The Effect of Three Dimensional Shaping on the Ballooning Stability Properties of Stellarator Equilibria and Tokamak Equilibria with Resonant Magnetic Perturbations

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Abstract

A detailed understanding of how the local shear and normal curvature affect the stability of various highly localized instabilities in 3D configurations is an enticing prospect for several important lines of research in the field of magnetic confinement fusion. There is evidence that Kinetic Ballooning Mode turbulence limits the pedestal pressure gradient in H-mode Tokamak experiments, and a better understanding of the effect of 3D Resonant Magnetic Perturbations (RMPs) on H-modes is crucial for the successful operation of future Tokamaks. There has also been an effort recently to mitigate turbulent transport in Stellarator designs through 3D shaping. A detailed understanding of how the local shear and normal curvature can be manipulated to modify the stability properties of microinstabilities is difficult to obtain given the computational cost and complexity associated with global 3D MHD equilibrium calculations. Local 3D equilibrium theory is implemented numerically to provide a detailed description of the local shear and normal curvature at little computational cost. Equilibrium calculations are coupled with an ideal MHD ballooning stability code to study the physics of local mode stability for both of these applications. By examining local 3D equilibria which model the use of RMP fields, it is shown that ideal MHD ballooning modes are sensitive to the 3D structure of the geodesic curvature and the rotational transform. Resonant components of the geodesic curvature can dominate the Pfirsch-Schluter spectrum and lead to strong modulation of the local shear, which was found to be destabilizing for large pressure gradients for the equilibria examined here. The local 3D equilibrium model is also used to generate a helical equilibrium that can be geometrically deformed to cancel out the toroidal component of the normal curvature, reproducing one property of Quasi-Helically Symmetric (QHS) Stellarators. This modulation of the normal curvature has a significant impact on the magnetic field spectrum and has the effect of uniformly increasing the local shear. Helical modulation of the normal curvature was found to be stabilizing for ideal MHD ballooning modes at high pressure for analysis of one field line. The calculations performed demonstrate the usefulness of the local 3D equilibrium model as a complement to existing efforts to optimize Stellarator configurations for turbulent transport. There is an enticing prospect of using local 3D equilibrium calculations to develop a cost function for use in
present day optimization efforts. These studies lay the groundwork for several useful applications of local 3D equilibrium theory.
## Contents

### Abstract

i

### 1 Background

1.1 Motivation for Nuclear Fusion ................................................. 1
1.2 Magnetic Confinement Fusion ............................................. 2
1.3 The Tokamak ................................................................. 3
  1.3.1 The High Confinement Mode ........................................ 4
  1.3.2 Edge Localized Modes ............................................. 4
  1.3.3 Pedestal Modelling ................................................ 5
  1.3.4 The effect of resonant magnetic perturbations on ELMs ............ 9
1.4 The Stellarator ............................................................ 10
  1.4.1 Transport Optimization of Stellarator Equilibria .................. 10

### 2 Methods

2.1 Magnetohydrodynamics ..................................................... 14
  2.1.1 Ideal MHD ........................................................... 14
  2.1.2 Applicability of the Ideal MHD Model ............................ 15
  2.1.3 MHD Equilibrium Theory ........................................ 16
  2.1.4 MHD Equilibrium in 3D .......................................... 18
2.2 Local 3D Equilibria ..................................................... 20
  2.2.1 Magnetic field line mapping .................................... 21
  2.2.2 Geometric properties of magnetic field lines .................... 22
  2.2.3 Plasma currents .................................................. 23
  2.2.4 Calculation of the jacobian ...................................... 24
  2.2.5 Profile quantities ................................................ 24
  2.2.6 The local shear .................................................. 25
2.2.7 Comparison to global 3D equilibrium calculations .......................... 26

2.3 Ideal MHD Stability Theory ......................................................... 27
  2.3.1 MHD Energy Principle .......................................................... 28
  2.3.2 Ballooning Mode Theory ...................................................... 29
  2.3.3 Periodicity ........................................................................... 31
  2.3.4 Ballooning instabilities in 3D configurations .............................. 33

3 Code Development ........................................................................... 35
  3.1 3DLEQ .................................................................................... 35
    3.1.1 Code overview ................................................................... 35
    3.1.2 Curvature expressions ......................................................... 36
    3.1.3 Numerical methods ............................................................. 38
    3.1.4 Benchmarking ..................................................................... 40
  3.2 Ballooning Stability Code ............................................................ 41
    3.2.1 Code overview .................................................................. 41
    3.2.2 Normalized Ballooning Equation ......................................... 42
    3.2.3 Calculation of Shear and Pfirsch Schluter Current ................. 43
    3.2.4 Numerical methods ............................................................. 44
    3.2.5 Benchmarking ..................................................................... 45

4 Tokamak Equilibria with Resonant Magnetic Perturbations ............. 47
  4.1 Equilibrium Calculations ............................................................ 47
    4.1.1 Motivation ......................................................................... 47
    4.1.2 Equilibrium parametrization ............................................... 48
    4.1.3 Radial magnetic field perturbation ....................................... 49
    4.1.4 Modifications to the MHD Equilibrium Quantities ............... 52
  4.2 Ballooning Stability Analysis ........................................................ 52
    4.2.1 Marginal Stability Diagrams ............................................... 52
    4.2.2 Comparison at fixed average shear, pressure gradient ............ 61
    4.2.3 Discussion ......................................................................... 64
5 Shaping of Stellarator Equilibria

