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Viscous Forces Due To Collisional Parallel Stresses For Extended MHD Codes

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Abstract

Various forms of the viscous force caused by collision-induced parallel viscous stresses within an inhomogeneous magnetized plasma are presented. New forms are proposed for initial value extended MHD codes to capture the "fast" (Braginskii) collisional viscous force effects on short time scales and multi-collisionality regime "residual" viscous forces on collision time scales and longer in axisymmetric toroidal plasmas. Collision-based viscosity coefficients are described in various collisionality regimes: high (Braginskii, Pfirsch-Schlüter), intermediate (plateau) and low (banana). Smoothed formulas for the residual viscous forces induced by electron and ion parallel stresses on the collision and longer time scales that encompass all these collisionality regimes are presented. Also, a generalized Ohm's law that includes both the fast Braginskii-type viscous effects and the slow residual effects that lead to the neoclassical parallel Ohm's law is proposed. Finally, suggestions are made for implementing and verifying viscous force effects in extended MHD codes.

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I. INTRODUCTION

Extended magnetohydrodynamic (MHD) models used in the CEMM project [1] seek to include all relevant physics needed for simulating fluid-type behavior of magnetically-confined toroidal plasmas. The M3D [2] and NIMROD [3] code projects under the CEMM umbrella have historically focused on using ideal and resistive MHD descriptions of axisymmetric equilibrium toroidal plasmas. Recently, two-fluid effects (e.g., diamagnetic flows, gyroviscosity) have been included. And explorations of plasma flows and their effects have begun.

In order to properly describe the evolution of flows, the anisotropic nature of the viscous force in a magnetized plasma needs to be taken into account. In particular, the viscous forces induced by collision-induced parallel stresses in the plasma need to be taken into account because: 1) they are the largest viscous forces; 2) they collisionally relax the electron and ion flows on their respective time scales; 3) in low collisionality plasmas they lead to important effects in the parallel neoclassical Ohm's law (trapped particle effects on the resistivity and bootstrap current) and the poloidal plasma flow (relaxation to an ion-temperature-gradient-determined value); and 4) it is important for numerical stability and convergence issues to properly treat dissipative effects in the M3D and NIMROD codes.

This report is organized as follows. Section II describes the parallel, cross and perpendicular stresses and in particular the collisional parallel viscous stresses in a magnetized plasma. Appendix A describes the collisional (Braginskii) parallel stresses in plasmas containing, as is typical in tokamaks, electrons, hydrogenic ions and impurity ions. The following section (III) presents various forms of parallel stresses and the resultant viscous forces. Thereafter, Section IV proposes specific forms for a combination of "fast" Braginskii-type and "residual" long-time-scale viscous forces induced by parallel stresses for inclusion in the M3D and NIMROD codes. Asymptotic residual parallel viscous force coefficients in high (Braginskii, Pfirsch-Schlüter), intermediate (plateau) and low (banana) collisionality regimes, and smoothed multi-collisionality formulas for electrons, hydrogenic and impurity ions are presented in Appendix B. Their effects are developed in Appendix C. The penultimate (V) section suggests verification tests of these viscous forces and their effects in the M3D and NIMROD codes. The ion viscous stresses and general Ohm's law in multi-collisionality regimes for tokamak plasmas that are suggested for use in extended MHD codes are summarized in Section VI. It also discusses possible limitations of the suggested forms.

II. COLLISIONAL STRESSES IN A MAGNETIZED PLASMA

The fluid-based viscous force density on a small volume of a plasma species is $-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$. The fact that this force density is a local differential of the local viscous stress tensor $\boldsymbol{\pi}$ implies that the physical processes that cause it are also local. This is unfortunately not true in low collisionality toroidal plasmas where the collision length $\lambda \equiv v_T/\nu$ is usually many times the toroidal circumference of the experimental device. Nonetheless, the following discussion and analysis seeks to capture the most important low collisionality physics within this local model of viscous forces — to facilitate inclusion of viscous force effects of parallel viscous stresses in the M3D and NIMROD extended-MHD-based initial value codes.

The collisional Braginskii [4] viscous stress tensors are given by

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{\parallel} + \boldsymbol{\pi}_{\wedge} + \boldsymbol{\pi}_{\perp}, \quad \text{parallel, cross (gyroviscous) and perpendicular stresses.} \tag{1}$$

Here, the subscripts $\parallel, \wedge, \perp$ indicate parallel, cross and perpendicular directions relative to the local magnetic field $\mathbf{B}(\mathbf{x})$. A strongly magnetized plasma species is defined as one that has a small collision frequency ν compared to the gyrofrequency ω_c and small gyroradius $\varrho \equiv v_T/\omega_c$ compared to cross and perpendicular gradient scale lengths of plasma properties and electromagnetic fields. For strongly magnetized toroidal plasmas of fusion interest a small gyroradius expansion is usually appropriate:

$$\delta \equiv \varrho/a \ll 1, \quad \text{small gyroradius expansion.} \tag{2}$$

Here, a is a characteristic macroscopic plasma dimension, typically the plasma minor radius. For arbitrary flow velocity magnitudes and properties, the characteristic scalings of the parallel, cross and perpendicular stresses can be written schematically for $R_0 q \gtrsim \lambda \gtrsim a$ as

$$\boldsymbol{\pi}_{\parallel} \sim \nu\lambda^2\, \boldsymbol{\nabla}_{\parallel}\mathbf{V}, \quad \boldsymbol{\pi}_{\wedge} \sim \nu\varrho\lambda\, \mathbf{B}\times\boldsymbol{\nabla}\mathbf{V}/B \sim \delta\, \boldsymbol{\pi}_{\parallel}, \quad \boldsymbol{\pi}_{\perp} \sim \nu\varrho^2\, \boldsymbol{\nabla}_{\perp}\mathbf{V} \sim \delta^2\, \boldsymbol{\pi}_{\parallel}, \quad \text{scalings.} \tag{3}$$

Thus, the parallel viscous stress tensor $\boldsymbol{\pi}_{\parallel}$ is usually dominant in small gyroradius, magnetized toroidal plasmas. The remainder of this report will concentrate on it. [For plasma perturbations extended long distances along field lines (i.e., $|\boldsymbol{\nabla}_{\parallel}| \lesssim 1/a$) but radially localized to a small fraction of the minor radius (i.e., $|\boldsymbol{\nabla}_{\perp}| \sim k_{\perp} \gg 1/a$), the cross and perpendicular stress effects can become larger than these scalings indicate by factors of $k_{\perp}a$ and $(k_{\perp}a)^2$; for such cases the cross (gyroviscous) force can exceed the parallel stress force — as is the case for many perturbed diamagnetic flow effects on MHD-type instabilities.]

The parallel viscous stresses for electrons and ions were originally written by Braginskii [4] for each species in the form (here, z is the local coordinate along $\mathbf{B}$)

$$\boldsymbol{\pi}_\parallel = -\,\eta_0\, W_{zz}, \quad W_{zz} \equiv 2\,\frac{\partial V_z}{\partial z} - \frac{2}{3}\,(\boldsymbol{\nabla}\!\cdot\!\mathbf{V}). \tag{4}$$

For an electron-ion plasma with a hydrogenic ion species (i.e., $Z_i\!=\!1$), the collisional viscosity coefficients for electrons and ions are [4] ($\tau \equiv 1/\nu$, $\lambda \equiv v_T/\nu$ and $v_T \equiv \sqrt{2T/m}$)

$$\eta_0^i = 0.96\, n_i T_i\, \tau_i = 0.48\, m_i n_i\, \nu_i \lambda_i^2, \quad \eta_0^e = 0.73\, n_e T_e\, \tau_e = 0.365\, m_e n_e\, \nu_e \lambda_e^2. \tag{5}$$

The parallel stress tensor can be written in general (for arbitrary collisionality in an inhomogeneous magnetized plasma) in the Chew-Goldberger-Low form as [5]

$$\boldsymbol{\pi}_\parallel \equiv \pi_\parallel\,(\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3), \quad \hat{\mathbf{b}}\boldsymbol{\cdot}\boldsymbol{\pi}_\parallel\boldsymbol{\cdot}\hat{\mathbf{b}} = (2/3)\,\pi_\parallel, \quad \text{parallel stress tensor.} \tag{6}$$

Here, $\pi_\parallel(\mathbf{x}, t) \equiv p_\parallel - p_\perp$ is the pressure anisotropy, which is a scalar function of space and time. Also, $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ is a unit vector along the local magnetic field $\mathbf{B}$ and $\mathbf{I}$ is the identity tensor (dyad). In the Braginskii high collisionality regime $\pi_\parallel$ is given for each species by

$$\boxed{\pi_\parallel \equiv -\,(3/2)\,\eta_0\,\hat{\mathbf{b}}\boldsymbol{\cdot}\mathbf{W}_V\boldsymbol{\cdot}\hat{\mathbf{b}},} \quad \text{collision-induced pressure anisotropy (a scalar),} \tag{7}$$

in which the rate of strain in the plasma species induced by the flow velocity $\mathbf{V}$ is

$$\mathbf{W}_V \equiv \boldsymbol{\nabla}\mathbf{V} + (\boldsymbol{\nabla}\mathbf{V})^{\mathsf{T}} - (2/3)\,\mathbf{I}\,(\boldsymbol{\nabla}\!\cdot\!\mathbf{V}), \quad \text{rate of strain induced by } \mathbf{V}. \tag{8}$$

The superscript T is the transpose of that tensor (dyad); thus, $\mathbf{W}_V$ is a symmetric tensor.

The Braginskii [4] closures for the parallel viscous stress tensor $\boldsymbol{\pi}_\parallel$ were developed for MHD-type applications where the flow velocity $\mathbf{V}$ is assumed to be large compared to the heat flow velocity $\mathbf{V}_q \equiv -\,2\mathbf{q}/5nT$ and higher order flow-type moments (energy-weighted heat flow etc.) — but still small compared to thermal speeds, i.e., $|\mathbf{V}|/v_T \ll 1$. However, in two-fluid treatments which include diamagnetic flows, the diamagnetic-type heat flows $\mathbf{V}_q$ are comparable to the diamagnetic flows $\mathbf{V}_* \equiv \mathbf{B}\!\times\!\boldsymbol{\nabla}p/(nqB^2)$ and cannot be neglected. Then, the rate of strain tensor is modified [6]: $\mathbf{W}_V \to \mathbf{W}_V + \mathbf{W}_q$, where the rate of strain tensor for heat flows is

$$\mathbf{W}_q \equiv (-2/5nT)\,[\,\boldsymbol{\nabla}\mathbf{q} + (\boldsymbol{\nabla}\mathbf{q})^{\mathsf{T}} - (2/3)\,\mathbf{I}\,(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{q})\,]. \tag{9}$$

Similarly, the stress tensor gets modified: $\boldsymbol{\pi} \to \boldsymbol{\pi}_V + \boldsymbol{\pi}_q$, in which $\boldsymbol{\pi}_q$ represents parallel heat stresses. Collisional viscosity coefficients including these heat flow effects and allowing for impure plasmas (i.e., for $Z_{\rm eff} \equiv \sum_i n_i Z_i^2/n_e > 1$) are discussed in Appendix A.

III. VISCOUS FORCE INDUCED BY COLLISIONAL PARALLEL STRESSES

Next, various geometric forms of the pressure anisotropy and the viscous force they induce will be explored. Using various vector and tensor identities and the definition of the local curvature of the magnetic field, $\boldsymbol{\kappa} \equiv (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})\hat{\mathbf{b}} = -\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}})$ with $\hat{\mathbf{b}} \equiv \mathbf{B}/B$, it can be shown that

$$\begin{aligned} \boxed{\mathbf{B}\cdot\mathsf{W}_V\cdot\mathbf{B}/2} &= \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{V}\cdot\mathbf{B} - (B^2/3)\,(\boldsymbol{\nabla}\cdot\mathbf{V}) \\ &= B\,(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\mathbf{V}/B) + [\mathbf{B}\times(\mathbf{B}\times\mathbf{V})]\cdot\boldsymbol{\kappa} - (B^2/3)\,\boldsymbol{\nabla}\cdot\mathbf{V} \\ &= \boxed{B\,\mathbf{V}\cdot\boldsymbol{\nabla}B + \mathbf{B}\cdot\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) + (2B^2/3)\,\boldsymbol{\nabla}\cdot\mathbf{V} - (\mathbf{B}\cdot\mathbf{V})(\boldsymbol{\nabla}\cdot\mathbf{B}).} \end{aligned} \quad (10)$$

The form on the last line will be used below — because, as discussed below, its first term is the only term that survives on longer than collision times scales, after the faster MHD-type compressional Alfvén and sound wave relaxation processes come into quasi-equilibrium. The representation of $\mathbf{B}\cdot\mathsf{W}_q\cdot\mathbf{B}$ is similar [6] with $\mathbf{V}\rightarrow\mathbf{q}$.

