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Unified theory of resistive and inertial ballooning modes in three-dimensional configurations

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Abstract

Analytic results for the stability of resistive ballooning modes (RBMs) and electron inertial ballooning modes are obtained using a two-scale analysis. This work generalizes previous calculations used for axisymmetric $\hat{s}-\alpha$ geometry [R.H. Hastie, J.J. Ramos and F. Porcelli *Phys.Plasmas* **10,** 4405 (2003)] to general three-dimensional geometry. A unified theory is developed for RBMs and inertial ballooning modes, in which the effects of both ideal magnetohydrodynamic (MHD) free energy (as measured by the asymptotic matching parameter Δ') and geodesic curvature drives in the non-ideal layer are included in the dispersion relation. This unified theory can be applied to determine the stability of drift-resistive-inertial ballooning modes in the low temperature edge regions of tokamak and stellarator plasmas where steep density gradients exist.

I. Introduction

The ion-temperature-gradient (ITG) and collisionless trapped electron (TE) modes are primary candidates for explaining the anomalous transport in the core of fusion plasmas. There are also indications that TE modes are responsible for core transport in the Helically Symmetric Experiment (HSX)[1]. The situation at the edge is different where plasma is influenced by collisions suggesting that resistive modes may be an important source of the turbulence. One of the known plasma modes believed to be responsible for the edge turbulence is the resistive ballooning mode (RBM), which appears to be quite a robust instability for typical plasma edge parameters. The RBM is a pressure gradient driven mode that is localized to regions with unfavorable magnetic field lines curvature and is unstable when the electron motion along the field line is strongly impeded by collisions. The RBM is likely to be unstable in the edge region, particularly in the vicinity of the separatrix or a magnetic divertor due to steep density gradients there. It is also natural for RBM to grow in the edge region of tokamak and stellarator plasmas because of the relatively low electron temperature and moderately high plasma density.

RBMs have been extensively studied in axisymmetric tokamaks using linear [2]-[8] and nonlinear theories [9]-[12]. However, the studies of these instabilities in fully three-dimensional geometries has been limited [13]-[18]. In Ref. [15], resistive ballooning modes were studied in general geometry employing the linearized equations of motion of resistive magnetohydrodynamics (RMHD). A multiple-length-scale expansion technique was used based on a small resistivity and growth rate expansion. RMHD equations computed in Ref. [18] for the Wendelstein 7-X configuration are compared and contrasted to the results with Wendelstein 7-AS using the Correra-Restrepo formulation given in Ref. [15]. A general theory that is applicable to 3-D configurations and that avoids the restrictive zero electron inertia assumption (and $\omega \gg \omega_{*en} \gg \omega_\kappa$, where ω is the mode frequency, ω_κ is the curvature drift frequency and ω_{*en} is the electron diamagnetic drift frequency) is not available.

In this paper, the stability criterion for non-ideal magnetohydrodynamic (MHD) ballooning modes is derived, for arbitrary three-dimensional ideal MHD stable electron-ion plasmas. In the presence of non-ideal effects, ballooning instabilities can be produced at plasma β levels far below the critical $\beta[\equiv p/(B^2/8\pi)]$ for the ideal ballooning instability. Electron inertia, diamagnetic flow effects, parallel ion dynamics, transverse particle diffusion and perpendicular gyro-viscous stress terms are included in the derivation. Temperature perturbations and equilibrium temperature gradients are neglected for simplicity. For parameters typical of the Helically Symmetric Experiment (HSX [19]), the characteristic growth rates exceed the electron collision frequency. In this regime, the electron inertial effects can dominate plasma resistivity and produce an instability whose growth rate scales with the electromagnetic skin depth.

A unified theory of RBM and inertial ballooning modes is developed, in which the effects of both the ideal MHD free energy (as measured by the asymptotic matching parameter Δ') and the geodesic curvature drive (in the non-ideal layer) are included in the dispersion relation. This unified theory can be applied to the low temperature steep density gradients edge regions of tokamaks and stellarators. The low temperature and the steep density gradient in the edge region enhance the growth rate of RBMs and overcome the stabilizing influence of magnetic shear.

The organization of this paper is as follows: In Section II, linearized ballooning equations are derived from Ohm's law, vorticity, continuity and total parallel momentum. In Section III, a pair of coupled second order ordinary differential equations are presented that describe shear Alfvén and drift acoustic waves. The shear Alfvén and drift acoustic equations in general 3-D geometry are then rewritten in Hamada coordinates. Section IV is devoted to an analytical study of these equations, using a multiple length scale expansion technique, and to a derivation of the ballooning mode dispersion relation. The conclusions are presented in Section V.

II. Dissipative drift ballooning equations

The reduced Braginskii fluid equations are used for a four-field model of drift resistive ballooning modes with high mode number [for details, see references [20] and [21]]. The equations for the parallel component of the generalized Ohm's law, vorticity, electron continuity and total parallel momentum in the low-β limit can be written in the following linearized form:

$$E_{\parallel} + T_e \nabla_{\parallel} n_e - \eta j_{\parallel} = \frac{m_e}{e^2 n_e} \frac{\partial j_{\parallel}}{\partial t}, \tag{1}$$

$$\nabla \cdot n \left(\mathbf{v}_{pi} + \mathbf{v}_{\pi i} \right) + \nabla \cdot n \left(\mathbf{v}_{Di} - \mathbf{v}_{De} \right) + \nabla \cdot \left(n \mathbf{v}_{pe} \right) + \frac{1}{e} \nabla_{\parallel} j_{\parallel} = 0, \tag{2}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot n\left(\mathbf{v}_E + \mathbf{v}_{De}\right) + \nabla \cdot n\left(\mathbf{v}_{pe} + \mathbf{v}_\eta\right) + \nabla \cdot \left(n\mathbf{v}_{\parallel e}\right) = 0, \tag{3}$$

$$m_i n \left(\frac{\partial}{\partial t}\right) v_{i\parallel} = -\widehat{\mathbf{b}} \cdot \nabla\left(p_i + p_e\right) - \widehat{\mathbf{b}} \cdot (\nabla \cdot \pi_i), \tag{4}$$

where

$$E_\parallel = -\nabla_\parallel \phi - \frac{1}{c}\frac{\partial A_\parallel}{\partial t}, \tag{5}$$

$$\mathbf{v}_E = -\frac{c}{B}\nabla\phi \times \widehat{\mathbf{b}}, \tag{6}$$

$$\mathbf{v}_{Dj} = -\frac{c}{q_j B n}\nabla p_j \times \widehat{\mathbf{b}}, \tag{7}$$

$$\mathbf{v}_{\pi i} = -\frac{c}{eBn}\nabla \cdot \pi_i \times \widehat{\mathbf{b}}, \tag{8}$$

$$\mathbf{v}_\eta = \frac{c}{eBn}\mathbf{R} \times \widehat{\mathbf{b}}, \tag{9}$$

and

$$\mathbf{v}_{pj} = \frac{1}{\omega_{cj}}\left(\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla\right)\mathbf{v}_j \times \widehat{\mathbf{b}}. \tag{10}$$