5.1 Equilibrium Calculations ................................................. 73
  5.1.1 Motivation ................................................................. 73
  5.1.2 Equilibrium Parametrization .......................................... 74
  5.1.3 Modifications to the MHD Equilibrium Quantities .............. 75

5.2 Ballooning Stability Analysis ........................................... 83
  5.2.1 Marginal Stability Diagrams .......................................... 83
  5.2.2 Comparison at fixed average shear, pressure gradient ......... 83
  5.2.3 Discussion ................................................................. 88

6 Conclusions ................................................................. 94
  6.1 Development of computational tools .................................. 94
  6.2 The physics of resonant magnetic perturbations ................. 94
  6.3 Optimization of stellarator equilibria .............................. 95

7 Bibliography ................................................................. 97
Chapter 1

Background

1.1 Motivation for Nuclear Fusion

The search for a long term clean, renewable energy source is one of the most important problems facing humankind today. Simply keeping pace with the rapid growth in energy consumption will be a challenge in and of itself. There is also a consensus amongst climate scientists that burning fossil fuels, the dominant source of energy production today, has already caused a substantial increase in global carbon dioxide levels which is beginning to drive a shift in the earth’s climate. Continued emission of these gasses by burning fossil fuels, even at present day rates, will continue to drive unprecedented changes in the global climate. This warming will likely have a substantial impact on humankind, leading to, for example, disruption of food production, increased frequency of extreme weather events, and flooding of low lying population centers due to sea level rise on the order of meters [1].

As a result, the development of renewable energy sources, free of carbon dioxide emissions, has become a major priority for scientific research across the globe. Nuclear fission plants and regenerative sources are possible choices for clean energy production in the near future. However, fission plants rely on a finite supply of fissile material and there are issues with the proliferation of nuclear fuel and the disposal of waste. Regenerative sources require more efficient energy storage than is presently available, and often are only available where there is favorable geography. As recently as 2007, the Renewable Energy Policy Network for the 21st Century found that regenerative sources such as wind and solar power accounted for only 3.4% of global power production.

Nuclear fusion is an attractive energy source for meeting the long term energy needs of mankind. Power generation from controlled nuclear fusion would fuse together light nuclei in order to release a fraction of the nuclear binding energy. The most favorable fusion reaction for use in a man-made reactor
is the fusion of the hydrogen isotopes Deuterium and Tritium, which releases a neutron, alpha particle, and 17.59 MeV of energy. Deuterium compromises approximately 0.015% of hydrogen on earth, making it readily available. Tritium is radioactive with a half-life of 12.3 years, but can be generated through neutron activation of lithium. Estimates based on present-day energy consumption predict that deposits of Deuterium and Lithium (for generation of Tritium) should last on the order of 10,000 years at least [2]. Although the scientific and engineering challenges associated with the construction of a nuclear fusion power plant make it one of the most challenging endeavors undertaken by humanity, the rewards would be similarly spectacular.

1.2 Magnetic Confinement Fusion

In order to sustain fusion reactions, deuterium and tritium nuclei must have enough kinetic energy to tunnel through the Coulomb barrier surrounding each nucleus. For a steady-state population of deuterium and tritium atoms which has settled into a Maxwellian distribution, the peak reaction rate occurs at around 100 keV. Fortunately, reactors utilizing D-T fusion should only need to operate at temperatures around 10 keV [2]. At these temperatures, the reactants will be completely ionized and exist in what is called the plasma state [3].

To confine a reactant mixture of ionized deuterium and tritium at fusion temperatures, a simple material container is not feasible. Fusion experiments on earth must use innovative methods to confine the hot plasma, such as inertial or magnetic forces. This work examines the confinement capabilities of the two prominent types of magnetic confinement devices, the Tokamak and the Stellarator. Stable confinement of ionized nuclei requires a magnetic field with toroidal topology where magnetic field lines helically rotate around the magnetic axis. There are several methods of producing such a configuration.

A Stellarator achieves this using only external magnetic coils which produce a fully three dimensional magnetic field configuration with helical shaping. A Tokamak uses external magnetic coils which are toroidally axisymmetric to produce a strong toroidal magnetic field. The magnetic field lines are given a helical pitch by a poloidal magnetic field which is generated by toroidally flowing currents within the plasma itself. The toroidal current is driven inductively by driving a time-dependent magnetic flux through the center of the torus. A comparison of the magnetic field coils used for each configuration is
shown in Figure 1. Tokamaks typically offer simplicity in design but complexity, as significant plasma currents provide a source of free energy which can excite instabilities leading to decreased confinement or a termination of the discharge. Stellarators have a more complex design but typically allow for easier operation due to favorable geometry and the lack of a net plasma current. This work examines the role of three dimensional magnetic field shaping on both of these configurations.

1.3 The Tokamak

First realized in 1956 at the Kurchatov Institute in Moscow, the Tokamak has since become one of the most prominent and successful methods of magnetic confinement. Only recently have other methods (transport optimized Stellarators) been able to achieve comparable confinement quality, which the Tokamak achieves through its axisymmetry. The cost of axisymmetry is that the helical pitch of magnetic field lines must be maintained by a poloidal magnetic field generated within the plasma by toroidal plasma currents. While the superior confinement quality of an axisymmetric device allows for higher temperatures for a given heating input, the Tokamak is susceptible to current driven instabilities which can be a significant barrier to operation at fusion relevant parameters.