In order of appearance in the last line of (10), contributions to $\mathbf{B}\cdot\mathsf{W}_V\cdot\mathbf{B}$ have the following effects. The first, $\mathbf{V}\cdot\boldsymbol{\nabla}B$ term indicates parallel strain induced by flow in directions in which the magnitude of the magnetic field varies. Since the lowest order flows are within a flux surface, this term mainly produces poloidal flow damping, at a rate proportional to the collision frequency ν (for each species). The second term represents MHD-type advection of the parallel component of the magnetic field, as is evident from its linearized form $\mathbf{B}_0\cdot\boldsymbol{\nabla}\times(\tilde{\mathbf{V}}\times\mathbf{B}_0)\sim B_0\,\partial\tilde{B}_\parallel/\partial t$. Together with part of $\boldsymbol{\nabla}\cdot\mathbf{V}$, it provides viscous damping of "fast" compressional Alfvén waves, which to lowest order relax $\tilde{P}+B_0\tilde{B}_\parallel/\mu_0$. The third, $\boldsymbol{\nabla}\cdot\mathbf{V}$ term represents plasma compressibility. Its residual after the fast relaxation of compressional Alfvén waves provides viscous damping of sound waves on the ion collision time scale. Because there are no magnetic monopoles in the universe, the final, $\boldsymbol{\nabla}\cdot\mathbf{B}$ term vanishes; however, this term could be kept in extended MHD codes to assist in "divergence $\mathbf{B}$ cleaning" — i.e., for relaxing away via viscous damping any numerical errors that cause $\boldsymbol{\nabla}\cdot\mathbf{B}\neq 0$. For simplicity, the analogous effects due to heat flows $\mathbf{V}_q \equiv -(2\mathbf{q}/5nT)$ will be neglected on the MHD compressional Alfvén and sound wave time scales; however, heat flow effects will be retained on the "slow" collision (poloidal flow damping) time scales.

The M3D and NIMROD extended MHD codes use semi-implicit numerical algorithms to take time steps longer than MHD wave time scales while capturing the constraints imposed by these processes. Thus, for analytic analyses, on collision time scales the last form of (10) can be simplified by taking $\boldsymbol{\nabla}\cdot\mathbf{V} = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. On the collision time scale perpendicular flows are $\mathbf{E}\times\mathbf{B}$ (with $\mathbf{E} \simeq -\boldsymbol{\nabla}\Phi$) plus diamagnetic flows in the form $\mathbf{V}_\perp = (1/B^2)\mathbf{B}\times\boldsymbol{\nabla} f$, in which $f = f(\psi_p)$ is a scalar flux function. Thus, one has $\mathbf{B}\cdot\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) = B(\mathbf{B}\cdot\boldsymbol{\nabla} f)(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}) \sim (k_\| a)\,\beta$. As indicated, this term is small for typical plasma responses that are highly extended along field lines and for low β plasmas; this quite small "residual" contribution will be neglected in the following discussion. With all these simplifications, on the collision time scale the "residual" pressure anisotropy induced by flows and heat flows is (see Appendix A for how the viscosity coefficients are determined for impure plasmas)

$$\pi_\| \simeq -\frac{3}{B}\left(\eta_{00}\mathbf{V}\cdot\boldsymbol{\nabla} B + \eta_{01}\,\frac{-2}{5\,nT}\,\mathbf{q}\cdot\boldsymbol{\nabla} B\right), \quad \text{on collision or longer time scales.} \tag{11}$$

The viscous force density caused by the parallel stress tensor $\boldsymbol{\pi}_\|$ defined in (6) is in general

$$\boxed{\mathbf{F}_\pi \equiv -\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\| = -\,\pi_\|\,[\,\boldsymbol{\kappa} - \mathbf{B}\,(\mathbf{B}\cdot\boldsymbol{\nabla} B)/B^3] - (1/B^2)\,\mathbf{B}\,(\mathbf{B}\cdot\boldsymbol{\nabla})\pi_\| + (1/3)\,\boldsymbol{\nabla}\pi_\|.} \tag{12}$$

And the parallel ($\mathbf{B}\cdot$) component of this viscous force is

$$\mathbf{B}\cdot\mathbf{F}_\pi \equiv -\,\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\,\boldsymbol{\pi}_\| = \pi_\|\,(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla} B) - (2/3)\,(\mathbf{B}\cdot\boldsymbol{\nabla})\,\pi_\|, \quad \text{parallel viscous force.} \tag{13}$$

The last term will be annihilated below by averaging this parallel force over a flux surface.

Up to now neither the magnetic field structure nor a coordinate system have been specified. However, they are needed to connect these results with axisymmetric neoclassical transport theory [7, 8]. The axisymmetric equilibrium magnetic field $\mathbf{B}_0 \equiv \mathbf{B}_t + \mathbf{B}_p$ has toroidal and poloidal components. It is written in terms of the poloidal magnetic flux ψ_p:

$$\mathbf{B}_0(\psi_p, \theta) = I\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi_p = \boldsymbol{\nabla}\psi_p\times\boldsymbol{\nabla}[\,q(\psi_p)\,\theta - \zeta\,], \quad I(\psi_p) \equiv RB_t. \tag{14}$$

The radial, poloidal straight-field-line and toroidal axisymmetry coordinates will be taken to be ψ_p, θ, ζ for which the poloidal rotation of a field line per unit toroidal rotation is $d\theta/d\zeta = 1/q(\psi_p) \equiv \mathbf{B}_0\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta$. The Jacobian for transforming from the laboratory ($\mathbf{x}$) to these (non-orthogonal) curvilinear coordinates is $\sqrt{g} \equiv 1/(\boldsymbol{\nabla}\psi_p\cdot\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta) = 1/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta$. The flux surface average (FSA) of a scalar function $f(\mathbf{x})$ on a ψ_p flux surface is defined by

$$\langle f(\mathbf{x})\rangle \equiv \frac{\int_0^{2\pi} d\zeta \int_0^{2\pi} f(\mathbf{x})\; d\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta}{2\pi\int_0^{2\pi} d\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta}, \quad \text{flux surface average of } f(\mathbf{x}). \tag{15}$$

The FSA is an annihilator for parallel derivatives of scalar functions: $\langle\mathbf{B}_0\cdot\boldsymbol{\nabla} f\rangle = 0$.

On the collision and longer time scales the flow and heat flow are incompressible [7, 8]: $\boldsymbol{\nabla}\cdot\mathbf{V} = 0$, $\boldsymbol{\nabla}\cdot\mathbf{q} = 0$. Since to first order in the small gyroradius expansion ($\delta \ll 1$) the flows and heat flows lie within flux surfaces, using the general relation for the divergence of a vector in an axisymmetric system one can show for each plasma species that [8]

$$0 = \boldsymbol{\nabla}\cdot\mathbf{V} = (\mathbf{V}\cdot\boldsymbol{\nabla}\theta)\,\frac{\partial}{\partial\theta}\left(\frac{\mathbf{V}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta}\right) \implies \boxed{U_\theta(\psi_p) \equiv \frac{\mathbf{V}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta},} \text{ poloidal flow function.} \quad (16)$$

Similarly, $\boldsymbol{\nabla}\cdot\mathbf{q} = 0$ yields $Q_\theta(\psi_p) \equiv (-2/5nT)\,\mathbf{q}\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta$. Thus, to lowest order in δ the FSA of (13) yields the residual parallel viscous force for each species (here, $\hat{\mathbf{b}}_0 \equiv \mathbf{B}_0/B_0$):

$$\langle \mathbf{B}_0\cdot\mathbf{F}_\pi\rangle \equiv -\,\langle \mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\|\rangle \simeq -\,3\,\langle\,(\hat{\mathbf{b}}_0\cdot\boldsymbol{\nabla}B_0)^2\rangle\,[\,\eta_{00}\,U_\theta + \eta_{01}\,Q_\theta\,]. \quad (17)$$

IV. VISCOUS FORCES FOR EXTENDED MHD CODES

Extensions of the flux surface average (FSA) collisional parallel viscous force in (17) to the low collisionality regimes of axisymmetric tokamak plasmas have been developed [8]; these "neoclassical" results are discussed, extended and summarized in Appendix B. And their effects on residual ion flows and the parallel Ohm's law are discussed in Appendix C. It is convenient to specify the FSA of the residual ion parallel viscous force as a damping force on the poloidal flow U_θ to an "intrinsic" or "offset" ion flow velocity $U_{i\theta}^0$ in the form

$$\boxed{\langle \mathbf{B}_0\cdot\mathbf{F}_{i\pi}\rangle \equiv -\,\langle \mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{i\|}\rangle = -\,m_i n_i\,\mu_i\,\langle B_0^2\rangle(U_{i\theta} - U_{i\theta}^0),} \quad \text{FSA } \| \text{ ion viscous force,} \quad (18)$$

$$\boxed{U_{i\theta}^0(\psi_p) \simeq k_i\,\frac{I(\psi_p)}{q_i\langle B_0^2\rangle}\,\frac{dT_{i0}(\psi_p)}{d\psi_p},} \quad \text{offset ion poloidal flow } \mathbf{V}_i^0\cdot\boldsymbol{\nabla}\theta = U_{i\theta}^0(\psi_p)\,\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta. \quad (19)$$

General formulas for the poloidal viscous damping frequency μ_i and offset poloidal flow coefficient k_i, which depend on the species and collisionality regime in tokamak plasmas composed of electrons, hydrogenic and impurity ions are specified in Appendix B. The usual asymptotic banana regime neoclassical coefficients for $Z_i = 1, \nu_{*i} \ll 1, \sqrt{\epsilon} \ll 1$ and no impurities are $\mu_i \simeq 2.24\sqrt{\epsilon}\,\nu_i$ and $k_i \simeq 1.17/(1 + 0.67\sqrt{\epsilon}\,)$.

In general, the offset poloidal ion flow $U_{i\theta}^0$, which is proportional to the poloidal ion heat flow (see Appendix C), does not depend solely on the temperature gradient of the ions being considered. Rather, in tokamak plasmas with small admixtures of impurities in addition to the dominant hyrogenic species, it depends on impurity density and temperature gradients, and the impurity collisionality regime [9]. In transport codes $U_{i\theta}^0$ is often evaluated using the

NCLASS code [10]. However, as discussed in [9], if the impurities are in the intermediate (plateau) or high (Braginskii, Pfirsch-Schlüter) collisionality regime, $U_{i\theta}^0$ depends predominantly on the ion temperature gradient, as indicated in (19). Thus, as discussed further in Appendix C, it will be assumed that the form of (19) provides a sufficient representation of the effects of small admixtures of impurities in extended MHD codes.

The general form of the viscous force induced by pressure anisotropy is given in (12). The key question is: what should be used for the pressure anisotropy $\pi_\parallel \equiv p_\parallel - p_\perp$? The answer for MHD-type responses is the combination of $\pi_\parallel$ given by (7) with the $\mathbf{B}\cdot\mathsf{W}_V\cdot\mathbf{B}/2$ given by (10) since on the fast compressional Alfvén and sound wave relaxation time scales the heat flows $\mathbf{V}_q$ can be neglected. After dissipation of the fast MHD-type compressional Alfvén and sound waves with the Braginskii parallel viscous damping coefficient η_{00}, the residual contribution to (10) on collision and longer time scales is the first term, $B\,\mathbf{V}\cdot\boldsymbol{\nabla}B$, as indicated in (11). For that term the general multi-collisionality regime result for the flux surface average of the residual parallel viscous force can be written in the form given in (18), which is a generalization of the corresponding Braginskii form given in (17). To capture all these properties within a single viscous force it is proposed to leave the fast (superscript f) viscous force effects in (7) on compressional Alfvén and sound waves unchanged with the Braginskii η_{00} coefficient, but to modify the coefficient of the $B\,\mathbf{V}\cdot\boldsymbol{\nabla}B$ term in (10) so the residual (superscript r) flux surface average it produces on time scales longer than collision times is given by (18). Thus, a pressure anisotropy for each species that includes both fast Braginskii-type and residual viscous stresses is

$$\boxed{\begin{aligned} \pi_\parallel \;&=\; \pi_\parallel^{\rm f} + \pi_\parallel^{\rm r} \\ &\pi_\parallel^{\rm f} \equiv -\,3\,\eta_{00}\left(\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B})}{B^2} + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{V} - \frac{(\mathbf{B}\cdot\mathbf{V})(\boldsymbol{\nabla}\cdot\mathbf{B})}{B^2}\right), \\ &\pi_\parallel^{\rm r} \equiv -\,mn\,\mu\,\langle B_0^2\rangle\frac{\hat{\mathbf{b}}_0\cdot\boldsymbol{\nabla}B_0}{\langle\,(\hat{\mathbf{b}}_0\cdot\boldsymbol{\nabla}B_0)^2\rangle}\left(U_\theta - U_\theta^0\right), \quad \hat{\mathbf{b}}_0 \equiv \mathbf{B}_0/B_0, \end{aligned}} \quad \text{proposed form.} \tag{20}$$

Hence, the suggested procedure for introducing ion viscous force effects due to ion collisional parallel stresses into extended MHD codes is to implement the viscous force given by (12) with the pressure anisotropy $\pi_{i\parallel}$ given in (20) using μ_i from (B14) and (B19) and k_i from (C27), plus (C26) (B15), (B16) and/or (B20). Formulas for the needed Braginskii-type collisional ion viscosity coefficients η_{00}^i for ions in impure plasmas are given in (A17).