Here $\widehat{\mathbf{b}} = \mathbf{B}/|B|$ is the unit vector along the magnetic field line, $\omega_{ci} = eB_0/m_i c$ is the ion cyclotron frequency evaluated at the magnetic axis, $j_\parallel = en(v_{\parallel i} - v_{\parallel e})$ is the plasma current parallel to the magnetic field, ϕ is the electrostatic potential, $p = n(T_i + T_e)$ is the isotropic pressure, T_j is the equilibrium temperature of species j (which is assumed to be constant), and π_i is the anisotropic ion stress tensor. The velocity $\mathbf{v}_{\pi i}$ is due to the stress tensor π, which contains a viscosity part and a finite Larmor radius part, (cf. p.20 in reference [6]) and $\mathbf{R}$ denotes the frictional force. The electron stress tensor is neglected and electron and ion densities are both denoted by n. To the lowest order ($\omega \ll \omega_{ci}$), $\mathbf{v}_j = \mathbf{v}_E + \mathbf{v}_{Dj}$ is substituted into the above polarization flow $\mathbf{v}_{pj}$ for species j. However, the total ion and electron velocities used in the Eqs. (2) and (3) are

$$\mathbf{v}_i = \mathbf{v}_E + \mathbf{v}_{Di} + \mathbf{v}_{pi} + \mathbf{v}_{\pi i} + \mathbf{v}_{\parallel i}, \tag{11}$$

$$\mathbf{v}_e = \mathbf{v}_E + \mathbf{v}_{De} + \mathbf{v}_{pe} + \mathbf{v}_\eta + \mathbf{v}_{\parallel e}. \tag{12}$$

Note that the perpendicular electric field in Eq. (6) is electrostatic in the limit of low β. Electromagnetic effects, other than Ohm's law, are included in these model equations through the parallel gradient $\nabla_\parallel = \widehat{\mathbf{b}} \cdot \nabla = \widehat{\mathbf{b}}^{(0)} \cdot \nabla + \widehat{\mathbf{b}}^{(1)} \cdot \nabla$ where $\widehat{\mathbf{b}}^{(0)}$ is the direction of the unperturbed magnetic field, and $\widehat{\mathbf{b}}^{(1)} = \nabla \times \widetilde{\mathbf{A}}_{\parallel}/B = \nabla\widetilde{\mathbf{A}}_{\parallel} \times \widehat{e}_{\parallel}/B$ is the magnetic perturbation associated with field line bending, $A_{\parallel}$. It follows that, when $\widehat{\mathbf{b}}^{(1)}\cdot\nabla$, operates on variable the equilibrium S_0, it has the linearized form, -$iS_0\omega_{*S}\widehat{A}_{\parallel}$, where $\omega_{*s} = -\left(cT_e/eB\right)\mathbf{k}\cdot \widehat{e}_{\parallel} \times \nabla \ln S_0$.

When quasi-neutrality is assumed, $n = n_e = n_i$, Ampére's law reduces to

$$\nabla_{\perp}^2 \widetilde{A}_{\parallel} = -\frac{4\pi}{c}\widetilde{j}_{\parallel} \tag{13}$$

Equations (1)-(4) can be written, using $\nabla_{\perp} = ik_{\perp}$, $\partial/\partial t = -i\omega t$, in the following linearized form in $\omega \sim \omega_s \sim \omega_{*j} \sim \omega_{\eta}$ maximal ordering

$$\left(\omega - \omega_{*en} + \omega H + ic^2 k_{\perp}^2 \eta_{\parallel}/4\pi\right)\widehat{\Psi} = -ic_s \nabla_{\parallel}^{(0)}\left(\widehat{\Phi} - \widehat{n}\right), \tag{14}$$

$$\omega k_{\perp}^2\rho_i^2\left(\widehat{n} + \tau\widehat{\Phi}\right) = \omega_{\kappa}\widehat{n} + k_{\perp}^2\rho_e^2\left(\omega - \omega_{*en}\right)\widehat{\Phi} - i\mu_{\perp}k_{\perp}^4\rho_i^2\left(\widehat{n} + \tau\widehat{\Phi}\right) - i\frac{\tau v_A^2}{c_s}\nabla_{\parallel}^{(0)}\left(k_{\perp}^2\rho_i^2\widehat{\Psi}\right), \tag{15}$$

$$\omega\widehat{n} - \omega_{*en}\widehat{\Phi} = \omega_{\kappa e}\left(\widehat{\Phi} - \widehat{n}\right) + k_{\perp}^2\rho_e^2\left(\omega - \omega_{*en}\right)\widehat{\Phi} - ic_s\nabla_{\parallel}^{(0)}\widehat{v}_{\parallel} + i\frac{\eta_{\perp}}{\eta_{\parallel}}\frac{c_s^2}{v_A^2}k_{\perp}^2\eta^*\widehat{n} + i\frac{\tau v_A^2}{c_s}\nabla_{\parallel}^{(0)}\left(k_{\perp}^2\rho_i^2\widehat{\Psi}\right), \tag{16}$$

$$\left(\omega + \omega_{\kappa i}\right)\widehat{\mathbf{v}}_{\parallel} + \omega_{*en}\widehat{\Psi} = -ic_s\nabla_{\parallel}^{(0)}\widehat{n} - 4i\mu_{\perp}k_{\perp}^2\widehat{v}_{\parallel}. \tag{17}$$

Here, $\widehat{\Psi} \equiv ec_s\widetilde{A_{\parallel}}/cT_e$, $\widehat{\Phi} \equiv e\widetilde{\phi}/T_e$, $\widehat{v}_{\parallel} \equiv \widetilde{v_{\parallel i}}/c_s$, and $\widehat{n} \equiv \widetilde{n}/n$ are the dimensionless perturbed parallel component of vector potential, electrostatic potential, parallel ion flow, and density, respectively. Also, ω is the mode frequency, $\eta^* = \left(c^2/4\pi\right)\eta_{\parallel}$, $H \equiv k_{\perp}^2\delta_e^2$, $\delta_e^2 \equiv c^2/\omega_{pe}^2$ is the electromagnetic skin depth, c is the speed of light, $\omega_{pe}^2 \equiv 4\pi ne^2/m_e$ is the electron plasma frequency, e is the electron charge, m_j is the mass of species j, $\mu_{\perp} \equiv 0.3\,\nu_{ii}\,\rho_i^2$ is the classical perpendicular viscosity, $\rho_i \equiv v_{ti}/\omega_{ci}$ is the ion Larmor radius, $\nu_{ii} = (4/3)(\sqrt{\pi}ne^4\lambda)/\sqrt{m_i}\,T_i^{3/2})$, λ is the Coulomb logarithm, $v_{ti} \equiv \sqrt{T_i/m_i}$ is the ion thermal velocity, $\omega_{ci} \equiv eB/m_ic$ is the ion cyclotron frequency, m_i is the ion mass, $\tau \equiv T_e/T_i$ is the ratio of electron to ion temperature, $\rho_e \equiv v_{te}/\omega_{ce}$ is the electron Larmor radius, $v_A^2 \equiv B^2/4\pi nm_i$ is the Alfvén speed, $c_s \equiv \left[\left(T_e + T_i\right)/m_i\right]^{1/2}$ is the sound speed, and $\eta_{\parallel}$ and $\eta_{\perp}$ are the longitudinal and transverse electrical resistivities. The frequency ω_{*en} in Eq. (14) is the diamagnetic drift frequency, $\omega_{*en} = -\left(cT_e/eB\right)\mathbf{k}\cdot\mathbf{e}_{\parallel} \times \nabla \ln n$, and the frequency ω_{κ} in

Eq. (15) is the curvature drift frequency, $\omega_\kappa \equiv \omega_{\kappa i} + \omega_{\kappa e}$ in which $\omega_{\kappa j} \equiv (2cT_j/eB)\, \mathbf{k} \cdot \mathbf{e}_\parallel \times \kappa$, where $\kappa = (\mathbf{e}_\parallel \cdot \nabla)\mathbf{e}_\parallel$ is the curvature vector, and $\mathbf{e}_\parallel = \mathbf{B}/B$ is the unit vector along the magnetic field line.