However, significant progress has been made in mitigating current-driven instabilities through control of plasma profiles and also direct suppression using external magnetic coils and electromagnetic
waves. As a result, in the 1990’s several Tokamak experiments (the Tokamak Fusion Test Reactor, or TFTR, at Princeton and the Joint European Torus, or JET, at Culham in the United Kingdom) were able to generate substantial fusion power, albeit over limited time scales. Operation of a Tokamak in steady state at fusion relevant conditions will first be attempted with the ITER device, presently under construction in Cadarache, France. Successful operation of the ITER device at the conditions required for sustained nuclear fusion power generation will require stable, steady-state operation in the high confinement mode [4].

1.3.1 The High Confinement Mode

In 1982, experiments on the ASDEX Tokamak discovered a fundamentally new mode of operation wherein the confinement quality of the discharge abruptly increased when the heating power was raised above a threshold value. [5]. This is in stark contrast to typical Tokamak operation, where confinement quality degrades with increased heating power, due to turbulent heat flows which limit confinement. The improved confinement (high confinement, or H-mode) occurs when strongly sheared poloidal flows spontaneously arise in the edge of the plasma, suppressing turbulence driven by micro-instabilities and producing a transport barrier. While there are a number of known mechanisms which can generate these sheared flows, there is still no comprehensive theory describing the onset of the H-mode.

While the underlying physics of the H-mode is still a very active topic of research to this day, it has been reliably achieved across a wide range of Tokamak and Stellarator experiments over the past decades. H-mode confinement will be required for ITER to achieve its goal of a self sustaining, fusion heated plasma. Therefore it is absolutely crucial that ITER is capable of stable H-mode operation at fusion-relevant parameters. In order to achieve this goal, a method of suppressing the periodic loss of stored energy to the device wall driven by Edge Localized Modes (ELMs) needs to be successfully implemented [4].

1.3.2 Edge Localized Modes

Suppression of turbulence over a localized region of the plasma edge during H-mode operation results in a sharp increase in pressure gradients across this transport barrier. This pressure gradient serves
as a direct free energy source for pressure-curvature driven instabilities, and also drives an appreciable edge current through the bootstrap effect. The resulting current gradients also serve as a source of free energy for current driven instabilities. The primary mechanism which limits the achievable stored energy in a H-mode discharge is intermediate-wavelength peeling-ballooning instabilities called Edge Localized Modes, or ELMs. These instabilities arise from a coupling of pressure-driven ballooning modes to current driven peeling modes in the plasma edge [4].

With sufficient heating power, the pressure profile in an H-mode discharge will continue to rise until the ELM stability boundary is reached. Peeling-ballooning instabilities initially grow with linear, exponential growth. The ballooning structure of the instability draws filaments of hot plasma out toward the device walls. Typically, 5% of the stored energy and plasma particle content is ejected during an ELM, resulting in lowered temperature and density gradients immediately afterwards. The altered plasma profiles are then stable to peeling-ballooning modes, and the cycle repeats with the pressure gradient again rises until the stability boundary is reached. This repeated ejection of high temperature plasma into the wall will be a serious problem for ITER, where 20 Megajoules of energy are expected to be released during each ELM, which would result in an accelerated erosion of wall materials [4]. A successful method of mitigating ELM instabilities will be discussed after a description of predictive H-mode modelling.

1.3.3 Pedestal Modelling

The region over which turbulence transport is suppressed is referred to as the pedestal, with key characteristics being the plasma pressure just inside the transport barrier (the top of the pedestal) and the radial size of the transport barrier (pedestal width). Figure 2 shows experimental profiles, with the width and height labeled. A predictive model of the peak pressure at the pedestal top (pedestal height) based on ELM stability and Kinetic Ballooning Mode (KBM) turbulence has been developed and validated with a series of dedicated experiments on the DIII-D Tokamak. The EPED1 model predicts the width and height of the pedestal given a set of operational parameters \( (B_T, I_p, R, a, \kappa, \delta, \text{global } \beta) \) from two hypotheses [6].
Figure 2: Temperature and density profiles for a H-mode discharge on the DIII-D Tokamak demonstrating the pedestal width and height nomenclature. The blue line is a fit to the experimental data, which is shown in red [7].
The first hypothesis is that the pedestal height in H-mode plasmas is constrained by peeling-ballooning stability. This is well supported given that high resolution diagnostics have allowed for detailed, quantitative tests of the peeling-ballooning model across a range of experiments. The EPED1 model uses a series of equilibria with increasing pedestal pressure, until the peeling-ballooning stability boundary is breached. However, the pedestal height also depends strongly on the width of the pedestal, giving a second condition.

The second hypothesis is that the mechanism which limits the pressure gradient within the pedestal, therefore setting the pedestal width, is the onset of KBM turbulence. The edge transport barrier which leads to the formation of the pedestal is thought to suppress long wavelength drift turbulence through sheared ExB flow. However, local pressure gradients within the pedestal reach a limit well before the peeling-ballooning boundary is reached. The onset of KBM turbulence is only weakly affected by sheared flows, making it a candidate for the limiting mechanism for pressure gradients within the pedestal. There is experimental evidence that pedestal pressure gradients in typical H-mode experiments are large enough to drive KBM turbulence. The criteria for the onset of KBM turbulence is well correlated with the stability boundary (without the region of second stability) for high-n ideal MHD ballooning modes, which forms the basis for EPED1 pedestal width predictions [9].