The introduction of electron viscous force effects due to electron parallel viscous stresses is more complicated — because their effects on the electron heat flow as well electron flows must be taken into account to produce the neoclassical parallel Ohm's law. The salient points of their effects are discussed in Appendix C between (C1) and (C21). The net effect of the residual parallel electron viscous forces is to modify the parallel electron momentum equation and thereby produce the residual noclassical parallel Ohm's law [11, 12]:

$$\boxed{\langle \mathbf{B}_0 \cdot \mathbf{E}^A \rangle = \eta_\parallel^{\rm nc} \, (\, \langle \mathbf{B}_0 \cdot \mathbf{J} \rangle - \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm drives} \rangle \,).} \quad \text{neoclassical parallel Ohm's law.} \tag{21}$$

Here, $\eta_\parallel^{\rm nc}$ is the neoclassical parallel resistivity [11, 12] defined in (C16); it includes both the electron heat flow effects ($\rightarrow$ Spitzer conductivity for homogeneous $|\mathbf{B}|$) and the viscosity effects (for $\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla} B \neq 0$). The $\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm drives} \rangle$ term includes currents driven by bootstrap and non-inductive current drive sources [12]: $\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm drives} \rangle = \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle + \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm CD} \rangle + \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm dyn} \rangle$. Thus, analogous to the ions, as discussed in Appendix C, the proposed general Ohm's law form that incorporates both the fast (superscript f) and residual viscous force effects is

$$\boxed{\begin{aligned} \mathbf{E} \; = \; & - \mathbf{V} \times \mathbf{B} + \frac{\mathbf{J} \times \mathbf{B} - \boldsymbol{\nabla} p_e - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\parallel}^{\rm f} - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\wedge}}{n_e e} - \frac{m_e}{e} \frac{d\mathbf{V}_e}{dt} \\ & + \eta_\perp \left(\mathbf{J}_\perp - \frac{3}{2} \frac{n_e \mathbf{B} \times \boldsymbol{\nabla} T_e}{B^2} \right) + \eta_\parallel^{\rm nc} \left(\mathbf{J}_\parallel - \mathbf{J}_{\parallel \rm drives} \right), \end{aligned}} \quad \text{Ohm's law.} \tag{22}$$

Here, $\mathbf{J}_\perp \equiv - \mathbf{B} \times (\mathbf{B} \times \mathbf{J})/B^2$, $\mathbf{J}_\parallel \equiv \mathbf{B} \, (\mathbf{B} \cdot \mathbf{J}) B^2$ and $\mathbf{J}_{\parallel \rm drives} \equiv \mathbf{B} \, \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm drives} \rangle / \langle B_0^2 \rangle$. The first line in (22) indicates the usual MHD advection, Hall-type terms and inertial effects with the only difference from normal being that only the fast part of the viscous force due to parallel electron stresses, $-\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\parallel}^{\rm r}$, is included there. The second line of (22) represents the anisotropic electrical resistivity effects that yield the usual perpendicular electric field from $\eta_\perp \mathbf{J}_\perp$ as well as the neoclassical parallel Ohm's law given in (21). To obtain neoclassical tearing modes (NTMs), the dominant bootstrap current drive [12] $\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle \sim - I \, dP_0 / d\psi_p$ has to be adapted to a more primitive form along perturbed magnetic field lines as follows:

$$I \, \frac{dP_0}{d\psi_p} \equiv B_0^2 \, \frac{\mathbf{J}_\perp \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} \quad \Longrightarrow \quad B^2 \, \frac{\mathbf{J}_\perp \cdot \boldsymbol{\nabla}\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \quad \text{with} \quad \mathbf{J}_\perp \equiv \frac{\mathbf{B} \times \boldsymbol{\nabla} P}{B^2}. \tag{23}$$

Note also that (22) includes non-inductive current-drive sources via $\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm CD} \rangle$ and dynamo currents induced by fluctuations via $\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm dyn} \rangle$ [12]; these current sources should not be added separately to avoid "double-counting."

Various residual viscous force models have been implemented and tested in reduced MHD codes [13–15]. Heuristic forms tested in the NIMROD code are discussed in [16]. The most successful heuristic form used in NIMROD is [16] $\mathbf{F}_\pi = -mn\mu\langle B^2\rangle(\mathbf{V}\cdot\mathbf{e}_\theta)\,\mathbf{e}_\theta/(\mathbf{B}\cdot\mathbf{e}_\theta)^2$, in which $\mathbf{e}_\theta = \sqrt{g}\,\boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi_p = \mathbf{B}_p/(\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta)$ is the covariant base vector. This form is obviously simpler than the form proposed in (18)–(22). However, the proposed ion viscous force in (12), (6), (20) and Ohm's law in (22) have the advantage that they capture the usual Braginskii viscous force as well as the residual multi-collisionality effects, offset ion flows $U^0_{i\theta}$ and neoclassical parallel Ohm's law on time scales longer than collision times.

V. VERIFICATION TESTS FOR VISCOUS FORCE EFFECTS

In implementing this suggested procedure for including the viscous force effects due to multi-collisionality parallel stresses in extended MHD codes, three categories of verification tests are suggested: fast MHD-type processes, relaxation of flows and neoclassical parallel Ohm's law, and transport-time-scale effects — as discussed in the succeeding paragraphs.

Fast MHD-type processes: In general, viscous force effects should be negligible and have no significant effects on ideal-MHD-type plasma responses. Thus, the first verification test is to make sure adding the viscous force effects "does no harm" to these responses.

Relaxation of flows and neoclassical parallel Ohm's law: The electron viscous force effects add to the magnetic field evolution equation a fourth order derivative, diffusive- (parabolic-) type term (via $\mathbf{V}_e = \mathbf{V}_i - \mathbf{J}/n_e e$ with $\mu_0\mathbf{J} \equiv \boldsymbol{\nabla}\times\mathbf{B}$), which is sometimes called a "hyper-resistivity" effect. While this is useful in dissipating magnetic field structures that are highly localized radially, it should not affect other physical processes much. Similarly, the ion viscous force adds a diffusive effect to the momentum equation, primarily to its parallel component. These effects should become significant on the electron and ion collision time scales, respectively; they should dissipate compressional Alfvén and sound waves on the ion collision time scale. On time scales longer than their respective collision time scales, the electron and ion flows should relax to being: incompressible, of first order in the gyroradius, and flowing within an equilibrium or average flux surface. Specifically, they should be described by (C3) and (C5), Eqs. (18)–(47) in [11], or Eqs. (11)–(38) in [12]. Suggested verification tests would be to check that, on time scales longer than their respective collision times: 1) electron and ion flows are incompressible to $\mathcal{O}\{\delta^2\}$ as indicated in (16) above and Eq. (22) in [11]; 2) the poloidal flow function $U_\theta \equiv \mathbf{V}\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta$ is approximately constant on flux surfaces (to order $\mathcal{O}\{\delta\}$); 3) the current density $\mathbf{J}$ is as given in Eqs. (26) and (29) in [11]; 4) the FSA neoclassical Ohm's law given in (21), Eq. (39) in [11] or Eq. (26) in [12] is obtained with the bootstrap current given by (C17) or Eq. (29) in [12]; 5) the poloidal ion flow is as indicated in (C27), Eqs. (44), (45) in [11] or Eqs. (34)–(37) in [12]; 6) the FSA radial flows $\langle \mathbf{V}\cdot\boldsymbol{\nabla}\psi_p \rangle$ for both electrons and ions are of order δ smaller than the corresponding poloidal and toroidal flows within flux surfaces; and 7) for an axisymmetric equilibrium $\mathbf{B}_0$ field and a pressure anisotropy $\pi_\|$ that is independent of the toroidal angle, the viscous force does not cause a toroidal torque on either plasma species (i.e., $\langle R^2 \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_\| \rangle = 0$). Ultimate verification and validation of the proposed form for the $\pi_\|$ given by (20) plus the viscous force it causes and the general Ohm's law in (22) will be through detailed comparisons for a wide range of applications with results from kinetic-based approaches, such as those being developed by Held et al. [17] and Ramos [18].

Transport-time-scale effects: The main residual viscous force effects induced by collision-induced parallel viscous stresses on the long transport time scale are (analytically) to enforce ambipolarity through first order in the gyroradius (via Eqs. (42)–(45) of [11]) and to cause the banana-plateau (ambipolar) neoclassical radial particle flux, as indicated in Eq. (89) in [11]. In addition, their inclusion facilitates the analytic derivation of a toroidal flow (rotation) equation on the transport time scale, Eq. (119) of [11]. Suggested verification tests would be to: 1) obtain the banana-plateau radial particle flux indicated in Eq. (89) of [11] and check that it is ambipolar; 2) show that the toroidal flow obeys the toroidal rotation evolution equation given by Eq. (119) in [11], after irrelevant terms there are eliminated; and 3) obtain NTMs using the adaptation of the bootstrap current drive in (23).

VI. DISCUSSION AND SUMMARY

As indicated by the boxed equations above and the paragraph after (20), a new procedure has been proposed for including viscous force effects caused by collision-induced parallel viscous stresses in high (Braginskii [4], Pfirsch-Schlüter), intermediate (plateau) and low (banana) collisionality regimes. The proposed procedure uses standard Braginskii collisional viscous forces with coefficients η_0 for relaxing the fast MHD-type responses, but a new multi-collisionality regime residual viscous force on collision time scales. Specifically, the new procedure uses the normal CGL pressure tensor form in (6) and viscous force definition in (12), but suggests a new form for the pressure anisotropy given by (20). The multi-collisionality regime poloidal flow viscous damping frequency μ_s for each species s is given in (B14), (B17), (B19) and the constant k_i for the intrinsic poloidal flow $U^0_{i\theta}$ defined in (19) is given by (C27). The relevant collisional (Braginskii) regime viscosity coefficients η^s_{00} for electrons and ions in impure plasmas, as is typical in tokamaks, are defined in (A11) and (A17), respectively. The general Ohm's law for including all these effects is given in (22).

Some issues and limitations regarding use of this procedure to represent viscous force effects due to collisional parallel viscous stresses in various collisionality regimes are: 1) Are the forms of the viscous force and its effects given by the combination of (12), (20) and (22) really the best or most appropriate forms? 2) Are long scale ($\gg \pi R_0 q$) parallel variations in flow components appropriately and adequately relaxed with the Braginskii η_{00} viscosity coefficients even in low collisionality regimes where "collisionless" closures [17, 19] become relevant? 3) While the exponential temporal decay of poloidal flows resulting from (18) is not precisely correct in low collisionality regimes [20], is this residual viscous force sufficient, except perhaps for applications where the poloidal flow dynamics is critical, since it produces the correct equilibrium flows? 4) Are the approximations that lead to the "offset" poloidal ion flow $U^0_{i\theta}$ sufficiently accurate for extended MHD modeling in the M3D and NIMROD codes? and 5) While the poloidal variation of the viscous force in the banana collisionality regime [21] is not the same as that implied by the combination of (12) and (20), is this residual viscous force sufficient, except perhaps for applications that depend critically on the poloidal variation of viscous force effects, since its flux-surface average is correct?

New extended MHD applications that should be facilitated by the inclusion of these proposed viscous stress forms fall into two categories analogous to the verification tests discussed in the preceding section. First, on time scales comparable to or slightly longer than the collision time scales of electrons and ions they should facilitate obtaining the proper currents and poloidal flows in low collisionality toroidal plasmas. This will allow, for example, inclusion of: the neoclassical rather than Spitzer resistivity with appropriate trapped particle and impurity effects (via $Z_{\rm eff}$), bootstrap current effects in H-mode pedestals, and diamagnetic flow-type effects on MHD instabilities (e.g., ELMs). On time scales longer than the ion collision time scale they will facilitate inclusion of proper poloidal and toroidal flows, radial electric field evolution in concert with the toroidal flow evolution, and bootstrap current effects. These effects will enable simulations of neoclassical tearing modes (NTMs) and resistive wall modes (RWMs) that include the appropriate low collisionality toroidal physics.

The ultimate fate of the proposed viscous forces and Ohm's law in extended MHD codes will depend on the practicality of their implementation and their usefulness in capturing the most important viscous force effects in low collisionality toroidal plasmas in extended MHD codes such as M3D and NIMROD.