III. Shear-Alfvén and Drift acoustic equation in 3-D geometry

Equations (14)-(17) can be reduced to a coupled system of a two second order differential equations, the shear-Alfvén equation and the drift-acoustic equation:

$$(\mathbf{B} \cdot \nabla) \left[\frac{(\omega - \omega_{*en})\, k_\perp^2 \left(\mathbf{e}_\parallel \cdot \nabla\right) U}{B \left(\omega - \omega_{*en} + \omega k_\perp^2 \delta_e^2 + i c^2 k_\perp^2 \eta_\parallel / 4\pi\right)} \right] + \frac{\omega_\kappa \omega_{*in}}{v_A^2 \rho_i^2} (U + V)$$

$$= -\frac{k_\perp^2}{v_A^2} \left(\omega + i\mu_\perp k_\perp^2\right) \left[(\omega - \omega_{*in})\, U - (1 + \tau)\, \omega_{*in} V\right], \tag{18}$$

and

$$\left[\left(\mathbf{e}_\parallel \cdot \nabla\right)^2 - 2 \left(\mathbf{e}_\parallel \cdot \nabla\right) \ln B \left(\mathbf{e}_\parallel \cdot \nabla\right)\right] V + \frac{(\omega - \omega_{*en}) \left(\omega + 4i\mu_\perp k_\perp^2\right)}{c_s^2} V$$

$$= \left[\frac{\left(\omega + 4i\mu_\perp k_\perp^2\right)}{c_s^2} \left(\frac{\omega k_\perp^2 \rho_i^2}{\omega_{*in}} - \frac{\omega_{\kappa i}}{\omega_{*in}} \right) + \frac{\left(\omega \delta_e^2 + i\eta^*\right) \left(\omega + i\mu_\perp k_\perp^2\right)}{\omega - \omega_{*en}} \frac{k_\perp^2}{v_A^2} \right] \tag{19}$$

$$\times \Big[(\omega - \omega_{*in})\, U - (1 + \tau)\, \omega_{*in} V \Big]$$

$$+ \left[\left(\frac{\omega \delta_e^2 + i\eta^*}{\omega - \omega_{*en}} \right) \left(\frac{\omega_\kappa \omega_{*in}}{v_A^2 \rho_i^2} \right) - \frac{\eta_\perp}{\eta_\parallel} i\eta^* \left(\omega + 4i\mu_\perp k_\perp^2\right) \frac{k_\perp^2}{v_A^2} \right] (U + V)$$

$$+ \Big[\left(\mathbf{e}_\parallel \cdot \nabla\right) \ln \left(\omega + 4i\mu_\perp k_\perp^2\right) \Big]$$

$$\times \left[\frac{\left(\omega \delta_e^2 + i\eta^*\right) k_\perp^2}{\omega - \omega_{*en} + \left(\omega \delta_e^2 + i\eta^*\right) k_\perp^2} \left(\mathbf{e}_\parallel \cdot \nabla\right) U + \left(\mathbf{e}_\parallel \cdot \nabla\right) V \right], \tag{20}$$

where

$$U = \widehat{\Phi} - \widehat{n}, \tag{21}$$

and

$$V = \frac{\omega}{\omega_{*en}}\widehat{n} - \widehat{\Phi}. \tag{22}$$

The equilibrium magnetic field in the Hamada coordinate system (v, θ, ζ) is

$$\mathbf{B} = \nabla v \times \left(\dot{\psi}\nabla\theta - \dot{\chi}\nabla\zeta\right), \tag{23}$$

where v is the volume enclosed within the flux surface, which is used to label the flux surface; θ and ζ are angle-like coordinates that increase by unity after one turn around the torus, the short way and the long way, respectively; ψ and χ are flux surface functions corresponding to longitudinal and transverse fluxes. In this notation, a dot over quantities indicates a derivative with respect to the volume. The Hamada coordinate system has a unit Jacobian

$$\nabla v \cdot \nabla\theta \times \nabla\zeta = 1, \tag{24}$$

and

$$q = \frac{\dot{\psi}}{\dot{\chi}} = \frac{\mathbf{B}\cdot\nabla\zeta}{\mathbf{B}\cdot\nabla\theta} = \frac{d\psi}{d\chi}, \tag{25}$$

is the safety factor. Since magnetic field lines lie on flux surfaces, the contravariant component in the radial direction vanishes, that is $\mathbf{B}\cdot\nabla v = 0$.

The coordinates $(v, \theta, \varphi = \zeta - q\theta)$ are introduced by transforming $(v,\ \theta,\ \zeta) \rightarrow (v,\ y,\ \varphi)$. The transformed basis vectors satisfy

$$\mathbf{B}\cdot\nabla v = 0, \quad \mathbf{B}\cdot\nabla y = \dot{\chi}, \quad \mathbf{B}\cdot\nabla\varphi = 0. \tag{26}$$

In this new set of coordinates, there is only one nonzero contravariant component of $\mathbf{B}$,

$$\mathbf{B} = \dot{\chi}\nabla\varphi \times \nabla v. \tag{27}$$

Since $q = q(v)$, the property of unit Jacobian is retained:

$$\nabla v \cdot \nabla y \times \nabla\varphi = 1. \tag{28}$$

The shear-Alfvén and drift-acoustic equations can be expressed as second order differential equations in the parallel ballooning coordinate y by using Hamada coordinates and a ballooning mode formalism:

$$\frac{d}{dy}\left[\frac{(\omega-\omega_{ne})\,K^2}{B^2\left(\omega-\omega_{ne}+\left(\omega\delta_e^2+i\eta^*\right)a^2K^2\right)}\frac{dU}{dy}\right]+\frac{8\pi\dot{p}\left(\kappa_v+\dot{q}y\kappa_\varphi\right)}{\dot{\chi}^4}\left(U+V\right)$$

$$=-\frac{K^2}{\dot{\chi}^2v_A^2}\left(\omega+i\mu_\perp a^2K^2\right)\left[(\omega-\omega_{ni})\,U-(1+\tau)\,\omega_{ni}V\right], \tag{29}$$

and

$$\frac{d}{dy}\left(\frac{\dot{\chi}^2}{B^2}\frac{dV}{dy}\right)+\frac{(\omega-\omega_{ne})\left(\omega+4i\mu_\perp a^2K^2\right)}{c_s^2}V$$

$$=\left[-\frac{\left(\omega+4i\mu_\perp a^2K^2\right)}{c_s^2}\left(2L_{nv}\left(\kappa_v+\dot{q}y\kappa_\varphi\right)+\frac{\tau\omega a^2K^2\rho_i^2}{\omega_{ne}}\right)\right.$$