The poloidal beta at the top of the pedestal is related to the average of the normalized pressure gradient \( <\alpha> \) across the pedestal as

\[
\Delta \sim \beta_{\theta,ped} / <\alpha> \quad (1.1)
\]

where \( \Delta \) is the average of the width of the density and temperature gradients in units of normalized poloidal flux. The relationship between the flux surface averaged local shear, \( s \), and the critical pressure gradient for the onset of ballooning instability \( \alpha_c \) is estimated as

\[
\alpha_c \sim 1/s^{1/2} \quad (1.2)
\]

based off of typical marginal stability diagrams for these equilibria. The surface averaged shear is expected to be dominated by pressure-driven bootstrap currents, giving the relation \( <s> \sim 1/\beta_{\theta,ped} \sim 1/\beta_{\theta,ped} \). Therefore the pedestal width scales as \( \Delta = c\beta_{\theta,ped}^{1/2} \) where typical marginal stability boundaries indicate \( c \sim 0.1 \). A value of \( c = 0.076 \) was determined by examining 4122 time slices from
Figure 3: EPED1 modeling of a DIII-D discharge. The top, solid curve, gives the Peeling-Ballooning stability constraint. The bottom, dotted curve is the KBM pedestal width constraint. The intersection of the two gives the EPED1 prediction, which is in good agreement with the experimental measurement [6].

120 discharges on the DIII-D Tokamak [6]. The EPED1 model predicts the pedestal width as:

$$\Delta = 0.076 \beta_{\theta,ped}^{1/2}.$$  \hfill (1.3)

Pedestal predictions are carried out by constructing a series of 2D MHD equilibria where the pedestal height is raised until the peeling-ballooning stability criteria is reached for a range of pedestal widths, as calculated by the ELITE code. Then, the pedestal width model is evaluated for these equilibria. The pedestal height also scales with the pedestal width, so the pedestal width calculation also provides another prediction for the pedestal height. The result is a set of two curves representing each constraint, with the pedestal prediction given by their overlap. An EPED1 calculation is shown in Figure 3 for a DIII-D discharge.

EPED1 has been benchmarked against data from sets of DIII-D and JT-60U discharges finding good agreement, and another validation was made using a set of dedicated experiments on DIII-D were carried out where EPED1 predictions were performed beforehand. The pedestal height can be measured very accurately in experiment, and the peeling-ballooning stability model is also very well validated. The success of the EPED1 model thus provides strong evidence that KBM turbulence is operative in
setting the pedestal pressure gradient [8].

1.3.4 The effect of resonant magnetic perturbations on ELMs

High performance, stationary H-mode discharges free of ELMs have been achieved in several experiments by applying three dimensional external Resonant Magnetic Perturbations (RMPs). In experiments on the DIII-D Tokamak, a set of coils external to the plasma can be used to apply $n = 3$ magnetic fields with a broad poloidal mode spectrum during ELMing H-mode discharges. With RMP fields of vacuum magnitude $\delta b_r/B_\phi \sim 10^{-4}$ a suppression of ELMs has been demonstrated in a number of configurations on the DIII-D Tokamak, and well as the ASDEX-U Tokamak. While this method has been repeatedly demonstrated experimentally, the details of ELM suppression vary greatly with shaping, collisionality, safety profile, rotation, etc [10].

When the external RMP coils are energized, a rapid pump-out of density from the plasma is observed. The density pedestal height is lowered, and the density pedestal width increases. The temperature pedestal height and gradient increase slightly. Peeling-Ballooning stability calculations assuming a 2D equilibrium show that ELM-free discharges with RMPs lie within the stable region due to the decrease in pressure at the top of the pedestal. One possible explanation for the change in plasma profiles with the addition of RMP fields is that they induce magnetic stochasticity. The large magnetic shear in the edge of Tokamak plasmas results in a large number of closely packed, low order rational surfaces. The magnitude of the RMP fields in a vacuum configuration is well known and suggests that they could create a series of resonant magnetic islands which overlap and destroy magnetic surfaces, leading to increased particle transport [10].

However, several characteristics of the plasma response to RMP fields cast doubt on this explanation. Predictions from quasilinear transport theory using vacuum values for the RMP fields have been unsuccessful in predicting the observed particle and heat transport. That the reduction in the edge pressure gradient is driven by increased particle transport, and not electron heat transport, contradicts our knowledge of stochastic electron heat transport [11]. In addition, there is evidence that the magnitude of the RMP fields within the plasma may be significantly smaller than vacuum levels due to shielding from rotation and the plasma response. Recent calculations incorporating toroidal rotation show that
the resonant component of the RMP fields may be suppressed by up to an order of magnitude, which may be enough to prevent island formation and overlap \[12,13\].