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Appendix A: Collisonal (Braginskii-type) Viscosities

The Braginskii [4] closures for the parallel viscous stress tensor $\boldsymbol{\pi}_\parallel$ were developed for collisional plasmas (i.e., $|\lambda \boldsymbol{\nabla} \mathbf{V}| \ll |\mathbf{V}|$) and MHD-type applications where the flow velocity $\mathbf{V}$ is of order the $\mathbf{E}\times\mathbf{B}$ flow velocity and large compared to the diamagnetic flows. In particular, $\mathbf{V}$ is assumed to be large compared to the heat flow velocity $\mathbf{V}_q \equiv (-2\mathbf{q}/5nT)$ and higher order flow-type moments (energy-weighted heat flow etc.) — but still small compared to thermal speeds, i.e., $|\mathbf{V}|/v_T \ll 1$. However, as noted at the end of Section II, since two-fluid treatments include diamagnetic flows, the diamagnetic-type heat flow $\mathbf{V}_q$ is comparable to the diamagnetic flow $\mathbf{V}_*$ and cannot be neglected. Then, the rate of strain tensor is modified: $\mathsf{W}_V \to \mathsf{W}_V + \mathsf{W}_q$ in which the rate of strain tensor for heat flows is given in (9). Similarly, the stress tensor gets modified: $\boldsymbol{\pi} \to \boldsymbol{\pi}_0 + \boldsymbol{\pi}_1 + \cdots \equiv \boldsymbol{\pi}_V + \boldsymbol{\pi}_q + \cdots$. Here, the subscript indicates the order j of the energy weighting Laguerre polynomial in the relevant moments of the distribution function ($\mathbf{v}' \equiv \mathbf{v} - \mathbf{V}$):

$$\boldsymbol{\pi}_j \equiv \int d^3v' \, m \left[\mathbf{v}'\mathbf{v}' - (v'^2/3)\,\mathsf{I}\right] L_j^{5/2}(mv'^2/2T)\, f(\mathbf{x},\mathbf{v},t), \tag{A1}$$

in which $L_j^{5/2}(x)$ are Laguerre polynomials: $L_0^{5/2} = 1$, $L_1^{5/2} = 7/2 - x, \cdots$.

Taking the $\int d^3v'\, m\, \hat{\mathbf{b}} \cdot [\mathbf{v}'\mathbf{v}' - (v'^2/3)\,\mathsf{I}\,] \cdot \hat{\mathbf{b}}\, L_j^{5/2}(mv'^2/2T)$ moments with $j = 0, 1$ of a Chapman-Enskog form of the collisional equilibrium ($\partial/\partial t < \nu_s$) plasma kinetic equation and neglecting higher order (in a collisional regime) $\hat{\mathbf{b}} \cdot (\boldsymbol{\pi} \cdot \boldsymbol{\nabla}\mathbf{V}) \cdot \hat{\mathbf{b}}$ viscous-dissipation-type terms yields a matrix equation for each plasma species:

$$n_s\, T_s \begin{bmatrix} \hat{\mathbf{b}} \cdot \mathsf{W}_V^s \cdot \hat{\mathbf{b}} \\ \hat{\mathbf{b}} \cdot \mathsf{W}_q^s \cdot \hat{\mathbf{b}} \end{bmatrix} = -\, \frac{6}{5\, \tau_{ss}}\, \mathsf{G}_s \cdot \begin{bmatrix} (2/3)\, \pi_{s0\|} \\ (2/3)\, \pi_{s1\|} \end{bmatrix}. \tag{A2}$$

Here, τ_{ss} is a reference self-collision frequency (s^{-1}) for a plasma species s defined by

$$\boxed{\frac{1}{\tau_{ss}} \equiv \frac{4}{3\sqrt{\pi}}\, \frac{4\pi\, n_s q_s^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2\, m_s^2\, v_{Ts}^3} = \frac{4}{3\sqrt{\pi}}\, \frac{4\pi\, n_s Z_s^4 e^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2\, m_s^{1/2}\, (2T_s)^{3/2}},} \quad \text{reference collision frequency.} \tag{A3}$$

The matrix G_s is a 2×2 matrix of Coulomb collisional "drag" coefficients on the stresses that result from the parallel stress moments of the collision operator. The parallel viscous stresses $\pi_{s0\|}$ and $\pi_{s1\|}$ will be obtained by multiplying this equation by the inverse G_s^{-1} of the matrix G_s for each species s. [If higher order energy moments are included (i.e., $j \geq 2$), they yield $(j{+}1) \times (j{+}1)$ matrices and $j{+}1$ equations; however, after inverting the larger G_s matrices the results obtained below change less than the $1/\ln\Lambda \sim 6$ % intrinsic accuracy of the Fokker-Planck Coulomb collision operator and hence are not warranted.)

The collisional matrix and its inverse can be written in general as [8, 23]

$$\mathsf{G}_s = Z \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{17}{4} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & \frac{205}{48} \end{bmatrix}, \quad \mathsf{G}_s^{-1} = \frac{\begin{bmatrix} \frac{17\,Z}{4} + \frac{205}{48\sqrt{2}} & -(\frac{3\,Z}{2} + \frac{3}{4\sqrt{2}}) \\ -(\frac{3\,Z}{2} + \frac{3}{4\sqrt{2}}) & Z + \frac{1}{\sqrt{2}} \end{bmatrix}}{2\,Z^2 + 301\,Z/48\sqrt{2} + 89/48}. \tag{A4}$$

Here, the first matrix in G_s represents collisions of a species s with a species s' of charge Z that has a much larger mass (i.e., $m_{s'} \gg m_s$) and the second matrix represents self-collisions within the s species. Dotting the inverse matrix G_s^{-1} with (A2) yields the collisional pressure anisotropy induced by the flow and heat flow rates of strain within the s species of

$$\pi_{s\|} = -\,(3/2)\,[\,\eta_{00}^s\, \hat{\mathbf{b}} \cdot \mathsf{W}_V^s \cdot \hat{\mathbf{b}} + \eta_{01}^s\, \hat{\mathbf{b}} \cdot \mathsf{W}_q^s \cdot \hat{\mathbf{b}}\,]. \tag{A5}$$

Here, the viscosity coefficients for each species s are

$$\eta_{00}^s = \frac{5}{6}\, \mathsf{G}_{s00}^{-1}\, n_s T_s\, \tau_{ss} = \frac{5}{6}\, \frac{17\,Z/4 + 205/48\sqrt{2}}{2\,Z^2 + 301\,Z/48\sqrt{2} + 89/48}\, n_s T_s\, \tau_{ss}, \tag{A6}$$

$$\eta_{01}^s = \frac{5}{6}\, \mathsf{G}_{s01}^{-1}\, n_s T_s\, \tau_{ss} = -\,\frac{5}{6}\, \frac{3\,Z/2 + 3/4\sqrt{2}}{2\,Z^2 + 301\,Z/48\sqrt{2} + 89/48}\, n_s T_s\, \tau_{ss}. \tag{A7}$$

For an electron-ion plasma with a hydrogenic ion species (i.e., $Z=1$), $\nu_e = 1/\tau_{ee}$ and the numerical coefficient in η^e_{00} is 0.73, in agreement with the Braginskii coefficient η^e_0 given in (5). For only one species of ions $Z \to 0$ in (A6), $\tau_{ii} \equiv \tau_i = 1/(\sqrt{2}\,\nu_i)$ and the numerical coefficient in η^i_{00} from (A6) is $(5/6)(205/89\sqrt{2})(1/\sqrt{2}) = 0.96$, in agreement with η^i_0 in (5).

Typical tokamak plasmas have small admixtures of impurity (non-hydrogenic) ions. For collisions of electrons with hydrogenic ions (subscript i, $Z_i=1$) and various types of impurity ions (subscript I, charge Z_I), the effective ion charge is

$$\boxed{Z_{\rm eff} \equiv \frac{n_i + \sum_I n_I Z_I^2}{n_e},} \quad \text{ion charge for electron collisions in an impure plasma.} \tag{A8}$$

The electron collision frequency and length are defined for an impure plasma by

$$\boxed{\nu_e \equiv \frac{Z_{\rm eff}}{\tau_{ee}} = \frac{4\sqrt{2\pi}\,(n_i + \sum_I n_I Z_I^2)\, e^4\, \ln\Lambda}{\{4\pi\epsilon_0\}^2\, 3\, m_e^{1/2} T_e^{3/2}} \simeq \frac{5\times 10^{-11}\, n_e({\rm m}^{-3})\, Z_{\rm eff}}{[T_e({\rm eV})]^{3/2}} \left(\frac{\ln\Lambda}{17}\right) {\rm s}^{-1},} \tag{A9}$$

$$\lambda_e = \frac{v_{Te}}{\nu_e} \simeq 1.2\times 10^{16} \frac{[T_e({\rm eV}]^2}{Z_{\rm eff}\, n_e({\rm m}^{-3})}\ {\rm m}. \tag{A10}$$

For an impure plasma the electron viscosity coefficients obtained from (A6) and (A7) are

$$\boxed{\eta^e_{00} = \frac{5}{12} \frac{17\, Z_{\rm eff}^2/4 + 205\, Z_{\rm eff}/48\sqrt{2}}{2\, Z_{\rm eff}^2 + 301\, Z_{\rm eff}/48\sqrt{2} + 89/48}\, m_e n_e\, \nu_e \lambda_e^2,} \tag{A11}$$

$$\eta^e_{01} = -\frac{5}{12} \frac{3\, Z_{\rm eff}^2/2 + 3\, Z_{\rm eff}/4\sqrt{2}}{2\, Z_{\rm eff}^2 + 301\, Z_{\rm eff}/48\sqrt{2} + 89/48}\, m_e n_e\, \nu_e \lambda_e^2. \tag{A12}$$

With only hydrogenic ions ($Z_{\rm eff} \to 1$), η^e_{00} reduces to the Braginskii η^e_0 in (5).

For collisions of hydrogenic ions (subscript i, $Z_i=1$) with various impurity ions (subscript I, charge Z_I) that are heavier than them ($m_I \gg m_i$), the effective ion charge is

$$\boxed{Z_* \equiv \frac{\sum_I n_I Z_I^2}{n_i},} \quad \text{ion charge for hydrogenic ion collisions in an impure plasma.} \tag{A13}$$

Note that the effective ion charge for electron collisions can be written in terms of this effective charge for hydrogenic ions via $Z_{\rm eff} = (n_i/n_e)(1 + Z_*)$ or $Z_* = (n_e/n_i)Z_{\rm eff} - 1$. The ion collision frequency and length can be defined for an impure plasma in terms of the conventional deuterium (mass m_D) ion collision frequency ($\nu_i \tau_{ii} = 1/\sqrt{2} + Z_*$):

$$\boxed{\nu_i \equiv \frac{4\sqrt{\pi}(n_i + \sqrt{2}\, \sum_I n_I Z_I^2) e^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2\, 3\, m_i^{1/2} T_i^{3/2}} \simeq \frac{5.8\times 10^{-13}\, n_i({\rm m}^{-3})\, (1+\sqrt{2}\, Z_*)}{(m_i/m_D)^{1/2}\, [T_i({\rm eV})]^{3/2}} \left(\frac{\ln\Lambda}{17}\right) {\rm s}^{-1},} \tag{A14}$$

$$\lambda_i = \frac{v_{Ti}}{\nu_i} \simeq 1.7\times 10^{16} \frac{[T_i(\mathrm{eV})]^2}{(1+\sqrt{2}Z_*)\, n_i(\mathrm{m}^{-3})} \ \mathrm{m}. \tag{A15}$$

Neglecting the small variations in the $\ln\Lambda$ coefficients, the ion collision frequency in an impure plasma can be written in terms of the electron collision frequency in (A9):

$$\nu_i = \frac{n_i}{n_e}\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2} \frac{1+\sqrt{2}\,Z_*}{\sqrt{2}\,Z_{\mathrm{eff}}}\,\nu_e \sim 10^{-2}\,\nu_e. \tag{A16}$$

For an impure plasma the ion viscosity coefficients obtained from (A6) and (A7) are

$$\boxed{\eta^i_{00} = \frac{5(1+\sqrt{2}Z_*)}{12\sqrt{2}} \, \frac{17\,Z_*/4 + 205/48\sqrt{2}}{2\,Z_*^2 + 301\,Z_*/48\sqrt{2} + 89/48}\, m_i n_i\, \nu_i \lambda_i^2,} \tag{A17}$$

$$\eta^i_{01} = -\frac{5(1+\sqrt{2}Z_*)}{12\sqrt{2}} \, \frac{3\,Z_*/2 + 3/4\sqrt{2}}{2\,Z_*^2 + 301\,Z_*/48\sqrt{2} + 89/48}\, m_i n_i\, \nu_i \lambda_i^2. \tag{A18}$$

With only hydrogenic ions ($Z_* \to 0$), η^i_{00} reduces to the Braginskii η^i_0 in (5).

Appendix B: Viscosity Coefficients For Multi-Collisionality Regimes

There is one fundamental approximation used in obtaining the Braginskii viscous stresses that is not appropriate for extended MHD descriptions of tokamak plasmas. Namely, the collision length λ is assumed to be shorter than parallel inhomogeneity scale lengths of the flow velocity $\mathbf{V}$ (i.e., $|\lambda\,\nabla_\| \mathbf{V}| \ll |\mathbf{V}|$). It is proposed here that this shortcoming be rectified by extending and utilizing a multi-collisionality form [9] of the neoclassical-based closures [8] for the residual parallel viscous force $\langle \mathbf{B}_0 \cdot \nabla \cdot \boldsymbol{\pi}_\| \rangle$. The general form of the flux-surface-average (FSA) parallel viscous force will be discussed first. Next, the lowest (banana) collisionality regime closure will be discussed. Thereafter, scalings of the viscosity coefficients for the low, intermediate and high collisionality regimes will be discussed. Then, multi-collisionality forms for the FSA of the residual parallel viscous force will be developed.