$$\left.+\frac{\left(\omega\delta_e^2+i\eta^*\right)\left(\omega+i\mu_\perp a^2K^2\right)}{\omega-\omega_{ne}}\frac{a^2K^2}{v_A^2}\right]\left[(\omega-\omega_{ni})\,U-(1+\tau)\,\omega_{ni}V\right]$$

$$+\left[\frac{\left(\omega\delta_e^2+i\eta^*\right)\left(\omega+i\mu_\perp a^2K^2\right)}{\omega-\omega_{ne}}\frac{a^2K^2}{v_A^2}\right]\left[(\omega-\omega_{ni})\,U-(1+\tau)\,\omega_{ni}V\right]$$

$$+\left[\left(\frac{\omega\delta_e^2+i\eta^*}{\omega-\omega_{ne}}\right)\frac{8\pi a^2\dot{p}\left(\kappa_v+\dot{q}y\kappa_\varphi\right)}{\dot{\chi}^2}-\frac{\eta_\perp}{\eta_\parallel}\frac{i\eta^*\left(\omega+4i\mu_\perp a^2K^2\right)a^2K^2}{v_A^2}\right]$$

$$\times\left(U+V\right)+\frac{\dot{\chi}^2}{B^2}\frac{d}{dy}\ln\left(\omega+4i\mu_\perp a^2k_\perp^2\right)\left[\frac{\left(\omega\delta_e^2+i\eta^*\right)a^2K^2}{\omega-\omega_{ne}+\left(\omega\delta_e^2+i\eta^*\right)a^2K^2}\frac{dU}{dy}+\frac{dV}{dy}\right], \tag{30}$$

where

$$K^2=|\nabla\varphi|^2-2\dot{q}y\nabla\varphi\cdot\nabla v+\dot{q}^2y^2|\nabla v|^2. \tag{31}$$

Here, $a=\partial S/\partial\varphi$ is the "mode number" that describes the component of the $\mathbf{k}$ vector that is perpendicular to the magnetic field and lies within the magnetic surface, $p=nT$ is the equilibrium pressure, $\omega_{ne}=(cT_ea/e\dot{\chi})L_{nv}$ is the electron diamagnetic frequency, $L_{nv}\equiv(d\ln n/dv)^{-1}$, $\kappa_v=\kappa\cdot\nabla\theta\times\nabla\varphi$ is the normal curvature, $\kappa_\varphi=\kappa\cdot\nabla v\times\nabla\theta=(-\dot{\chi}/2\dot{p})\mathbf{B}\cdot\nabla\sigma$ is the geodesic curvature, and $\sigma=\mathbf{j}\cdot\mathbf{B}/B^2$. The coordinate y is defined as labeling points along the magnetic field and, as such, $\mathbf{B}\cdot\nabla=\dot{\chi}(d/dy)$.

IV. Analysis of Resistive Ballooning Mode equations

Analytic progress can be made toward understanding the structure of non-ideal MHD ballooning modes by using a multiple scale analysis. This derivation generalizes the work of Hastie *et al.* [8] to general three-dimensional equilibria. A small parameter ϵ can be defined that accounts for the disparate timescales associated with the current diffusion and Alfvén times:

$$\epsilon \equiv (\frac{\omega_\eta}{\omega_A})^{1/3} \ll 1. \tag{32}$$

In the following derivation, a more general ordering is used:

$$\omega \sim \omega_s \sim \omega_{nj} \sim \epsilon\, \omega_A, \tag{33}$$

and viscosity is assumed to be comparable to resistivity, $\omega_\mu = \epsilon^3 \omega_A \sim \omega_\eta$.

Equations (29) and (30) can be solved using a two variable expansion procedure. The variables y and $z = \epsilon y$ are taken as two different length scales along the magnetic field and the Ansatz used is

$$U\,(y) = U_0\,(y,z) + \epsilon U_1\,(y,z) + \epsilon^2 U_2\,(y,z) + ..., \tag{34}$$

$$V\,(y) = V_0\,(y,z) + \epsilon V_1\,(y,z) + \epsilon^2 V_2\,(y,z) + ..., \tag{35}$$

$$\frac{dU}{dy} = \frac{\partial U}{\partial y} + \epsilon \frac{\partial U}{\partial z}, \tag{36}$$

and

$$U_i(y+N, z) = U_i(y,z), \quad i = 0, 1, 2, \ldots . \tag{37}$$

The function U depends upon the variable y that characterizes the variation of equilibrium quantities within a flux surface. The variable z accounts for the long envelope of the eigenfunction along the magnetic field line due to non-ideal MHD effects. For $|y| \sim 1$, an ideal MHD region can be identified where resistivity, electron inertia and viscosity can be neglected. The drift-acoustic and shear-Alfvén equations to the zeroth order in ϵ are independent and can be written as

$$\frac{d}{dy}\left(\frac{\dot{\chi}^2}{B^2}\frac{dV_0}{dy}\right)+\frac{\omega\left(\omega-\omega_{ne}\right)}{c_s^2}V_0=0, \tag{38}$$

$$\frac{d}{dy}\left[\frac{K^2}{B^2}\frac{dU_0}{dy}\right]+\frac{4\pi}{\dot{\chi}^4}\left(2\dot{p}\kappa_v-\dot{q}\dot{\chi}^2 y\frac{d\sigma}{dy}\right)U_0=0. \tag{39}$$

In the limit of large values of $|y|$, the shear Alfvén equation yields the asymptotic solution [13]

$$U=a_1|y|^s+a_2|y|^{-1-s},\quad |y|\to\infty, \tag{40}$$

where

$$s=-\frac{1}{2}+\left[\frac{1}{4}-E-F-H\right]^{1/2}. \tag{41}$$

The quantities E, F, and H depend on the equilibrium and are defined as [13]:

$$E=\frac{A\left\langle B^2/|\widehat{\nabla}v|^2\right\rangle}{\dot{q}^2\dot{\chi}^4}\left[\dot{I}\ddot{\Psi}-\dot{J}\ddot{\chi}-\dot{q}\dot{\chi}^2\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right], \tag{42}$$

$$F=\frac{A^2\left\langle B^2/|\widehat{\nabla}v|^2\right\rangle}{\dot{q}^2\dot{\chi}^4}\left[\left\langle\frac{B^2\sigma^2}{|\widehat{\nabla}v|^2}\right\rangle-\frac{\left\langle\sigma B^2/|\widehat{\nabla}v|^2\right\rangle^2}{\left\langle B^2/|\widehat{\nabla}v|^2\right\rangle}+\dot{p}^2\left\langle\frac{1}{B^2}\right\rangle\right], \tag{43}$$

$$H=\frac{A^2\left\langle B^2/|\widehat{\nabla}v|^2\right\rangle}{\dot{q}\dot{\chi}^2}\left[\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}-\frac{\left\langle\sigma B^2/|\widehat{\nabla}v|^2\right\rangle}{\left\langle B^2/|\widehat{\nabla}v|^2\right\rangle}\right]. \tag{44}$$