Understanding the mechanism by which RMP fields modify plasma profiles is an active area of research given the importance of successful ELM mitigation during operation of the ITER device. The sensitivity of ELM mitigation to the edge rotational transform intuitively supports the idea of overlapping resonant magnetic islands, but there are a number of reasons to doubt this explanation. It may be necessary to invoke other mechanisms by which RMP fields modify plasma profiles to achieve stabilization of ELMs. A better understanding of how much screening of the magnetic perturbations occurs is needed, and in turn the response of the equilibrium (i.e. changes to the spectrum of Pfirsch-Schluter currents) to these perturbations must be understood.

1.4 The Stellarator

An alternative class of magnetic confinement devices achieves the required helical pitch of magnetic field lines by directly imposing it with external 3D coils. While the simplicity in design of a Tokamak’s axisymmetric magnetic toroidal field coils is lost, no internal plasma currents are required to produce the confining magnetic field. Driving appreciable internal plasma currents requires a transient swing of magnetic flux through the center of the device, and provides a drive for an entire class of current driven instabilities. Therefore Stellarators are more amenable to steady state operation and are robust against disruptions.

1.4.1 Transport Optimization of Stellarator Equilibria

In general, classical Stellarator configurations do not confine the drift trajectories of particles trapped within wells of magnetic fields strength. Initial Stellarator experiments suffered from large particle and heat fluxes due to their poor neoclassical (referring to transport driven by the effect of particles being trapped within regions of weak magnetic field strength) transport properties relative to axisymmetric devices. In these initial designs, particles trapped within regions of weak magnetic field strength experience a net outward radial drift, creating a channel for direct particle loss, particularly at high temperatures. Beginning in the 1980’s, research has focused on overcoming this significant deficiency.
In 1983 Boozer showed that a 3D configuration only needed a magnetic field structure with symmetry in magnetic coordinates (not in physical space) to achieve comparable transport properties to a physically symmetric configuration [14]. In 1988, Nuhrenberg and Zille realized this idea by developing the first Quasi-Helical configuration [15].

The geometric flexibility of Stellarators has lead to the realization of several different methods of optimization with regard to neoclassical transport. These efforts have been quite successful, as modern day Stellarators exhibit neoclassical transport properties similar to axisymmetric devices, where the dominant transport is driven instead by turbulent fluctuations resulting from microinstabilities. Taking advantage of the inherent geometric flexibility of Stellarator configurations, attention is now being turned to the mitigation of turbulent transport [16]. In recent years there have been a number of computational tools developed which allow for the study of gyrokinetic microturbulence in Stellarator configurations, such as the GENE/GIST code package [17].

Simulations of turbulence driven by Ion Temperature Gradient (ITG) modes have demonstrated that the turbulent heat flux depends strongly on geometry, with physics similar to ideal MHD ballooning modes (which are the focus of this work.) Unstable eigenmodes tend to localize within regions where the normal curvature is negative. The local shear also plays an important role in the localization of these modes. It has been shown that by artificially doubling the magnitude of the local shear (and adjusting other equilibrium quantities to ensure consistency) for a given configuration, turbulent heat transport could be lowered by as much as 30%. Both the normal curvature and the local shear are intimately related to the physical geometry of flux surfaces, and thus turbulent heat transport is very sensitive to geometry. By combining the GENE/GIST code package with the STELLOPT optimization code and the VMEC 3D MHD equilibrium code, a method of optimizing Stellarator configurations for turbulent transport has been developed.

STELLOPT minimizes a cost function by varying the shaping of 3D MHD equilibria produced by running the VMEC code. While it would be ideal to calculate the turbulent heat flux directly with GENE/GIST during the optimization loop, this is too computationally intensive. A single flux tube simulation can require as much as 100 CPU days of computation time, and STELLOPT typically transitions through hundreds of equilibria in one run. Instead, an analytic approximation for turbulent heat flux derived from quasilinear theory has been used as a proxy function for the turbulent heat
flux during the optimization loop. Applying this method to the equilibrium for the National Compact Stellarator eXperiment (NCSX) resulted in a configuration with turbulent heat transport reduced by a factor of 2 to 2.5. However, the analytic proxy function significantly underestimates turbulent heat flux for other equilibria, thus making the optimization less effective. A comparison of the proxy function to the heat flux obtained from nonlinear gyrokinetic simulations with the GENE code is shown in Figure 4.

This optimization process has been applied to an axisymmetric Tokamak equilibrium as well. The optimization process reduced heat flux driven by ITG modes by inwardly deforming the flux surface on the inboard side, a feature which has been previously shown to stabilize ideal MHD ballooning modes as well. GENE simulations showed that the optimized configuration had lower levels of Electron Temperature Gradient (ETG) driven heat transport as well, even though the proxy function was derived from ITG physics [18]. The similar response of ITG, ETG, and MHD ballooning modes to optimization through shaping suggests that there are common aspects to the physics of these instabilities, and the well-understood physics of ballooning modes can be used to guide optimization efforts. Computationally, the growth rate and eigenmodes of ideal MHD ballooning modes can be evaluated at drastically lower cost than full nonlinear gyrokinetic simulations of ITG or ETG turbulence. Therefore it may even be useful to incorporate ballooning stability calculations into the STELLOPT optimization loop as an alternative proxy for ITG growth rates. The study of gyrokinetic microturbulence in Stellarator configurations is a developing field in and of itself, and the prospects for further optimization provide a strong motivation to further understand three dimensional shaping and its impact on various highly localized instabilities.
Figure 4: Comparison of the analytic proxy function for heat flux (red) and the heat flux obtained from nonlinear gyrokinetic simulation (black) for several configurations [17].
Chapter 2

Methods

2.1 Magnetohydrodynamics

2.1.1 Ideal MHD

Magnetohydrodynamics (MHD) is one of the simplest models for describing the macroscopic dynamics of a magnetically confined plasma. MHD provides a self-consistent description of the collective behavior of a plasma by treating it as a single-species conducting fluid. This combines the equations of fluid dynamics with Maxwell’s equations, resulting in a coupled, nonlinear system of partial differential equations [19].