In all collisionality regimes the residual FSA parallel viscous force $\langle \mathbf{B}_0 \cdot \nabla \cdot \boldsymbol{\pi}_{0\|} \rangle$ and parallel viscous heat force $\langle \mathbf{B}_0 \cdot \nabla \cdot \Theta_\| \rangle$ can be written [8] for each species in forms analogous to the corresponding FSA of Braginskii closure relations in (17):

$$\begin{bmatrix} \langle \mathbf{B}_0 \cdot \nabla \cdot \boldsymbol{\pi}_\| \rangle \\ \langle \mathbf{B}_0 \cdot \nabla \cdot \Theta_\| \rangle \end{bmatrix} = \frac{mn}{\tau} \langle B_0^2 \rangle \begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{01} & \mu_{11} \end{bmatrix} \begin{bmatrix} U_\theta \\ Q_\theta \end{bmatrix} \equiv \frac{mn}{\tau} \langle B_0^2 \rangle \, \mathsf{M} \cdot \begin{bmatrix} U_\theta \\ Q_\theta \end{bmatrix}, \tag{B1}$$

in which $mn/\tau \equiv m_s n_s/\tau_{ss}$ for the s species. The symmetric matrix M (with $\mu_{10} \equiv \mu_{01}$) of dimensionless viscous damping frequency coefficients and its inverse are

$$\mathsf{M} \equiv \begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{01} & \mu_{11} \end{bmatrix}, \quad \mathsf{M}^{-1} \equiv \frac{\begin{bmatrix} \mu_{11} & -\mu_{01} \\ -\mu_{01} & \mu_{00} \end{bmatrix}}{\mu_{00}\,\mu_{11} - \mu_{01}^2}, \quad \text{viscous damping coefficients.} \tag{B2}$$

The dimensionless damping frequencies μ_{ij} will be written in terms of a reference collision frequency $\nu_{\rm ref}$ that will depend on the collisionality regime and a matrix of dimensionless, positive-definite coefficients $\hat{K}_{ij}$ (see Eqs. (4.20)–(4.22) in [8]):

$$\boxed{\mathsf{M} \equiv \begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{01} & \mu_{11} \end{bmatrix} = \nu_{\rm ref}\,\tau \begin{bmatrix} \hat{K}_{00} & \frac{5}{2}\hat{K}_{00} - \hat{K}_{01} \\ \frac{5}{2}\hat{K}_{00} - \hat{K}_{01} & \hat{K}_{11} - 5\,\hat{K}_{01} + \frac{25}{4}\hat{K}_{00} \end{bmatrix}.} \tag{B3}$$

The very low collisionality physics that needs to be captured in a fluid description is the parallel stress and resultant residual parallel viscous force induced by collisions of untrapped (circulating) particles that carry parallel flows with the "immobile" trapped particles. In the "banana" low collisionality regime [7, 8] trapped particles circumnavigate their banana drift orbits before collisions scatter them out of the trapped particle region of velocity space. The banana (superscript B) collisionality regime collision frequency for each species s is

$$\boxed{\nu_{\rm ref}^B \equiv (f_t/f_p)\,\nu_s \sim \sqrt{\epsilon}\,\nu_s,} \quad \text{banana collisionality regime reference frequency.} \tag{B4}$$

The relevant collision frequencies for electrons and ions are given in (A9) and (A14). Here, the flow-weighted fraction of circulating particles f_c is defined by [8, 9]

$$\boxed{f_c \equiv \frac{3}{4}\langle B_0^2\rangle \int_0^{1/B_{\rm max}} \frac{\lambda\, d\lambda}{\langle \sqrt{1 - \lambda\, B_0(\theta)}\,\rangle} \simeq 1 - 1.46\,\sqrt{\epsilon} + 0.46\,\epsilon\sqrt{\epsilon},} \quad \text{circulating fraction.} \tag{B5}$$

Slightly more accurate approximate forms are given in [24, 25]. The complementary fraction of trapped particles is $f_t \equiv 1 - f_c \simeq 1.46\sqrt{\epsilon} - 0.46\,\epsilon\sqrt{\epsilon}$. In the approximate forms at the end of all these formulas the variation of the magnetic field strength on a magnetic flux surface has been approximated by $B_0 \simeq B_0 R_0/R \simeq B_{00}(1 - \epsilon\cos\theta)$ in which

$$\boxed{\epsilon \equiv \frac{B_{\rm max} - B_{\rm min}}{B_{\rm max} + B_{\rm min}} \simeq \frac{r}{R_0} \ll 1,} \quad \text{inverse aspect ratio.} \tag{B6}$$

In the banana collisionality regime the dimensionless poloidal flow damping coefficients $\hat{K}_{ij}^B$ can be written for a plasma composed of electrons, hydrogenic ions and impurity ions as indicated in the first column of Table 1 (see Eqs. (4.18), (4.19) and (4.61)–(4.64) in [8]).

TABLE I: ASYMPTOTIC DIMENSIONLESS VISCOSITY COMPONENTS.

For a pure electron-ion plasma $Z \to Z_i$ for electrons but $Z \to 0$ for the ions.

In impure plasmas $Z \to Z_{\rm eff}$ for electrons, $Z \to Z_*$ for ions and $Z \to 1/Z_*$ for impurities.

In the rightmost column $D \equiv (6/5)(2Z^2 + 301/48\sqrt{2} + 89/48) \simeq 2.40Z^2 + 5.32Z + 2.225$.

collisionality regime:	banana (B)	plateau (P)	Pfirsch-Schlüter (PS)
$\hat{K}_{00}$	$[Z + \sqrt{2} - \ln(1{+}\sqrt{2})]/(\nu_s \tau_{ss})$	$\sqrt{\pi}$	$(17Z/4 + 205/48\sqrt{2})/D$
	$\simeq (Z + 0.533)/(\nu_s \tau_{ss})$	$\simeq 1.77$	$\simeq (4.25Z + 3.02)/D$
$\hat{K}_{01}$	$[Z + 1/\sqrt{2}]/(\nu_s \tau_{ss})$	$3\sqrt{\pi}$	$(7/2)(23Z/4 + 241/48\sqrt{2})/D$
	$\simeq (Z + 0.707)/(\nu_s \tau_{ss})$	$\simeq 5.32$	$\simeq (20.13Z + 12.43)/D$
$\hat{K}_{11}$	$[2Z + 9/4\sqrt{2}]/(\nu_s \tau_{ss})$	$12\sqrt{\pi}$	$(49/4)(33Z/4 + 325/48\sqrt{2})/D$
	$\simeq (2Z + 1.591)/(\nu_s \tau_{ss})$	$\simeq 21.27$	$\simeq (101.06Z + 58.65)/D$

Collisionality regimes in tokamak plasmas are defined by the ratio of the effective collision frequency of trapped particles $\nu_{\rm eff} \sim \nu/f_t^2 \sim \nu/\epsilon$ to their bounce frequency $\omega_b \sim \sqrt{\epsilon}\, v_T/R_0 q$:

$$\boxed{\nu_* \equiv \frac{\nu_{\rm eff}}{\omega_b} = \frac{\nu}{\epsilon^{3/2}(v_T/R_0 q)} = \frac{R_0 q}{\epsilon^{3/2}\lambda}, \quad \text{collisionality regime parameter.}} \qquad \text{(B7)}$$

The three relevant collisionality regimes (for each species) are

$$\begin{aligned} &\nu_* \ll 1, && \text{low (banana, } B\text{) collisionality regime,} \\ &1 \ll \nu_* \ll \epsilon^{-3/2}, && \text{intermediate (plateau, } P\text{) collisionality regime,} \\ &\epsilon^{-3/2} \ll \nu_*, && \text{high (Braginskii, Pfirsch-Schlüter, } PS\text{) collisionality regime.} \end{aligned} \qquad \text{(B8)}$$

The FSA residual parallel viscous force in the banana regime was discussed in the preceding paragraph. The parameters for the plateau and Pfirsch-Schlüter collisionality regimes will be discussed in the next two paragraphs.

The plateau regime is an intermediate collisionality regime where typical untrapped particles are collisionless, but trapped and low parallel velocity particles drift radially off flux surfaces, which causes radial plasma transport and parallel viscous forces. The plateau regime reference frequency is independent of collision frequency. For each species it is

$$\nu_{\rm ref}^P \equiv \frac{\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla} B_0)^2\rangle}{\langle B_0^2\rangle}\,\frac{v_{Ts}^2}{\omega_{ts}} \simeq \frac{1}{2}\epsilon^2\omega_{ts}, \quad \text{plateau regime reference frequency.} \qquad \text{(B9)}$$

In the last approximate expression a large aspect ratio tokamak expansion which yields $\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle/\langle B_0^2\rangle \simeq \epsilon^2/(2R_0^2q^2)$ has been used and the characteristic transit frequency for each species of untrapped particles has been defined by

$$\omega_{ts} \equiv \frac{v_{Ts}}{R_0 q}, \quad \text{species transit frequency.} \tag{B10}$$

In the plateau collisionality regime the dimensionless poloidal flow damping coefficients $\hat{K}^P_{ij}$ can be written for a plasma composed of electrons, hydrogenic ions and impurity ions as indicated in the second column of Table 1 (see (4.65) in [8]).

The Pfirsch-Schlüter regime is [7, 8] the high collisionality Braginskii regime whose FSA parallel viscous force, which is in the form of (B1), was given in (17). Its reference collision frequency is

$$\nu^{PS}_{\rm ref} \equiv \frac{3}{2}\frac{\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle}{\langle B_0^2\rangle}\frac{v_{Ts}^2}{\nu_s} \simeq \frac{3}{4}\frac{\epsilon^2\omega_{ts}^2}{\nu_s} \quad \text{Pfirsch-Schlüter regime reference frequency,} \tag{B11}$$

which is inversely proportional to the collision frequency. In the Pfirsch-Schlüter collisionality regime the dimensionless poloidal flow damping coefficients $\hat{K}^{PS}_{ij}$ can be written for a plasma composed of electrons, hydrogenic ions and impurity ions as indicated in the third column of Table 1 (see (A6), (A7) above and Eqs. (4.31)–(4.40) in [8]).

In general, the collisionality parameter ν_* is solely a function of the poloidal flux ψ_p and can be specified for a general axisymmetric magnetic field geometry by [9] (for each species)

$$\boxed{\nu_{*s} \equiv \frac{f_t/f_c}{2.92}\frac{\nu_s\,\omega_{ts}}{v_{Ts}^2}\frac{\langle B_0^2\rangle}{\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle} \sim \frac{\nu_s}{\epsilon^{3/2}(v_{Ts}/R_0q)},} \quad \text{general collisionality parameter.} \tag{B12}$$

The collision frequencies for electrons ν_e and ions ν_i in an impure plasma are defined in (A9) and (A14). The impurity collision frequency ν_I is specified in (B21) below. As indicated at the end of (B12), in a large aspect ratio tokamak where $\sqrt{\epsilon} \ll 1$ one obtains $f_t/f_c \sim 1.46\sqrt{\epsilon}$ and again $\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle/\langle B_0^2\rangle \simeq \epsilon^2/(2R_0^2q^2)$, (B12) reduces to the approximate (B7).

Multi-collisionality forms of the parallel viscosity coefficients μ_e, μ_i for electrons, ions in the desired form given by (18) have been developed [7–9]. They can be written in the generic form [7] $\mu \sim \sqrt{\epsilon}\,\nu/[(1+\nu_*^{1/2}+\nu_*)(1+\epsilon^{3/2}\nu_*)]$, with various order unity numerical factors in front of each of the factors. Here, the $\nu_*^{1/2}$ factor in the denominator arises from [7] collisional boundary layer effects in the vicinity of the velocity-space boundary between trapped and untrapped (circulating) particles.

In this work smoothed formulas for the residual parallel viscous forces are desired that encompass all three collisionality regimes and asymptotically approach the low (banana) collisionality regime results for $\nu_* \ll 1$, the plateau results for $1 \ll \nu_* \ll \epsilon^{-3/2}$, and the high (Braginskii) collisionality regime when $\nu_* \gg \epsilon^{-3/2}$. Also, small admixtures of impurities should be allowed for since tokamak plasmas often have $Z_{\rm eff} \sim 2$–3. Such descriptions have been developed by Kim et al. [9] from general formulas presented in [8]. Those results will be used after taking account of the Errata in Ref. [9] and correcting the coefficient of the Pfirsch-Schlüter term in Table 1 of [9] in the denominator from 1/6 to 2/3 to obtain the correct high collisionality limit. Also, collisional boundary layer effects [7] will be added with an assumed coefficient of unity. Finally, the viscous damping frequencies in (B3) will be referenced to the banana regime reference frequency in (B4). Thus, the proposed dimensionless viscosity coefficients are, in the spirit of a Padé approximation, for each species s

$$\boxed{\hat{K}_{ij}^{\rm tot} \;=\; \frac{\hat{K}_{ij}^{B}}{\left[1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}_{ij}^{B}/\hat{K}_{ij}^{P}\right]\left[1+2\hat{K}_{ij}^{P}/(3\,\omega_{ts}\tau_{ss}\hat{K}_{ij}^{PS})\right]}.} \qquad \text{(B13)}$$

Here, the $\hat{K}_{ij}^{B}$, $\hat{K}_{ij}^{P}$ and $\hat{K}_{ij}^{PS}$ quantities are the dimensionless viscosity coefficients in the banana, plateau and Pfirsch-Schlüter collisionality regimes, respectively; they are given in Table I. The corresponding poloidal flow and heat flow damping frequencies μ_{ij} are obtained by multiplying the $\hat{K}_{ij}^{\rm tot}$ coefficients by the banana regime reference collision frequency in (B4) and combining them as indicated in (B3). As in the caption of Table I, $Z \to Z_{\rm eff}$ for electrons, $Z \to Z_*$ for ions and $Z \to 1/Z_*$ for impurities.