These quantities combine to yield the key resistive MHD instability parameter:

$$D_R=F+E+H^2. \tag{45}$$

Since ideal MHD stability is assumed in this analysis, the Mercier stability criterion is satisfied

$$-D_I=\frac{1}{4}-(E+F+H)=\frac{1}{4}+H^2-H-D_R\geqq 0. \tag{46}$$

For the solution in the outer region along the field lines ($|y|\gtrsim\epsilon^{-1}$), resistivity, inertia and viscosity must be taken into account. The shear-Alfvén wave equation in the lowest order in ϵ can be written as

$$\frac{d}{dy}\left[\frac{1}{B^2\left(1+\widehat{\omega}_{IR}\right)}\frac{dU_0}{dy}\right]=0, \tag{47}$$

whose solution is $U_0 = U_0\,(z)$. Thus, U_0 does not explicitly depend on y in this case. In first order, it follows, with the use of Eq. (37), that (tildes over quantities indicate variations of these quantities within flux surface)

$$\begin{aligned} U_1\,(y,z) &= U_1(z) + A_1 U_0(z)\left[\int \left(\widetilde{\frac{\sigma B^2}{|\,\widehat{\nabla} v|^{\,2}}}\right)dy + \dot{q}^2 z^2 \widehat{\omega}_{IR}\int \widetilde{\sigma B^2}dy\right] \\ &+ A_2(z)\left[\frac{1}{\dot{q}^2 z^2}\int \widetilde{\frac{B^2}{|\,\widehat{\nabla} v|^{\,2}}}dy + \widehat{\omega}_{IR}\int \widetilde{B^2}dy\right], \end{aligned} \tag{48}$$

where

$$A = 4\pi\left(\overline{a}/q\right)^2, \quad A_1 = \left(\frac{A}{\dot{q}z\dot{\chi}^2}\right), \quad A_2(z) = \frac{L}{M}, \quad \widehat{\omega}_{IR} = \frac{\widehat{\omega}\delta^2/\epsilon^2 + i}{\widehat{\omega}-\widehat{\omega}_{ne}},$$

$$L = \left[\frac{\partial U_0}{\partial z} - A_1\left(\left\langle \frac{\sigma B^2}{|\,\widehat{\nabla} v|^{\,2}}\right\rangle - \dot{q}^2\;z^2\widehat{\omega}_{IR}\left\langle \sigma B^2\right\rangle\right)U_0\right], \tag{49}$$

$$M = \frac{1}{\dot{q}^2 z^2}\left\langle \frac{B^2}{|\widehat{\nabla} v|^{\,2}}\right\rangle + \widehat{\omega}_{IR}\left\langle B^2\right\rangle, \tag{50}$$

$$\left\langle B^2\right\rangle = \left\langle B^2\right\rangle(v_0,\varphi_0) = \frac{\oint B^2 dy}{\oint dy}, \tag{51}$$

$$\widetilde{B^2} = B^2 - \left\langle B^2\right\rangle, \quad i.e., \quad \left\langle \widetilde{B^2}\right\rangle = 0, \tag{52}$$

and

$$\begin{aligned} \delta &= \frac{q}{\overline{a}}\left(\frac{c}{\omega_{pe}}\right)\left(\frac{\partial S}{\partial \varphi}\right), \quad \omega_\eta = \frac{c^2\eta_\parallel}{4\pi}\left(\frac{q}{\overline{a}}\right)^2\left(\frac{\partial S}{\partial\varphi}\right)^2, \quad \widehat{\nabla} = \frac{\overline{a}}{q}\nabla, \\ \widehat{\omega} &= \frac{\omega}{\epsilon\omega_A}, \quad \omega_A = \frac{V_A}{L_\parallel}, \quad L_\parallel = 2\pi q R_0, \quad R_0 \equiv \frac{\left\langle B^2\right\rangle^{1/2}}{2\pi q|\dot{\chi}|} = \frac{\left\langle B^2\right\rangle^{1/2}}{2\pi q|\dot{\psi}|} \end{aligned} \tag{53}$$

At second order, a solubility condition for U_2 is derived by taking into account Eq. (37).

The result is a differential equation for U_0 that depends upon integrals of U_1 and V_1:

$$\left\langle \frac{\partial}{\partial z}\frac{\dot{q}^2 z^2 |\widehat{\nabla} v|^2}{B^2 A_3}\frac{\partial U_1}{\partial y}\right\rangle = -\frac{A_1 \dot{q} z U_0}{\dot{\chi}^2}\left[\langle 2\dot{p}\kappa_v\rangle - \dot{q}z\dot{\chi}^2\left(\left\langle\frac{\partial \sigma}{\partial z}\right\rangle - \left\langle\left(\frac{\partial U_1}{\partial y}+\frac{\partial V_1}{\partial y}\right)\sigma\right\rangle\right)\right]$$

$$-\left\langle \frac{\partial}{\partial z}\frac{\dot{q}^2 z^2 |\widehat{\nabla} v|^2}{B^2 A_3}\frac{\partial U_0}{\partial z}\right\rangle - \dot{q}^2\ z^2\left(\,\widehat{\omega}-\widehat{\omega}_{ni}\right)U_0\left\langle\frac{|\widehat{\nabla} v|^2}{B^2}\left(\,\widehat{\omega}+i\widehat{\mu}\dot{q}^2 z^2\ |\widehat{\nabla} v|^2\right)\right\rangle, \tag{54}$$

where $A_3 = 1 + \dot{q}^2 z^2 \widehat{\omega}_{IR} |\widehat{\nabla} v|^2$, and $\widehat{\mu} = \mu/\eta_{\parallel}$.

a) Electrostatic limit

One special limit that can be pursued analytically is the electrostatic limit. This corresponds to a mode driven by geodesic curvature in the non-ideal MHD layer [3]. In this case, $\dot{q}^2 z^2 \widehat{\omega}_{IR} |\widehat{\nabla} v|^2 \gg 1$, and $V_1 = 0$ yields

$$\frac{d^2 U_0}{dz^2} + W_1\left[\langle 2\dot{p}\kappa_v\rangle + \dot{q}\dot{\chi}^2\left(\langle\sigma\rangle - \frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right)\right]U_0$$

$$+W_2\left[\left\langle\sigma^2 B^2\right\rangle - \frac{\left\langle\sigma B^2\right\rangle^2}{\langle B^2\rangle} - W_3\left\langle\frac{|\widehat{\nabla} v|^2}{B^2}\right\rangle\right]z^2 U_0 = 0, \tag{55}$$

where

$$W_1 = \left(\frac{A}{\dot{\chi}^4}\right)\widehat{\omega}_{IR}\left\langle B^2\right\rangle,\quad W_2 = \frac{\dot{q}^2\dot{\chi}^4}{\langle B^2\rangle}W_1^2,\quad W_3 = \widehat{\omega}\left(\widehat{\omega}-\widehat{\omega}_{ni}\right)\frac{\dot{q}^2\left\langle B^2\right\rangle}{W_2}. \tag{56}$$