In particular, this work utilizes the ideal MHD model which neglects any finite resistivity of the conducting fluid. The equations of ideal MHD are:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} &= \mathbf{J} \times \mathbf{B} - \nabla p, \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= 0, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}, \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \\
\frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) &= 0,
\end{align*}
\]

where \( \rho \) is the mass density of the plasma, \( \mathbf{v} \) is the fluid velocity, \( \mathbf{J} \) is the plasma current density, \( \mathbf{B} \) is the magnetic field, \( p \) is the plasma pressure, \( \gamma \) is the ratio of specific heats and \( \mathbf{E} \) is the electric field.
2.1.2 Applicability of the Ideal MHD Model

The limits of applicability for ideal MHD follow from the assumptions used to derive the model. MHD can be derived from a kinetic equation describing the exact evolution of a probability distribution function which provides a continuous representation of all the particles within the plasma (in three spatial dimensions in both physical space and velocity space). The assumption of a Maxwellian distribution implies that MHD is only applicable for sufficiently collisional plasmas, as binary collisions tend to drive the plasma towards a Maxwellian distribution. Any effects resulting from a deviation from thermodynamic equilibrium will not be described by ideal MHD. By assuming that the plasma has settled into a local thermodynamic equilibrium and the distribution function is a Maxwellian in velocity space, a set of fluid equations describing the electron and ion populations can be derived by taking moments of the kinetic equation for each species. These fluid equations can be further simplified to obtain the ideal MHD equations by ignoring electron momentum (as the electron mass is much smaller than the ion mass) and assuming a scalar pressure [19].

MHD is aimed at modeling long wavelength, low frequency dynamics. Specifically, it is assumed that $\omega \ll \omega_p = (n_e e^2/\epsilon_0 m_e)^{1/2}$ and $\lambda \gg \lambda_D = (\epsilon_0 k_B T_e/n_e e^2)^{1/2}$. $\omega_p$ is the frequency of electron plasma oscillations which are not described by MHD because they occur on timescales faster than the plasma reaches a Maxwellian equilibrium. $\lambda_D$ is the Debye wavelength, the length scale over which individual electrons will shield electric fields, behavior which MHD cannot describe as it does not explicitly capture electron dynamics. The result of these assumptions is that quasineutrality will be maintained (i.e. $\nabla \cdot \mathbf{J} = 0$, no separation of charges). At these time and length scales, the displacement current in Ampere’s law is negligible, so it is neglected in the derivation as well.

Ideal MHD in particular ignores any finite resistivity or viscosity of the plasma. Therefore ideal MHD does not include any dissipative mechanisms, and any behavior dependent on finite electrical resistivity (which is typically on longer time scales) is neglected. One important consequence of this assumption is that magnetic flux is frozen into fluid elements, and no changes in magnetic topology are allowed. As previously discussed, one important aspect of MHD is that the electron-ion fluid is treated as a single species fluid. Net flows in the plasma are assumed to be dominated by ions. The magnitude
of forces arising from separate electron dynamics can be characterized as:

\[
\frac{|F_{\text{2Fluid}}|}{|F_{\text{MHD}}|} \sim \frac{|\nabla \rho_e|}{n_e e} \frac{1}{|\mathbf{v} \times \mathbf{B}|} \sim \frac{\rho_i}{L} \tag{2.7}
\]

where \( L \) is the length scale of interest and \( \rho_i \) is the radius of ion gyromotion about magnetic field lines. This ratio is assumed to be \( \ll 1 \), which is satisfied for typical MHD dynamics. When the length scale of behavior of interest approaches \( \rho_i \), the assumptions behind MHD begin to break down and a model including separate electron dynamics is required.

Generally speaking, ideal MHD is applied to plasma dynamics where \( t \sim R_0/v_A \), with \( v_A = B/(\mu_0 \rho)^{1/2} \) being the Alfven velocity. For sufficiently collisional plasmas where dissipation is negligible at these scales, MHD provides an excellent description of plasma behavior which is substantially more tractable than a full kinetic description. Ideal MHD has been highly successful at explaining fast-growing, macroscopic instabilities in a range of magnetic confinement experiments. In the case of Tokamaks, ideal MHD has lead to significant improvements in performance through equilibrium shaping and external control of instabilities with external coils [19].

### 2.1.3 MHD Equilibrium Theory

Taking the Ideal MHD equations and setting all time derivatives and plasma flows to zero leads to the equations of MHD equilibrium in the magnetostatic limit.

\[
\mathbf{J} \times \mathbf{B} = \nabla p \quad \tag{2.8}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \tag{2.9}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \tag{2.10}
\]

This system describes the time-independent configurations which are the basis for magnetic confinement fusion reactors. By projecting the first equation onto \( \mathbf{B} \) it is shown that \( \mathbf{B} \cdot \nabla p = 0 \), implying that the pressure will be constant along magnetic field lines. Establishing a MHD equilibrium requires that magnetic field lines lie on nested, toroidal surfaces.