For example, the basic dimensionless poloidal flow damping frequency for each species is

$$\boxed{\mu_s \;\equiv\; \mu_{s00} \;=\; \nu_s\tau_{ss}\frac{f_t}{f_c}K_{s00}^{\rm tot} \;=\; \frac{(\nu_s\tau_{ss})\,(f_t/f_c)\,K_{s00}^{B}}{\left[1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}_{s00}^{B}/\hat{K}_{s00}^{P}\right]\left[1+2\hat{K}_{s00}^{P}/(3\,\omega_{ts}\tau_{ss}\hat{K}_{s00}^{PS})\right]}.} \qquad \text{(B14)}$$

In the asymptotic banana regime ($\nu_{*s} \to 0$) this μ_s becomes $(\nu_s\tau_{ss})(f_t/f_c)\hat{K}_{s00}^{B}$, which is the banana result implied by $\nu_{\rm ref}^{B}\tau_{ss}K_{s00}^{B}$. Similarly, in the asymptotic Pfirsch-Schlüter regime ($\nu_{*s} \gg \epsilon^{-3/2}$ or $\omega_{ts}\tau_{ss} \gg 1$) the μ_s in (B14) reduces to $(3T_s\tau_{ss}^2/m_s)\hat{K}_{s00}^{PS}\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle/\langle B_0^2\rangle$, which in turn yields $3\,\eta_{00}^{s}\tau_{ss}\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle/(m_sn_s\langle B_0^2\rangle)$ in which the η_{00}^{s} coefficients are the generalized Braginskii coefficients given in (A6).

For extended MHD simulations it is convenient to specify the FSA of the residual parallel viscous force in terms of the damping of the poloidal flow to an "offset" flow velocity U_θ^0

in the form (for each plasma species) indicated in (18) and (19). Thus, it is convenient to write the first row of the matrix equation in (B1) as

$$\langle \mathbf{B}_0\!\cdot\!\nabla\!\cdot\!\boldsymbol{\pi}\rangle = mn\mu_{00}\langle B_0^2\rangle\left(U_\theta - U_\theta^0\right), \quad \boxed{U_\theta^0 \equiv -\left(\mu_{01}/\mu_{00}\right) Q_\theta, \ \text{offset poloidal flow.}} \tag{B15}$$

A multi-collisionality formula for the relative coefficient μ_{01}/μ_{00} of the poloidal heat flow Q_θ can be constructed similarly for each species from the coefficients in Table 1:

$$\boxed{\frac{\mu_{s01}}{\mu_{s00}} = \frac{5}{2} - \frac{\hat{K}^{\rm tot}_{s01}}{\hat{K}^{\rm tot}_{s00}} = \frac{5}{2} - \frac{\hat{K}^B_{s01}}{\hat{K}^B_{s00}}\,\frac{1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}^B_{s00}/\hat{K}^P_{s00}}{1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}^B_{s01}/\hat{K}^P_{s01}}\,\frac{1+2\hat{K}^P_{s00}/(3\,\omega_{ts}\tau_{ss}\hat{K}^{PS}_{s00})}{1+2\hat{K}^P_{s01}/(3\,\omega_{ts}\tau_{ss}\hat{K}^{PS}_{s01})}.} \tag{B16}$$

To lowest order in $\sqrt{\epsilon}$, for hydrogenic ions and no impurities ($Z_{\rm eff}\to 1,\ Z_*\to 0$), the ratio μ_{i01}/μ_{i00} is 1.17, -0.5 and -1.6 in the asymptotic banana, plateau and Pfirsch-Schlüter (PS) collisionality regimes, respectively. The value of -1.6 disagrees with the ratio of -2.1 quoted just after Eq. (6.134) in [7], but is consistent with the values inferred from the ratio of the first two rows of Table I in [8] in the the PS limit where $\mu_{i01}/\mu_{i00} \equiv 5/2 - \hat{K}^{PS}_{i01}/\hat{K}^{PS}_{i00}$.

These coefficients for the poloidal flow damping frequency μ and ratio μ_{01}/μ_{00} in the offset poloidal flow yield numerically correct FSA parallel viscous forces (B1) and (17) in the asymptotic limits of the banana and Pfirsch-Schlüter collisionality regimes. However, they may overestimate them slightly in intermediate collisionality regimes — without the $\nu_*^{1/2}$ boundary layer effect the μ value can be a factor of order 1.4 too large in the plateau collisionality regime (see Fig. 1 in [8]).

In the large aspect ratio limit $\sqrt{\epsilon} \ll 1$ the viscous damping frequency μ and offset flow coefficient μ_{01}/μ_{00} can be simplified. Namely, using the definition in (B14) and the specifications of the coefficients in Table I, one obtains for electrons ($\nu_e\tau_{ee} = Z_{\rm eff}$)

$$\boxed{\mu_e} \simeq \frac{1.46\sqrt{\epsilon}(1+\frac{0.533}{Z_{\rm eff}})\,\nu_e}{\left[1+\nu_{*e}^{1/2}+1.65(1+\frac{0.533}{Z_{\rm eff}})\nu_{*e}\right]\left[1+1.18\frac{2.4Z_{\rm eff}^2+5.32Z_{\rm eff}+2.225}{Z_{\rm eff}(4.25Z_{\rm eff}+3.02)}\,\epsilon^{3/2}\nu_{*e}\right]}$$

$$\overset{Z_{\rm eff}=2.5}{\simeq} \boxed{\frac{1.77\sqrt{\epsilon}\,\nu_e}{(1+\nu_{*e}^{1/2}+2\,\nu_{*e})\left(1+1.06\,\epsilon^{3/2}\nu_{*e}\right)}.} \tag{B17}$$

The corresponding electron coefficient for the offset poloidal flow is

$$\boxed{\frac{\mu_{e01}}{\mu_{e00}}} \simeq \frac{5}{2} - \frac{Z_{\rm eff}+0.707}{Z_{\rm eff}+0.533}\,\frac{1+\nu_{*e}^{1/2}+1.65(1+\frac{0.533}{Z_{\rm eff}})\nu_{*e}}{1+\nu_{*e}^{1/2}+0.55(1+\frac{0.707}{Z_{\rm eff}})\nu_{*e}}\,\frac{1+1.18\frac{2.4Z_{\rm eff}^2+5.32Z_{\rm eff}+2.225}{Z_{\rm eff}(4.25Z_*+3.02)}\epsilon^{3/2}\nu_{*e}}{1+3.54\frac{2.4Z_{\rm eff}^2+5.32Z_{\rm eff}+2.225}{Z_{\rm eff}(20.13Z_{\rm eff}+12.43)}\epsilon^{3/2}\nu_{*e}}$$

$$\overset{Z_{\rm eff}=2.5}{\simeq} \boxed{\frac{5}{2} - 1.06\,\frac{1+\nu_{*e}^{1/2}+2\,\nu_{*e}}{1+\nu_{*e}^{1/2}+0.70\,\nu_{*e}}\,\frac{1+1.06\,\epsilon^{3/2}\nu_{*i}}{1+0.69\,\epsilon^{3/2}\nu_{*e}}.} \tag{B18}$$

The last approximate forms are appropriate for most tokamak plasmas where $Z_{\rm eff}$ is often in the range of 2–3. The electron collision frequency ν_e is given in (A9) and the electron collisionality parameter ν_{*e} is defined in (B7) with $\nu \to \nu_e \equiv Z_{\rm eff}/\tau_{ee}$, $v_T \to v_{Te} \equiv \sqrt{2T_e/m_e}$ and $\lambda \to \lambda_e \equiv v_{Te}/\nu_e$, which is given in (A10).

The corresponding ion viscous poloidal damping frequency is given by ($\nu_i \tau_{ii} = Z_* + 1/\sqrt{2}$)

$$\boxed{\mu_i} \;\simeq\; \frac{1.46\sqrt{\epsilon}\,\frac{Z_*+0.533}{Z_*+0.707}\,\nu_i}{\left[1+\nu_{*i}^{1/2}+1.65\frac{Z_*+0.533}{Z_*+0.707}\nu_{*i}\right]\left[1+1.18\frac{2.4Z_*^2+5.32Z_*+2.225}{(Z_*+0.707)(4.25Z_*+3.02)}\,\epsilon^{3/2}\nu_{*i}\right]}$$
$$\stackrel{Z_*=3}{\simeq} \boxed{\frac{1.32\sqrt{\epsilon}\,\nu_i}{(1+\nu_{*i}^{1/2}+1.49\,\nu_{*i})\left(1+0.80\,\epsilon^{3/2}\nu_{*i}\right)}}. \tag{B19}$$

The corresponding ion coefficient for the offset poloidal flow is

$$\boxed{\frac{\mu_{i01}}{\mu_{i00}}} \;\simeq\; \frac{5}{2}-\frac{Z_*+0.707}{Z_*+0.533}\,\frac{1+\nu_{*i}^{1/2}+1.65\frac{Z_*+0.533}{Z_*+0.707}\nu_{*i}}{1+\nu_{*i}^{1/2}+0.55\nu_{*i}}\,\frac{1+1.18\frac{2.4Z_*^2+5.32Z_*+2.225}{(Z_*+0.707)(4.25Z_*+3.02)}\epsilon^{3/2}\nu_{*i}}{1+3.54\frac{2.4Z_*^2+5.32Z_*+2.225}{(Z_*+0.707)(20.13Z_*+12.43)}\epsilon^{3/2}\nu_{*i}}$$
$$\stackrel{Z_*=3}{\simeq} \boxed{\frac{5}{2}-1.05\,\frac{1+\nu_{*i}^{1/2}+1.57\,\nu_{*i}}{1+\nu_{*i}^{1/2}+0.55\,\nu_{*i}}\,\frac{1+0.80\,\epsilon^{3/2}\nu_{*i}}{1+0.52\,\epsilon^{3/2}\nu_{*i}}}. \tag{B20}$$

The last approximate forms are appropriate for most tokamak plasmas where Z_* is often in the range of 2–4. The ion collision frequency ν_i is given in (A14) and the ion collisionality parameter ν_{*i} is defined in (B7) with $\nu \to \nu_i \equiv (Z_* + 1/\sqrt{2})/\tau_{ii}$, $v_T \to v_{Ti} \equiv \sqrt{2T_i/m_i}$ and $\lambda \to \lambda_i \equiv v_{Ti}/\nu_i$, which is given in (A15).

Finally, formulas for these quantities will be specified for an impurity (subscript I) ion species. To do so, consider a typical case where there is only one dominant impurity ion species, which is often carbon in present tokamak plasmas. Thus, a plasma that is composed of electrons, hydrogenic ions (e.g., deuterons with $Z_i = 1$ and $m_D/m_H = 2$) and one species of impurity ions (e.g., carbon with $Z_I = 6$ and $m_I/m_D = 6$) will be considered. For this type of plasma the impurity collision frequency is typically dominated by impurity-impurity collisions. [Impurity-electron collisions are negligible for $Z_* n_i/n_e \gg (2m_e/m_D)^{1/2} \sim 1/43$ and impurity-ion collisions are negligible for $Z_* \equiv n_I Z_I^2/n_i \gg (2m_D/m_I)^{1/2} \sim 0.6$.] Hence, the impurity collision frequency can be referenced to the ion collision frequency in (A14) as

$$\nu_I \;=\; Z_I^2\left(\frac{Z_*}{1/\sqrt{2}+Z_*}\right)\left(\frac{T_i}{T_I}\right)^{3/2}\left(\frac{m_i}{m_I}\right)^{1/2}\nu_i \;\simeq\; Z_I^2\left(\frac{m_i}{m_I}\right)^{1/2}\nu_i. \tag{B21}$$

And the impurity neoclassical collisionality parameter can be written in terms of ν_{*i} as

$$\nu_{*I} \;=\; Z_I^2\left(\frac{Z_*}{1/\sqrt{2}+Z_*}\right)\left(\frac{T_i}{T_I}\right)^{2}\nu_{*i} \;\simeq\; Z_I^2\,\nu_{*i}. \tag{B22}$$

Thus, impurity ions are usually much more collisional than the hydogenic ions — by a factor of $\nu_{*I}/\nu_{*i} \sim Z_I^2 \sim 36$ for carbon ions with $T_I \simeq T_i$ and $Z_* \gg 1$. With these specifications the impurity poloidal flow damping frequency μ_I and impurity ratio μ_{I01}/μ_{I00} coefficient for the offset poloidal flow are obtained from the corresponding ion formulas in (B19) and (B20) using $\nu_i \to \nu_I$, $\nu_{*i} \to \nu_{*I}$ and $Z_* \to 1/Z_*$ (see $Z \to 1/Z_*$ note in caption of Table I).

