$ is related to the guiding center flow velocity usually calculated in drift kinetics and gyrokinetics via [3] $nq\,(\mathbf V_E + \mathbf V_*) = nq\mathbf V_g + \boldsymbol\nabla\times\mathbf M$. Here, $\mathbf V_g \equiv \int d^3v\, q\mathbf v_d f_M/n$ is the flow velocity of particle guiding centers having drift velocities $\mathbf v_d$ and $\mathbf M \equiv -p\,\mathbf B/B^2$ is the magnetism produced by the magnetic moments of the charged particles. Since to lowest order in $\beta \equiv \mu_0 p/B_0^2 << 1$ and the gyroradius parameter δ, $\tilde{\mathbf v}_d \simeq \tilde{\mathbf E}\times\mathbf B_0/B_0^2$ and $\boldsymbol\nabla\times\tilde{\mathbf M} \simeq \mathbf B_0\times\boldsymbol\nabla\tilde p/B_0^2$, we obtain

$$\tilde{\mathbf V}_g = \int d^3v\, q\tilde{\mathbf v}_d f_M/n_0 \simeq \tilde{\mathbf V}_E, \qquad \text{(A1)}$$

$$\tilde{\mathbf V}_M = (1/n_0 q)\boldsymbol\nabla\times\tilde{\mathbf M} \simeq \tilde{\mathbf V}_*. \qquad \text{(A2)}$$

Thus, $\tilde{\mathbf V}_\wedge = \tilde{\mathbf V}_g + \tilde{\mathbf V}_* + \mathcal O\{\beta\delta,\delta^2\}$. And in general, since $\boldsymbol\nabla\cdot\boldsymbol\nabla\times\mathbf M = 0$, we have $(1/V')(\partial/\partial\rho)(V'\langle\overline{\tilde n_1\tilde{\mathbf V}_\wedge}\rangle) = \langle\boldsymbol\nabla\cdot\overline{\tilde n\tilde{\mathbf V}_\wedge}\rangle = \langle\boldsymbol\nabla\cdot\overline{\tilde n\tilde{\mathbf V}_g}\rangle \equiv (1/V')(\partial/\partial\rho)(V'\Gamma_g)$, where Γ_g is the fluctuation-induced particle flux evaluated in gyrokinetic codes — see for example [12]. Also, since $\mathbf E\times\mathbf B$ flows are the same for electrons and ions, the net particle fluxes Γ_g determined from guiding center flows are ambipolar to lowest order.

Appendix B: Gyroviscosity and its effects

The gyroviscous stress $\boldsymbol\pi_\wedge$ is a symmetric tensor that is caused by diamagnetic-type $\mathbf B\times\boldsymbol\nabla\mathbf V$ effects. It is described in detail in [30]. The gyroviscous stress tensor can be written to lowest order (in δ and neglecting heat flux effects) for illustrative purposes as [3]

$$\boldsymbol\pi_\wedge = (p/4\omega_c)[\mathsf W_\wedge + \mathsf W_\wedge^{\mathsf T}]. \qquad \text{(B1)}$$

Here, $\mathsf W_\wedge \equiv \hat{\mathbf b}\times\mathsf W\cdot(\mathsf I + 3\hat{\mathbf b}\hat{\mathbf b})$ in which $\hat{\mathbf b} \equiv \mathbf B/B$, $\mathsf I$ is the identity tensor, $\mathsf W = \boldsymbol\nabla\mathbf V + (\boldsymbol\nabla\mathbf V)^{\mathsf T} - (2/3)\,\mathsf I\,(\boldsymbol\nabla\cdot\mathbf V)$ and the superscript $\mathsf T$ indicates the transpose of that tensor. For this form of the gyroviscous stress where the heat flux effects are neglected, to lowest order in β and δ the gyroviscous force can be written as [30]

$$-\boldsymbol\nabla\cdot\boldsymbol\pi_\wedge \simeq mn\mathbf V_*\cdot\boldsymbol\nabla\mathbf V = (m/q)\,\boldsymbol\nabla\times\mathbf M\cdot\boldsymbol\nabla\mathbf V. \qquad \text{(B2)}$$

This gyroviscous force cancels the $\mathbf V_*\cdot\boldsymbol\nabla\mathbf V$ part of $\mathbf V\cdot\boldsymbol\nabla\mathbf V$ in (33) and causes it to become approximately $\mathbf V_E\cdot\boldsymbol\nabla\mathbf V$, which has been speculated [59] to be in general $\mathbf V_g\cdot\boldsymbol\nabla\mathbf V$.

Since the ζ-average fluctuation-induced gyroviscous stress $\overline{\boldsymbol\pi_\wedge}$ is a symmetric tensor, using (54) we find in general that

$$\langle\mathbf e_\zeta\cdot\boldsymbol\nabla\cdot\overline{\boldsymbol\pi_\wedge}\rangle = \langle\boldsymbol\nabla\cdot(\mathbf e_\zeta\cdot\overline{\boldsymbol\pi_\wedge})\rangle = \frac{1}{V'}\frac{\partial}{\partial\rho}\langle\mathbf e_\zeta\cdot\overline{\boldsymbol\pi_\wedge}\cdot\boldsymbol\nabla\rho\rangle. \qquad \text{(B3)}$$

To lowest order in β, neglecting heat flux effects and gradients of equilibrium quantities, the gyroviscous stress induced by fluctuations is

$$\overline{\boldsymbol\pi_\wedge} \simeq \overline{(1/4\omega_{c0})\,\tilde p\,[\tilde{\mathsf W}_\wedge + \tilde{\mathsf W}_\wedge^{\mathsf T}]}\,. \qquad \text{(B4)}$$

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