In a toroidal coordinate system, the poloidal flux \( \psi \) can be defined as the integral of the magnetic field through a poloidal slice \( \psi = \int dS_\theta \cdot \mathbf{B} \). With the toroidal angular coordinate being \( \zeta \), the
axisymmetric magnetic field can be written as

\[
\mathbf{B} = I(\psi)\nabla \zeta + \nabla \zeta \times \nabla \psi
\]  

(2.11)

where \(I(\psi')\) is the toroidal current contained inside the surface where \(\psi = \psi'\). For fully three dimensional equilibria, the magnetic field cannot generally be written this way. If the configuration is axisymmetric, the equilibrium equations can be simplified into one partial differential equation for \(\psi\)

\[
\Delta^* \psi = -\mu_0 R^2 \frac{dp(\psi)}{d\psi} - I(\psi) \frac{dI(\psi)}{d\psi}
\]  

(2.12)

where the \(\Delta^*\) operator is an elliptic differential operator given by

\[
\Delta^* \psi = R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}
\]  

(2.13)

Equation 2.12 is the Grad-Shafranov equation, a nonlinear elliptic partial differential equation describing axisymmetric MHD equilibria [16]. For solutions to this equation, the magnetic field lines lie entirely on a set of nested 'flux surfaces', i.e. isosurfaces of \(\psi\). In addition to pressure being constant on flux surfaces,

\[
\mathbf{J} \cdot \nabla p = 0,
\]  

(2.14)

i.e. no plasma currents flow normal to flux surfaces. This property will be central to the derivation of the local 3D equilibrium model. One chooses two of the three flux functions \([p(\psi), I(\psi), q(\psi) = d\psi_{\text{tor}}/d\psi_{\text{pol}}]\) and appropriate boundary conditions. Solving for \(\psi\) as a function of the physical space coordinates determines the remaining flux function and gives a solution to the ideal MHD equilibrium equations [19].

The flux function \(q(\psi)\) has a more intuitive, physical meaning as well. An equivalent definition is given by

\[
q(\psi) = \left( \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \Delta \theta_i \right)^{-1},
\]  

(2.15)

where \(\Delta \theta_i\) is the change in poloidal angle experienced by a magnetic field line for one toroidal transit. The safety factor describes the number of poloidal transits a magnetic field line makes during one toroidal transit.

Another important property of the magnetic fields in an MHD equilibrium configuration is that straight field line coordinates can be utilized. In a coordinate system where the angular variables are
the typical, geometric angles, the path of magnetic field lines is often a complicated function of both angles. With the proper choice of angular variables (say $\psi, \Theta, \Phi$) the magnetic field can be written simply as
\[
B = \nabla \psi \times \nabla \Phi + q(\psi) \nabla \psi \times \nabla \Theta, \tag{2.16}
\]
where
\[
d\Theta/d\Phi = 1/q(\psi). \tag{2.17}
\]
In a straight field line coordinate system, the path of a magnetic field line is described by a linear combination of $\Theta$ and $\Phi$. Instead of integrating an ordinary differential equation to obtain the path of magnetic field lines, they can be obtained simply through solutions to Equation 2.17. Straight field line coordinates are also extremely useful when solving differential equations which contain differentials along magnetic field lines, as those differentials can be separated into a linear combination of differentials in $\Theta$ and $\Phi$.

2.1.4 MHD Equilibrium in 3D

The Grad-Shafranov equation provides an eloquent and practical description of ideal MHD equilibrium in axisymmetric systems. An analogous partial differential equation can be derived for a three dimensional system with some rigorous physical symmetry (i.e. helical symmetry), but for systems which are 'fully three dimensional' (i.e. there is no physical symmetry), Grad-Shafranov theory does not provide a rigorous description of the equilibrium.

To demonstrate this, assume that some fully three dimensional MHD equilibrium configuration with nested flux surfaces exists. With the proper choice of angular coordinates, the magnetic field can be written as in Equation 2.16. A non-zero $dp/d\psi$ implies the existence of a diamagnetic current providing confinement, $J_\perp = (B \times \nabla p)/B^2$. Quasineutrality gives a relationship between the parallel and perpendicular currents,
\[
\nabla \cdot J = 0 = \nabla \cdot (J_\parallel B + J_\perp), \tag{2.18}
\]
\[\nabla \cdot \left( \frac{J_\parallel B}{B} \right) = (B \cdot \nabla) \frac{J_\parallel}{B}. \tag{2.19}\]
In a straight field line coordinate system the differential along \( \mathbf{B} \) becomes
\[
(B \cdot \nabla) \sim \left( \frac{\partial}{\partial \Theta} + q(\psi) \frac{\partial}{\partial \Phi} \right).
\] (2.20)