Appendix C: Collisional Friction And Viscous Forces And Their Effects

In order to determine the offset poloidal flow U_θ^0 defined in (B15), consider next the determination of the poloidal heat flow Q_θ, for electrons, hydrogenic ions and impurities. In general this requires [8, 9] the simultaneous solution of the FSA parallel momentum and heat momentum (flux) equations for electrons and separately for hydrogenic and impurity ions. In transport codes these are often evaluated using the NCLASS code [10]. Here, approximate analytic results will be presented to illustrate how these heat flows are determined and their consequences. The key asssumption made [9] to obtain these approximate results is that the flow velocities of the impurities are the same as those of the hydrogenic ions. The conditions for validity of this approximation are discussed at the end of this appendix.

Analogous to (B1), the friction and heat friction forces for a species s of particles flowing with velocity $\mathbf{V}_s$ colliding with a heavier ($m_{s'} \gg m_s$) species s' flowing with velocity $\mathbf{V}_{s'}$ can be written in the matrix form [8, 26]

$$\begin{bmatrix} \mathbf{R}_{\rm V}^{s/s'} \\ \mathbf{R}_{\rm q}^{s/s'} \end{bmatrix} \simeq -\frac{m_s n_s}{\tau_{ss}} \left(\frac{n_{s'} Z_{s'}}{n_s} \right) \begin{bmatrix} Z_{s'} & \frac{3}{2} Z_{s'} \\ \frac{3}{2} Z_{s'} & \sqrt{2} + \frac{13}{4} Z_{s'} \end{bmatrix} \begin{bmatrix} \mathbf{V}_s - \mathbf{V}_{s'} \\ \frac{-2}{5nT} \mathbf{q}_s \end{bmatrix}. \qquad \text{(C1)}$$

The total collisional friction and heat friction forces on a given species s are determined by summing over all species s', including the species s:

$$\mathbf{R}_{s\rm V} \equiv \sum_{s'} \mathbf{R}_{\rm V}^{s/s'}, \quad \mathbf{R}_{s\rm q} \equiv \sum_{s'} \mathbf{R}_{\rm q}^{s/s'}. \qquad \text{(C2)}$$

Consider next the poloidal flow $U_{s\theta}$ and heat flow $Q_{s\theta}$. At first order in the gyroradius the flows lie within a flux surface. They are a combination of the parallel flow velocity $\mathbf{V}_{s\|} \equiv \mathbf{B}_0\,(\mathbf{B}_0 \cdot \mathbf{V}_s)/B_0^2$ and the $\mathbf{V}_E = \mathbf{E} \times \mathbf{B}_0/B_0^2 = \mathbf{B}_0 \times \boldsymbol{\nabla}\Phi_0/B_0^2$ and diamagnetic $\mathbf{V}_{s*} \equiv \mathbf{B}_0 \times \boldsymbol{\nabla} p_s/(n_s q_s B_0^2)$ flow velocities. Thus, for a species s with pressure $p_s \equiv n_s T_s$ the poloidal flow function defined in (16) is (see Eqs. (3.43), (3.45), (3.47) in [8] and (A13), (A16) in [9])

$$U_{s\theta} \equiv \frac{\mathbf{V}_s \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{(\mathbf{V}_{s\|} + \mathbf{V}_E + \mathbf{V}_{s*}) \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{\mathbf{B}_0 \cdot \mathbf{V}_s}{B_0^2} + \frac{I}{B_0^2} \left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_s q_s} \frac{dp_s}{d\psi_p} \right). \qquad \text{(C3)}$$

Multiplying this equation by $n_s q_s B_0^2$, summing over species using the quasineutrality condition $\sum_s n_s q_s = 0$, and averaging over a flux surface yields a relation for the FSA parallel current $\mathbf{J} \equiv \sum_s n_s q_s \mathbf{V}_s$ in the plasma:

$$\langle \mathbf{B}_0 \cdot \mathbf{J} \rangle = \langle B_0^2 \rangle \sum_s n_s q_s U_{s\theta} - I \, \frac{dP}{d\psi_p}, \tag{C4}$$

in which $P \equiv \sum_s p_s$ is the total plasma pressure. Assuming that the ion and impurity flow velocities are approximately equal and using the fact that from quasineutrality $n_i + \sum_I n_I Z_I = n_e$, this last relation yields a relation for the poloidal electron flow function:

$$n_e e \langle B_0^2 \rangle U_{e\theta} = - \, \langle \mathbf{B}_0 \cdot \mathbf{J} \rangle - I \, \frac{dP}{d\psi_p} + n_e e \langle B_0^2 \rangle U_{i\theta}. \tag{C5}$$

The poloidal heat flow function is $Q_\theta \equiv (-2/5nT) \, \mathbf{q} \cdot \boldsymbol{\nabla}\theta / \mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta$. The total heat flow within the flux surface is $\mathbf{q} \equiv \mathbf{B}_0 \, (\mathbf{B}_0 \cdot \mathbf{q})/B_0^2 + \mathbf{q}_\wedge$ in which the diamagnetic heat flow is $\mathbf{q}_\wedge \equiv (5nT/2) \mathbf{B} \times \boldsymbol{\nabla} T / q B_0^2$. Thus, the FSA of B_0^2 times the poloidal heat flow for each species s is (see Eqs. (3.44), (3.46), (3.48) in [8], (A13), (A15) in [9]) or Eq. (13) in [12]

$$\langle B_0^2 \rangle Q_{s\theta}(\psi_p) \equiv \frac{-2}{5 n_s T_s} \langle B_0^2 \rangle \frac{\mathbf{q}_s \cdot \boldsymbol{\nabla}\theta}{\mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta} = \frac{-2}{5 n_s T_s} \langle \mathbf{B}_0 \cdot \mathbf{q}_s \rangle - \frac{I}{q_s} \frac{dT_s}{d\psi_p}. \tag{C6}$$

These formulas will be used to determine the poloidal electron heat flow $Q_{e\theta}$ and parallel neoclassical Ohm's law in a plasma composed of electrons, hydrogenic ions and impurity ions. The coupled FSA parallel momentum and heat flow equations for electrons are

$$0 \; = \; - \, n_e e \langle \mathbf{B}_0 \cdot \mathbf{E}^A \rangle + \langle \mathbf{B}_0 \cdot \mathbf{R}_{e\mathrm{V}} \rangle - \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\|} \rangle, \tag{C7}$$

$$0 \; = \; \langle \mathbf{B}_0 \cdot \mathbf{R}_{e\mathrm{q}} \rangle - \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \Theta_{e\|} \rangle. \tag{C8}$$

Here, the electron collisional friction $\mathbf{R}_{e\mathrm{V}}$ and heat friction $\mathbf{R}_{e\mathrm{q}}$ are caused by collisions of electrons with electrons, hydrogenic ions and impurity ions, as defined in (C2). Thus, assuming again that the impurity flow velocity is approximately equal to the hydrogenic ion flow velocity, the friction and heat friction forces can be written in the matrix form

$$\begin{bmatrix} \mathbf{R}_{e\mathrm{V}} \\ \mathbf{R}_{e\mathrm{q}} \end{bmatrix} = - \, \frac{m_e n_e}{\tau_{ee}} \, \mathsf{N}_e \cdot \begin{bmatrix} \frac{-1}{n_e e} \mathbf{J} \\ \frac{-2}{5 n_e T_e} \mathbf{q}_e \end{bmatrix}, \quad \mathsf{N}_e \equiv \begin{bmatrix} \nu_{e00} & \nu_{e01} \\ \nu_{e01} & \nu_{e11} \end{bmatrix} = \begin{bmatrix} Z_{\mathrm{eff}} & \frac{3}{2} Z_{\mathrm{eff}} \\ \frac{3}{2} Z_{\mathrm{eff}} & \sqrt{2} + \frac{13}{4} Z_{\mathrm{eff}} \end{bmatrix}. \tag{C9}$$

The corresponding FSA electron parallel viscous forces from (B1) are

$$\begin{bmatrix} \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\|} \rangle \\ \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \Theta_{e\|} \rangle \end{bmatrix} \equiv \frac{m_e n_e}{\tau_{ee}} \, \mathsf{M}_e \cdot \begin{bmatrix} \langle B_0^2 \rangle U_{e\theta} \\ \langle B_0^2 \rangle Q_{e\theta} \end{bmatrix}, \tag{C10}$$

in which the matrix M_e of viscosity coefficients from (B3) and (B13) for electrons is

$$\mathsf{M}_e \equiv \begin{bmatrix} \mu_{e00} & \mu_{e01} \\ \mu_{e01} & \mu_{e11} \end{bmatrix} = (\nu_e \tau_{ee}) \frac{f_t}{f_c} \begin{bmatrix} K^{\rm tot}_{e00} & \frac{5}{2} K^{\rm tot}_{e00} - K^{\rm tot}_{e01} \\ \frac{5}{2} K^{\rm tot}_{e00} - K^{\rm tot}_{e01} & \hat{K}^{\rm tot}_{e11} - 5\,\hat{K}^{\rm tot}_{e01} + \frac{25}{4} \hat{K}^{\rm tot}_{e00} \end{bmatrix}$$
$$\sim\ 1.46\sqrt{\epsilon} \begin{bmatrix} 0.533 + Z_{\rm eff} & 0.625 + \frac{3}{2} Z_{\rm eff} \\ 0.625 + \frac{3}{2} Z_{\rm eff} & 1.386 + \frac{13}{4} Z_{\rm eff} \end{bmatrix}. \qquad \text{(C11)}$$

The last form indicates the form of the M_e coefficients in the asymptotic low collisionality ($\nu_{*e} \to 0$) and large aspect ratio ($\sqrt{\epsilon} \ll 1$) limit.

Using (C3)–(C6) and (C9), (C10) in (C7), (C8) and $q_e = -\,e$, the matrix equation to be solved for the FSA parallel current $\langle \mathbf{B}_0 \cdot \mathbf{J} \rangle$ and electron heat flow $\langle \mathbf{B}_0 \cdot \mathbf{q}_e \rangle$ becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = -\,n_e e \begin{bmatrix} \langle \mathbf{B}_0 \cdot \mathbf{E}^A \rangle \\ 0 \end{bmatrix} + \frac{m_e n_e}{\tau_{ee}} \left([\mathsf{N}_e + \mathsf{M}_e] \cdot \begin{bmatrix} (1/n_e e)\, \langle \mathbf{B}_0 \cdot \mathbf{J} \rangle \\ (2/5 n_e T_e) \langle \mathbf{B}_0 \cdot \mathbf{q}_e \rangle \end{bmatrix} \right.$$
$$\left. + \, \mathsf{M}_e \cdot \begin{bmatrix} (I/n_e e)\, dP/d\psi_p - \langle B_0^2 \rangle U_{i\theta} \\ -\,(I/e)\, dT_e/d\psi_p \end{bmatrix} \right). \qquad \text{(C12)}$$

This matrix equation is solved for the FSA parallel plasma current and electron heat flow by taking the inner product of the inverse matrix $[\mathsf{N}_e + \mathsf{M}_e]^{-1}$ with it:

$$\begin{bmatrix} \langle \mathbf{B}_0 \cdot \mathbf{J} \rangle \\ (2e/5T_e) \langle \mathbf{B}_0 \cdot \mathbf{q}_e \rangle \end{bmatrix} = \frac{n_e e^2 \tau_{ee}}{m_e} [\mathsf{N}_e + \mathsf{M}_e]^{-1} \cdot \begin{bmatrix} \langle \mathbf{B}_0 \cdot \mathbf{E}^A \rangle \\ 0 \end{bmatrix}$$
$$-\ \big[[\mathsf{N}_e + \mathsf{M}_e]^{-1} \cdot \mathsf{M}_e \big] \cdot \begin{bmatrix} I\, dP/d\psi_p - n_e e \langle B_0^2 \rangle U_{i\theta} \\ -\,(n_e I)\, dT_e/d\psi_p \end{bmatrix}. \qquad \text{(C13)}$$

The first row of this matrix equation yields the parallel neoclassical Ohm's law. It will be written in terms of an equation for the parallel inductive electric field induced by the parallel current because MHD "owns" $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}/\mu_0$ since it evolves the magnetic field $\mathbf{B}$:

$$\boxed{\langle \mathbf{B}_0 \cdot \mathbf{E}^A \rangle = \eta_\parallel^{\rm nc} \left(\langle \mathbf{B}_0 \cdot \mathbf{J} \rangle - \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle \right),} \quad \text{neoclassical parallel Ohm's law.} \qquad \text{(C14)}$$

If there are current-drive sources from external sources (e.g., for electron cyclotron current drive) or fluctuations (dynamo) they can be added to this equation via [12]

$$\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle \to \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm drives} \rangle \equiv \langle \mathbf{B}_0 \cdot (\mathbf{J}_{\rm bs} + \mathbf{J}_{\rm CD} + \mathbf{J}_{\rm dyn}) \rangle. \qquad \text{(C15)}$$

The current-drive (subscript CD) and dynamo (subscript dyn) currents are defined in [12].