The condition for existence of a solution is $2\dot{p}\left\langle\kappa_v\right\rangle + \dot{q}^2\dot{\chi}^2\left(\langle\sigma\rangle - \left\langle\sigma B^2\right\rangle/\left\langle B^2\right\rangle\right) > 0$. The solutions of Eq. (55) in this case result in the following eigenvalue expression

$$U_{0n} = \exp\left(-z_1^2\right)H_n\left(z_1\right), \tag{57}$$

where

$$z_1^2 = \left[W_2\left(\frac{\left\langle\sigma B^2\right\rangle^2}{\langle B^2\rangle} - \left\langle\sigma^2 B^2\right\rangle - W_3\left\langle\frac{|\widehat{\nabla} v|^2}{B^2}\right\rangle\right)\right]z^2, \tag{58}$$

and where H_n are Hermite polynomials of order n. The resulting dispersion relation is

$$\omega\left(\omega-\omega_{ne}\right)\left(\omega-\omega_{ni}\right) = -\frac{\omega_A^2\left(\omega\delta^2+i\omega_\eta\right)}{\left\langle |\widehat{\nabla}v|^{2}/B^2\right\rangle}\left(\frac{4\pi\left(\overline{a}/q\right)^2}{\dot{\chi}^2}\right)^2\left[\left\langle\sigma^2B^2\right\rangle-\left\langle\sigma B^2\right\rangle^2/\left\langle B^2\right\rangle\right.$$

$$\left.+\frac{\langle B^2\rangle}{\left(\dot{q}^2\dot{\chi}^4\left(2n+1\right)^2\right)}\times\left\{\langle 2\dot{p}\kappa_v\rangle+\dot{q}\dot{\chi}^2\left(\langle\sigma\rangle-\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\right)\right\}^2\right]. \quad (59)$$

Solutions to this electrostatic dispersion relation lead to an infinite sequence of modes with growth rates scaling as the resistive ballooning mode, $\gamma\sim{\omega_\eta}^{1/3}$ (for $\delta=0$ and $\omega_{nj}=0$), or the electron inertia ballooning mode, $\gamma\sim\delta$ (for $\omega_\eta=0$ and $\omega_{nj}=0$). In the absence of drift effects the modes are purely growing.

b) Coupling of visco-resistive ballooning modes to drift-acoustic waves

When the value of U_1 from Eq. (48) is used in the solubility equation (54), one obtains the following equation in the $\omega\sim\omega_s$ ordering:

$$\frac{\partial}{\partial X}\frac{X^2}{1+X^2}\frac{\partial U_0}{\partial X} + \frac{H\left(1-H\right)}{\left(1+X^2\right)^2}\left(U_0+V_0\right)-\frac{H\left(1+H\right)X^2}{\left(1+X^2\right)^2}\left(U_0+V_0\right)+D_R\left(U_0+V_0\right)$$

$$= \left(Q_3V_0+Q_1U_0\right)X^2+\left(Q_2U_0+Q_4V_0\right)X^4$$

$$- \frac{AN_1X}{\dot{q}\dot{\chi}^2}\left[\left\langle\frac{\sigma}{\sqrt{o_1}}\frac{\partial V_1}{\partial y}\right\rangle+\langle\sigma\rangle\frac{\partial V_0}{\partial X}-\frac{M_1}{N_1\left(1+X^2\right)}\frac{\partial V_0}{\partial X}\right], \quad (60)$$

where

$$X={o_1}^{1/2}z, \quad o_1=\frac{\dot{q}^2\langle B^2\rangle}{N_1}\widehat{\omega}_{IR}, \quad N_1=\left\langle\frac{B^2}{|\widehat{\nabla}v|^2}\right\rangle, \quad M_1=\left\langle\frac{\sigma B^2}{|\widehat{\nabla}v|^2}\right\rangle+N_1\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}X^2, \quad (61)$$

$$\frac{\partial}{\partial y}\frac{1}{B^2}\frac{\partial V_1}{\partial y} = -\frac{\left(\widehat{\omega}-\widehat{\omega}_{ne}\right)\left(\widehat{\omega}+4i\widehat{\mu}_\perp\dot{q}^2z^2|\widehat{\nabla}v|^{2}\right)}{\langle B^2\rangle\,\widehat{\omega}_s^2}V_1\left(y,z\right)$$

$$+ \dot{q}z\frac{d\sigma}{dy}\left[\frac{\dot{\chi}^2}{4\pi\dot{p}^2}\left(\widehat{\omega}+4i\widehat{\mu}_\perp\dot{q}^2z^2|\widehat{\nabla}v|^{2}\right)\left(\widehat{\omega}-\widehat{\omega}_{ni}\right)-\frac{A\widehat{\omega}_{IR}}{\dot{\chi}^2}\right]U_0, \quad (62)$$

$$Q_1 = -\frac{A^2 N_1^2}{\dot{q}^2 \dot{\chi}^4 \langle B^2 \rangle} \left[\frac{\dot{q}}{A} \frac{\omega (\omega - \omega_{ni}) (\omega - \omega_{ne})}{\omega_A^2 (\omega \delta^2 + i\omega_\eta)} \left(\frac{1}{A} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle \right) + \left\langle \sigma^2 B^2 \right\rangle - \frac{\left\langle \sigma B^2 \right\rangle^2}{\langle B^2 \rangle} \right], \quad (63)$$

$$Q_2 = -\frac{i\omega_\mu (\omega - \omega_{ni}) (\omega - \omega_{ne})^2 N_1^3}{\dot{q}^2 \langle B^2 \rangle \omega_A^2 (\omega \delta^2 + i\omega_\eta)^2} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle^2, \quad (64)$$

$$Q_3 = \frac{A^2 N_1^2}{\dot{q}^2 \dot{\chi}^4 \langle B^2 \rangle} \left[\frac{\dot{\chi}^4}{A} \frac{\omega (1 + \tau) \omega_{ni} (\omega - \omega_{ne})}{\omega_A^2 (\omega \delta^2 + i\omega_\eta)} \left(\frac{1}{A} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle \right) - \left\langle \sigma^2 B^2 \right\rangle + \frac{\left\langle \sigma B^2 \right\rangle^2}{\langle B^2 \rangle} \right], \quad (65)$$

and

$$Q_4 = \frac{i\omega_\mu (1 + \tau) \omega_{ni} (\omega - \omega_{ne})^2 N_1^3}{\dot{q}^2 \langle B^2 \rangle \omega_A^2 (\omega \delta^2 + i\omega_\eta)^2} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle^2. \quad (66)$$

Here, $\mu_\perp = 0$ is assumed. Additionally, a single harmonic approximation is employed to describe the Pfirsch-Schulter current spectrum $\sigma \propto \sin(M - Nq)y$, here M is the poloidal mode number and N is the toroidal mode number. Using these approximations the solubility condition for U_2 given by Eq. (60), takes the following form:

$$\frac{\partial}{\partial X} \frac{X^2}{1 + X^2} \frac{\partial U_0}{\partial X} + \frac{H (1 - H)}{(1 + X^2)^2} U_0 - \frac{H (1 + H) X^2}{(1 + X^2)^2} U_0 + D_R U_0 - Q_{1V} U_0 X^2 = 0, \quad (67)$$

where

$$Q_{1V} = -\frac{A^2 N_1^2}{\dot{q}^2 \dot{\chi}^4 \langle B^2 \rangle} \left[\frac{\dot{\chi}^4}{A} \frac{\omega (\omega - \omega_{ni}) (\omega - \omega_{ne})}{\omega_A^2 (\omega \delta^2 + i\omega_\eta)} \left(\frac{1}{A} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle + P_S \frac{\langle \sigma^2 B^2 \rangle}{4\pi \dot{p}^2} \right) \right.$$
$$\left. + \left\langle \sigma^2 B^2 \right\rangle (1 - P_S) - \frac{\left\langle \sigma B^2 \right\rangle^2}{\langle B^2 \rangle} \right], \quad (68)$$

and

$$P_S = \frac{(M - Nq)\omega_s^2}{(M - Nq)^2 \omega_s^2 - \omega (\omega - \omega_{ne})}. \quad (69)$$

Equation (67) describes the coupling between the drift resistive ballooning mode and the drift acoustic mode in the limit that the mode frequency is comparable to the sound wave frequency.