In (21) the neoclassical parallel electrical resistivity $\eta_\parallel^{\rm nc}$ is defined by

$$\boxed{\eta_\parallel^{\rm nc} \equiv \frac{m_e}{n_e e^2 \tau_{ee}} \frac{1}{[\mathsf{N}_e + \mathsf{M}_e]^{-1}_{00}} \overset{|\mathsf{N}_e| \gg |\mathsf{M}_e|}{\Longrightarrow} \frac{m_e}{n_e e^2 \tau_{ee}} \frac{1}{[\mathsf{N}_e]^{-1}_{00}} = \frac{m_e \nu_e}{n_e e^2} \frac{\sqrt{2} + Z_{\rm eff}}{\sqrt{2} + (13/4) Z_{\rm eff}} \equiv \frac{1}{\sigma_\parallel^{\rm Sp}}.} \tag{C16}$$

As indicated at the end, when viscosity effects are negligible the parallel neoclassical resistivity reduces to the inverse of the Spitzer electrical conductivity. This formula for the Spitzer conductivity is typically accurate to within about 1 % for $Z_{\rm eff} \sim$ 1–4, but incorrect by about 5 % for $Z_{\rm eff} \to \infty$. Greater accuracy can be obtained by including "energy-weighted heat flow" effects and inverting the resultant 3×3 matrix equation. However, such an expanded treatment is usually not warranted because the intrinsic uncertainty in the Coulomb collision operator is $\sim 1/\ln\Lambda \sim 1/17 \simeq 6$ %, which is larger than the errors in (C16). Neglecting electron heat flow effects, the N_e and M_e matrices reduce to just their 00 elements; in this limit the neoclassical parallel resistivity becomes simply $\eta_\parallel^{\rm nc} \simeq \eta_\perp (1 + \mu_{e00}/\nu_{e00})$, in which $\eta_\perp \equiv (m_e \nu_e)/(n_e e^2)$ is the perpendicular plasma resistivity and $\mu_{e00} \sim 1.46\sqrt{\epsilon}\,(0.533 + Z_{\rm eff})$.

The FSA bootstrap current in (C14) is defined by

$$\boxed{\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle = \big[\, [\mathsf{N}_e + \mathsf{M}_e]^{-1} \cdot \mathsf{M}_e \big]_{00} \left(-I \frac{dP}{d\psi_p} + n_e e \langle B_0^2 \rangle U_{i\theta} \right) + \big[\, [\mathsf{N}_e + \mathsf{M}_e]^{-1} \cdot \mathsf{M}_e \big]_{01} \left(n_e I \frac{dT_e}{d\psi_p} \right).} \tag{C17}$$

Neglecting electron heat flow effects so the N_e and M_e matrices simplify to their 00 elements, the poloidal ion flow $U_{i\theta}$, and electron temperature gradient effects, to lowest order in the asymptotic banana collisionality and large aspect ratio regime the bootstrap current becomes

$$\langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm bs} \rangle \sim -\frac{\mu_{e00}}{\nu_{e00} + \mu_{e00}} I \frac{dP}{d\psi_p} \sim -\frac{1.46\sqrt{\epsilon}\,(0.533 + Z_{\rm eff})}{Z_{\rm eff} + 1.46\sqrt{\epsilon}\,(0.533 + Z_{\rm eff})} \frac{B_0}{B_\theta} \frac{dP}{dr}. \tag{C18}$$

Now that the FSA neoclassical parallel Ohm's law has been specified, the Braginskii-derived Ohm's law can be modified so that it obtains the form given in (C14) for time scales longer than the electron collision time. The total electron momentum equation is given by

$$m_e n_e \, d\mathbf{V}_e/dt = -\, n_e e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) - \boldsymbol{\nabla} p_e - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e + \mathbf{R}_{e\rm V}. \tag{C19}$$

Using $\mathbf{V}_e \equiv \mathbf{V}_i - \mathbf{J}/n_e e$ with $\mathbf{V}_i \simeq \mathbf{V}$ (plasma flow velocity) and (1) for representing the components of the electron viscous force $\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e$, the electron momentum equation can be rewritten as a general Ohm's law:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{\mathbf{J} \times \mathbf{B} - \boldsymbol{\nabla} p_e - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\wedge}}{n_e e} + \frac{\mathbf{R}_{e\rm V} - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{e\parallel}}{n_e e} - \frac{m_e}{e} \frac{d\mathbf{V}_e}{dt}. \tag{C20}$$

Here, the perpendicular electron viscous force $\nabla\cdot\boldsymbol{\pi}_{e\perp}$ has been neglected because it is always negligible. For time scales longer than the electron collision time $1/\nu_e$ the inertia term can neglected. Then, for $\mathbf{B}\simeq\mathbf{B}_0$ the dominant contributions to the FSA of the parallel ($\mathbf{B}_0\cdot$) component of this equation involves the inductive electric field $\mathbf{E}^A$, friction force $\mathbf{R}_{e\mathrm{V}}$ and viscous force due to electron parallel stresses $\nabla\cdot\boldsymbol{\pi}_{e\|}$. (The electron gyroviscous stress force $\nabla\cdot\boldsymbol{\pi}_{e\wedge}$ is higher order in the gyroradius expansion and can be neglected.) The resultant FSA parallel equation is simply (C7), which yields the parallel neoclassical Ohm's law given by (21). However, the perpendicular friction force yields the usual result

$$\mathbf{R}_{e\mathrm{V}\perp}=n_e e\,\eta_\perp\left(\mathbf{J}_\perp-\frac{3}{2}\frac{\mathbf{B}\times\nabla T_e}{eB^2}\right),\quad \mathbf{J}_\perp\equiv-\frac{\mathbf{B}\times(\mathbf{B}\times\mathbf{J})}{B^2}. \tag{C21}$$

These effects are all assembled in the general Ohm's law given in (22).

The ion parallel momentum and heat flow equations will be considered next in order to determine the poloidal ion heat flow $Q_{i\theta}$ en route to determining the "offset" ion poloidal flow function $U^0_{i\theta}$. As indicated in the caption of Table I, the ion friction and viscosity matrices N_i and M_i can be obtained by changing Z_{eff} to Z_* in the corresponding electron matrices. However, when the hydrogenic and impurity ions are assumed to have the same flow velocities there is no frictional force $\mathbf{R}_{i\mathrm{V}}$ between the ion species, although there is an ion heat friction force $\mathbf{R}_{i\mathrm{q}}$; this is taken into account by setting $Z\rightarrow Z_*=0$ in the friction matrix. Thus, the ion friction and viscosity coefficient matrices become simply

$$\mathsf{N}_i=\begin{bmatrix}\nu_{i00} & \nu_{i01}\\ \nu_{i01} & \nu_{i11}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & \sqrt{2}\end{bmatrix}, \tag{C22}$$

$$\mathsf{M}_i=\begin{bmatrix}\mu_{i00} & \mu_{i01}\\ \mu_{i01} & \mu_{i11}\end{bmatrix}\sim(\nu_i\tau_{ii})\,\frac{f_t}{f_c}\begin{bmatrix}0.533+Z_* & 0.625+1.5Z_*\\ 0.625+1.5Z_* & 1.386+3.25Z_*\end{bmatrix}. \tag{C23}$$

As usual, the last form of M_i indicates the banana collsionality regime ($\nu_{*i}\ll 1$) coefficients.

The total FSA plasma parallel force balance equation [8, 11] has no parallel friction force $\mathbf{R}_{i\mathrm{V}}$ and hence to lowest order (in $\sqrt{m_e/m_D}\sim 1/60\ll 1$) is simply

$$0=-\left\langle\mathbf{B}_0\cdot\nabla\cdot\boldsymbol{\pi}_{i\|}\right\rangle=-\left(m_i n_i/\tau_{ii}\right)\left\langle B_0^2\right\rangle\left(\mu_{i00}U_{i\theta}+\mu_{i01}Q_{i\theta}\right). \tag{C24}$$

The FSA parallel heat flow equation for the hydrogenic ions is

$$0=\left\langle\mathbf{B}_0\cdot\mathbf{R}_{i\mathrm{q}}\right\rangle-\left\langle\mathbf{B}_0\cdot\nabla\cdot\boldsymbol{\pi}_{i\|}\right\rangle=-\frac{m_i n_i}{\tau_{ii}}\left[\nu_{i11}\left(\frac{-2}{5n_iT_i}\right)\left\langle\mathbf{B}_0\cdot\mathbf{q}_i\right\rangle+\left\langle B_0^2\right\rangle\left(\mu_{i01}U_{i\theta}+\mu_{i11}Q_{i\theta}\right)\right]. \tag{C25}$$

Using the definition of the FSA parallel ion heat flow in (C6) and the solution from (C24) of $U_{i\theta} = -(\mu_{i01}/\mu_{i00})\, Q_{i\theta}$, this equation can be solved for the poloidal ion heat flow function:

$$Q_{i\theta} = c_{\#} \frac{I}{q_i \langle B_0^2 \rangle} \frac{dT_i}{d\psi_p}, \quad \boxed{c_{\#} \equiv \frac{1}{1 + (\mu_{i11} - \mu_{i01}^2/\mu_{i00})/\nu_{i11}}.} \tag{C26}$$

Various limits of the numerical coefficient $c_{\#}$ are of interest. When viscosity effects are negligible, $c_{\#} = 1$. For a pure electron-ion plasma (i.e., no impurities, $Z_* \to 0$) in the $\nu_{*i} \ll 1$ collisionality regime, one obtains $c_{\#} = 1/[1 + 0.46(f_t/f_c)]$. When ion-impurity collisions are dominant (i.e., when $Z_* \gg 1$) in the $\nu_{*i} \ll 1$ collisionality regime, $c_{\#} = 1/[1 + 0.71(f_t/f_c)]$.

Since the poloidal ion heat flow has been determined, the offset poloidal ion flow function defined in (B15) can now be specified:

$$\boxed{U_{i\theta}^0 = k_i \frac{I}{q_i \langle B_0^2 \rangle} \frac{dT_i}{d\psi_p}, \quad k_i \equiv c_{\#} \frac{\mu_{i01}}{\mu_{i00}},} \quad \text{final offset poloidal ion flow.} \tag{C27}$$

The constant k_i is the usual neoclassical factor [7] for the poloidal ion flow. For a pure electron-ion plasma (i.e., $Z_* = 0$) in the asymptotic banana collisionality ($\nu_{*i} \ll 1$) and large aspect ratio ($\sqrt{\epsilon} \ll 1$, $f_t/f_c \simeq 1.46\sqrt{\epsilon} \ll 1$) regime it is $1.17/(1 + 0.67\sqrt{\epsilon}\,)$.

The preceding analysis has assumed that impurity flow velocities are approximately equal to the hydrogenic ion flow velocities. However, in general one should solve simultaneously for the impurity flow velocities and the hydrogenic ion flow velocities from their respective FSA parallel momentum and heat flow equations, allowing for differences in them. This is what the NCLASS code [10] does. In doing so one finds in general that the hydrogenic and impurity ion flows and hence the offset $U_{i\theta}^0$ depend on the impurity density and temperature gradients. However, since impurity collision frequencies are usually much larger than ion collision frequencies, impurities have much higher collisionalities ($\nu_{*I}/\nu_{*i} \sim Z_I^2 \gg 1$); hence, they are often in the plateau or even Pfirsch-Schlüter collisionality regimes. There, their viscous damping effects become negligible; then, it can be shown [9] that to lowest order the offset poloidal ion flow $U_{i\theta}^0$ is solely proportional to the ion temperature gradient. An approximate criterion for the validity of (19) and (C27) with the coefficient k_i determined mainly by the collisionality regime of the hydrogenic ions results from requiring [9] that the impurity viscous force be negligible compared to the impurity-ion friction force:

$$|\mathsf{N}_I| \gg |\mathsf{M}_I|, \quad \text{which for plateau regime impurities is} \quad \nu_{*I} \gtrsim f_t/f_c \sim 1.46\sqrt{\epsilon}. \tag{C28}$$

Thus, impurities effectively must be in the plateau or Pfirsch-Schlüter collisionality regime.

Very deep in the ion banana collisionality regime (i.e., $\nu_{*i} \ll 1$) one can have $\nu_{*I} < 1$; then, the impurity ion density and temperature gradient effects should be taken into account by using NCLASS [10] to obtain the offset flows $U^0_{i\theta}$. However, this very low hydrogenic ion collisionality regime is only barely reached in most present tokamak plasmas. When impurity flows become important one should really be solving three-fluid equations that include the impurity density, momentum, energy and heat flow equations. For the purpose of extended MHD codes it will be assumed that the offset poloidal ion flows $U^0_{i\theta}$ can be represented in terms of the ion temperature gradients as indicated in (19), (B15) and (C27).

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