If the frequency of the mode is assumed to be smaller than the sound frequency ($\omega_s \gg \omega$), then $V_0(z) \neq 0$, and Eq. (62) can be written as

$$\begin{aligned} V_1 \ = \ & \frac{\partial V_0}{\partial z}\frac{\int \widetilde{B^2}dy}{\langle B^2\rangle} + \dot{q}\dot{\chi}^2 z\left[\frac{\widehat{\omega}\left(\widehat{\omega}-\widehat{\omega}_{ni}\right)U_0-\left(1+\tau\right)\widehat{\omega}\widehat{\omega}_{ni}V_0}{4\pi\dot{p}^2} - \frac{A\widehat{\omega}_{IR}\left(U_0+V_0\right)}{\dot{\chi}^4}\right] \\ & \times\left[\int\widetilde{\sigma B^2} - \frac{\langle\sigma B^2\rangle}{\langle B^2\rangle}\int\widetilde{B^2}\right]dy + 4i\widehat{\mu}\dot{q}^2 z^2\left[\left(\widehat{\omega}-\widehat{\omega}_{ni}\right)U_0 - \left(1+\tau\right)\widehat{\omega}_{ni}V_0\right] \\ & \times\left[\int\widetilde{|\widehat{\nabla}v|^2\sigma B^2} - \frac{\left\langle|\widehat{\nabla}v|^2\sigma B^2\right\rangle}{\langle B^2\rangle}\int\widetilde{B^2}\right]dy + V_1\left(z\right). \end{aligned} \tag{70}$$

The following second order differential equation for the shear-Alfvén mode is derived by using V_1 in Eq. (60):

$$\begin{aligned} & \frac{\partial}{\partial X}\frac{X^2}{1+X^2}\frac{\partial U_0}{\partial X} + \frac{H\left(1-H\right)}{\left(1+X^2\right)^2}\left(U_0+V_0\right) - \frac{H\left(1+H\right)X^2}{\left(1+X^2\right)^2}\left(U_0+V_0\right) + \frac{HX}{1+X^2}\frac{\partial V_0}{\partial X} \\ = \ & -D_R\left(U_0+V_0\right) + \left(Q_{1M}U_0+Q_{3M}V_0\right)X^2 + \left(Q_{2M}U_0+Q_{4M}V_0\right)X^4. \end{aligned} \tag{71}$$

Similarly, a solubility equation for V_2 can be obtained for drift-acoustic-modes:

$$\begin{aligned} \frac{\partial^2 V_0}{\partial X^2} \ + \ & \left(1+Q_*\right)\frac{H^2}{1+X^2}V_0 + \left(Q_* + \frac{\Omega\dot{\chi}^4}{A4\pi\dot{p}^2}\right)\left[\frac{H^2X^2}{1+X^2}\frac{A^2N_1}{\dot{q}^2\dot{\chi}^4}I_2\right]V_0 + \left(Q_5U_0 - Q_6V_0\right)X^2 \\ = \ & \left(1+Q_*\right)D_RV_0 + \left(1-\frac{\Omega\dot{\chi}^4}{A4\pi\dot{p}^2}\right)\left[D_R - \frac{H^2}{1+X^2} - \frac{HX}{1+X^2}\frac{\partial}{\partial X}\right]U_0, \end{aligned} \tag{72}$$

where $Q_1$to Q_4 in Eqs. (63) to (66) are modified as follows:

$$\begin{aligned} Q_{1M} \ = \ & -\frac{A^2N_1^2}{\dot{q}^2\dot{\chi}^4\langle B^2\rangle}\left[\frac{\dot{\chi}^4}{A}\frac{\omega\left(\omega-\omega_{ni}\right)\left(\omega-\omega_{ne}\right)}{\omega_A^2\left(\omega\delta^2+i\omega_\eta\right)}\right. \\ & \left.\times\left(\frac{1}{A}\left\langle\frac{|\widehat{\nabla}v|^2}{B^2}\right\rangle + \frac{\langle\sigma^2B^2\rangle - \langle\sigma B^2\rangle^2/\langle B^2\rangle}{4\pi\dot{p}^{\,2}}\right)\right], \end{aligned} \tag{73}$$

$$Q_{2M} = -\frac{i\omega_\mu \left(\omega - \omega_{ni}\right) \left(\omega - \omega_{ne}\right)^2 N_1^3}{\dot{q}^2 \left\langle B^2 \right\rangle \omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)^2} \left[\left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle^2 + \frac{A I_1}{\pi \dot{p}^{\,2}} \right], \tag{74}$$

$$\begin{aligned} Q_{3M} \;=\; & \frac{A^2 N_1^2}{\dot{q}^2 \dot{\chi}^4 \left\langle B^2 \right\rangle} \left[\frac{\dot{\chi}^4}{A} \frac{\omega \left(1+\tau\right) \omega_{ni} \left(\omega - \omega_{ne}\right)}{\omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)} \right. \\ & \left. \times \left(\frac{1}{A} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle + \frac{\left\langle \sigma^2 B^2 \right\rangle - \left\langle \sigma B^2 \right\rangle^2 / \left\langle B^2 \right\rangle}{4\pi \dot{p}^{\,2}} \right) \right], \end{aligned} \tag{75}$$

$$Q_{4M} = \frac{i\omega_\mu \left(1+\tau\right) \omega_{ni} \left(\omega - \omega_{ne}\right)^2 N_1^3}{\dot{q}^2 \left\langle B^2 \right\rangle \omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)^2} \left[\left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle^2 + \frac{A I_1}{\pi \dot{p}^{\,2}} \right], \tag{76}$$

and where

$$\begin{aligned} Q_5 \;=\; & \frac{A^2 N_1^2}{\dot{q}^2 \dot{\chi}^4 \left\langle B^2 \right\rangle} \left[\frac{\dot{\chi}^4}{A} \left\{ \left(1 + a^2 Q_*\right) \Omega - i \frac{\eta_\perp}{\eta_\parallel} \frac{\omega \omega_\eta \left(\omega - \omega_{ne}\right)^2}{\omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)^2} \right\} \frac{1}{A} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle \right. \\ & \left. - \; Q_* \left(1 - \frac{\Omega \dot{\chi}^4}{A 4\pi \dot{p}^{\,2}} \right) \left(\left\langle \sigma^2 B^2 \right\rangle - \frac{\left\langle \sigma B^2 \right\rangle^2}{\left\langle B^2 \right\rangle} \right) \right], \end{aligned} \tag{77}$$

$$\begin{aligned} Q_6 \;=\; & \frac{A^2 N_1^2}{\dot{q}^2 \dot{\chi}^4 \left\langle B^2 \right\rangle} \left[\frac{\dot{\chi}^4}{A} \left\{ \left(1 + a^2 Q_*\right) Q_* + i \frac{\eta_\perp}{\eta_\parallel} \frac{\omega \omega_\eta \left(\omega - \omega_{ne}\right)^2}{\omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)^2} \right\} \frac{1}{A} \left\langle \frac{|\widehat{\nabla} v|^2}{B^2} \right\rangle \right. \\ & \left. + \left\{ Q_* (Q_* + 2) + \frac{\Omega \dot{\chi}^4}{A 4\pi \dot{p}^{\,2}} \right\} \left(\left\langle \sigma^2 B^2 \right\rangle - \frac{\left\langle \sigma B^2 \right\rangle^2}{\left\langle B^2 \right\rangle} \right) \right], \end{aligned} \tag{78}$$

$$I_1 = \frac{\left\langle |\widehat{\nabla} v|^2 \sigma^2 B^2 \right\rangle}{\left\langle B^2 \right\rangle} - \frac{\left\langle \sigma B^2 \right\rangle \left\langle |\widehat{\nabla} v|^2 \sigma B^2 \right\rangle}{\left\langle B^2 \right\rangle^2}, \qquad I_2 = \left\langle \frac{B^2 \sigma^2}{|\nabla v|^2} \right\rangle - \frac{\left\langle \sigma B^2 / |\nabla v|^2 \right\rangle^2}{\left\langle B^2 / |\nabla v|^2 \right\rangle}, \tag{79}$$

and

$$Q_* = \frac{\left(1+\tau\right) \omega \omega_{ni} \left(\omega - \omega_{ne}\right)}{\omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)} \frac{\dot{\chi}^4}{A 4\pi \dot{p}^2}, \qquad \Omega = \frac{\omega \left(\omega - \omega_{ni}\right) \left(\omega - \omega_{ne}\right)}{\omega_A^2 \left(\omega \delta^2 + i\omega_\eta\right)}. \tag{80}$$

Equations (71)–(72) are a fourth order coupled set of flux-surface-averaged differential equations with the effects of electron inertia, drifts, viscosity and transverse particle diffusion included. These effects are not included in Ref. [15]. The above equations describe coupling between visco-resistive ballooning modes and drift-acoustic waves for a general ordering of ω relative to ω_s.

c) Visco-resistive-inertial ballooning modes

In the limit $\omega \gg \omega_s$, $V_0 = V_1 = 0$, sound wave propagation is neglected and the visco-resistive-inertial ballooning mode equation can be shown to be given by

$$\frac{\partial}{\partial X}\frac{X^2}{1+X^2}\frac{\partial U_0}{\partial X} + \frac{H\left(1-H\right)}{\left(1+X^2\right)^2}U_0 - \frac{H\left(1+H\right)X^2}{\left(1+X^2\right)^2}U_0 + D_R U_0 - Q_1 U_0 X^2 - Q_2 U_0 X^4 = 0. \quad (81)$$

The limit of zero viscosity, $\omega_\mu = 0$, implies $Q_2 = 0$ in Eq. (81), and the drift-resistive-inertial ballooning equation is recovered. This equation has the same form as the resistive MHD case considered in Ref. [15] except that the diamagnetic and electron inertia corrections are included in the coefficients Q_1 and Q_0 [defined below in Eq. (83)]. A valid solution can be constructed in the ideal and resistive regions by matching the ideal solution for $|y| \longrightarrow \infty$ to the resistive solution for $|X| \longrightarrow 0$. The resulting general dispersion relation, $\Delta = \Delta'$, is obtained, where Δ' can be calculated by using the conventional definition $\Delta' \equiv a_2/a_1$, the ratio of coefficients of the large and small solutions of the asymptotic form of the ideal solution. The expression for Δ is given by

$$\Delta \equiv \frac{4y_0^{1+2s}Q_1^{(5-2s)/4}}{Q_1 - \left(1+s-H\right)^2}\frac{\Gamma\left[1/2+s\right]}{\Gamma\left[-1/2-s\right]}\frac{\Gamma\left[(1/4)\left(Q_1^{1/2}+3-2s-D_R/Q_1^{1/2}\right)\right]}{\Gamma\left[(1/4)\left(Q_1^{1/2}+1+2s-D_R/Q_1^{1/2}\right)\right]}, \quad (82)$$

where

$$Q_1 = \frac{\omega\left(\omega-\omega_{ni}\right)\left(\omega-\omega_{ne}\right)}{Q_0}, \quad Q_0 = \frac{\dot{q}^2\left\langle B^2\right\rangle\omega_A^2\left(\omega\delta^2+i\omega_\eta\right)}{AN_1M}, \quad (83)$$

$$X^2 = \frac{\dot{q}^2\left\langle B^2\right\rangle\left(\omega\delta^2+i\omega_\eta\right)}{N_1\left(\omega-\omega_{ne}\right)}Z^2 = \frac{Z^2}{y_0^2 Q_1}, \quad y_0^2 = \frac{\omega_A^2}{AM\omega\left(\omega-\omega_{ni}\right)}, \quad (84)$$

$$M = \left\langle\frac{B^2}{|\widehat{\nabla}v|^2}\right\rangle\left[\frac{1}{A}\left\langle|\widehat{\nabla}v|^2/B^2\right\rangle + \frac{1}{4\pi\dot{p}^2}\left\{\left\langle\sigma^2B^2\right\rangle - \frac{\left\langle\sigma B^2\right\rangle^2}{\left\langle B^2\right\rangle}\right\}\right]. \quad (85)$$

The Γ in Eq. (82) is the gamma function. For the special case when $D_R > 0$, this derivation reproduces the stability criterion derived in Ref. [15] with electron inertia and diamagnetic corrections:

$$\omega\left(\omega-\omega_{ne}\right)\left(\omega-\omega_{ni}\right) = -\frac{\omega_A^2\left(\omega\delta^2+i\omega_\eta\right)}{AN_1M}\dot{q}^2\left\langle B^2\right\rangle \times \left[\left\{\left(\frac{1}{2}+s+2n\right)^2+D_R\right\}^{1/2}-\left(\frac{1}{2}+s+2n\right)\right]^2. \quad (86)$$

In the absence of electron inertia and diamagnetic effects, resistive ballooning modes are found with a growth rate proportional to the cube root of the resistivity.

V. Conclusions

A unified theory of resistive and electron inertial ballooning modes is developed in this paper. Linearized ballooning equations are derived from the combination of Ohm's law, the vorticity equation, the continuity equation, and total momentum equation. Coupled second order ordinary differential equations that describe the shear-Alfvén and drift-acoustic modes in general 3-D geometry are presented. A two-scale analysis for resistive and inertial ballooning modes (RIBMs) stability is developed in which the effects of ideal MHD free energy (as measured by the asymptotic matching parameter Δ') as well as the effects of geodesic curvature drives in the non-ideal layer are included in the dispersion relation. In the electrostatic limit, solutions to the dispersion relation of the shear-Alfvén mode lead to an infinite sequence of modes with growth rates scaling as the resistive ballooning mode, $\gamma \sim {\omega_\eta}^{1/3}$ (for $\delta = 0$ and $\omega_{nj} = 0$), or the electron inertia ballooning mode, $\gamma \sim \delta$ (for $\omega_\eta = 0$ and $\omega_{nj} = 0$). In the absence of diamagnetic flow effects, the modes are found to be purely growing. Also, developed in this paper is the coupling of visco-resistive ballooning modes to drift-acoustic waves in the limits of large and small sound wave frequency compared with the mode frequency.

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