Fourier decomposing the magnetic field and parallel current as
\[
\frac{1}{B^2} = \sum_{mn} \left( \frac{1}{B^2} \right)_{mn} \cos(m\Theta - n\Phi),
\] (2.21)
\[
\frac{J_B}{B} = \sum_{mn} \left( \frac{J_B}{B} \right)_{mn} \cos(m\Theta - n\Phi),
\] (2.22)
and inserting into Equation 2.19 leads to
\[
\sum_{mn} \left( \frac{J_B}{B} \right)_{mn} [m - nq(\psi)] \sin(m\Theta - n\Phi) \sim \frac{\partial p(\psi)}{\partial \psi} \sum_{mn} \left( \frac{1}{B^2} \right)_{mn} \sin(m\Theta - n\Phi). \] (2.23)

Herein lies the problem of 3D MHD equilibrium. Dividing each side by \((m - nq(\psi))\), at every surface in the plasma where \(m = nq(\psi)\), Equation 2.23 implies there is an infinite parallel current unless \((1/B^2)_{mn}\) is zero. In other words, with the ideal MHD model, the quasineutrality constraint cannot generally be satisfied at every rational surface for a fully three dimensional configuration.

While the previous calculation has shown that constructing general solutions to the MHD equilibrium equations in 3D is nontrivial, it is know from experience that such equilibria do exist. A number of codes have been developed to obtain numerical solutions to \(\mathbf{J} \times \mathbf{B} = \nabla p\) in fully 3D configurations. The Variational Moments Equilibrium Code (VMEC) is one of the most widely used tools for calculating solutions to the MHD equilibrium equations in 3D, having been used to design a number of experiments. VMEC assumes the existence of a set of nested flux surfaces and then minimizes the total plasma energy,
\[
W = \int \left( \frac{|B|^2}{2\mu_o} + \frac{p}{\gamma - 1} \right) d^3x.
\] (2.24)

The position of magnetic flux surfaces (denoted by the radial coordinate \(s\)) is written as a Fourier series in the angular variables,
\[
R(s, \theta, \zeta) = \sum_{m=0}^{m=M} \sum_{n=-N}^{n=N} R_{mn}(s) \cos(m\theta - n\zeta)
\] (2.25)
\[
Z(s, \theta, \zeta) = \sum_{m=0}^{m=M} \sum_{n=-N}^{n=N} Z_{mn}(s) \sin(m\theta - n\zeta).
\] (2.26)

A system of nonlinear, coupled ordinary differential equations in \(s\) for \(R_{mn}(s)\) and \(Z_{mn}(s)\) are solved using a steepest descent iteration method to compute the equilibrium which minimizes \(W\) for the given
boundary conditions. The shape of the outermost flux surface can be explicitly chosen, or the shape of
the device boundary can be chosen allowing the outermost flux surface to evolve freely [20].

The computational cost associated with calculating solutions to the MHD equilibrium equations
over an entire 3D plasma volume is significant, especially for shaping studies where a large number of
equilibria are calculated in succession. Exercising control over the flux surface shape across the plasma
volume is also nontrivial. While global 3D MHD equilibrium calculations are the basis for the design
of Stellarator experiments, this work seeks to complement global equilibrium calculations by using an
alternative method to obtain solutions to the 3D MHD equilibrium equations.

2.2 Local 3D Equilibria

The Local 3D Equilibrium model used in this work owes its origins to the work of Greene and Chance,
who developed a method of plasma profile variation for a given equilibrium flux surface in axisymmetry.
Their work allows one to self consistently vary the surface averaged local shear, pressure gradient, or
surface averaged parallel current on a flux surface without recomputing the entire equilibrium, and
was initially used to study the marginal stability boundary of ideal MHD ballooning modes in a space
parametrized by the pressure gradient and average shear. This has since become the standard method
of examining the ballooning stability properties of a given equilibrium, and is utilized in this work as
well. Miller and coworkers extended the work of Greene and Chance to allow self consistent modifica-
tions to the flux surface geometry as well. The model developed by Miller and coworkers has lead to
fruitful insights into the relationship between equilibrium geometry and the stability of highly localized
instabilities such as ballooning modes and ion temperature gradient (ITG) modes [21].

The work of Miller and coworkers was then extended to general 3D configurations by Hegna and
Nakajima in Reference [22] and was then summarized in Reference [23]. The local 3D equilibrium
model from [23] is the basis for this work. In axisymmetry, the MHD equilibrium in the vicinity of one
magnetic surface can be perturbed self consistently by using the Grad Shafranov equation to constrain
the perturbation to equilibrium quantities. In 3D, there is no equivalent Grad Shafranov equation, so
the full set of ideal MHD equilibrium equations must be used. After choosing a parametrization for
the flux surface shape in physical space, the rotational transform and two profile quantities, the MHD
equilibrium equations are used to construct a self-consistent set of magnetic fields and currents at a particular magnetic surface. A constraint comes in the form of a linear partial differential equation derived using the machinery of differential geometry and the property that no plasma currents flow normal to flux surfaces. This ensures that the equilibrium parametrization satisfies the equations of ideal MHD equilibrium in the vicinity of the surface of interest, though there is no guarantee that the solution can be realized within a global equilibrium.

In this section the procedure by which one constructs a local 3D equilibrium is described. As an aside, it is important to note that the following theory does not solve the problem of infinite currents at rational surfaces. A number of magnetic differential equations (i.e. \((\mathbf{B} \cdot \nabla) h = \ldots\)) are solved in the process of constructing a local 3D equilibrium solution. In practice, one can either choose an irrational surface or constrain the resonant component of the parallel currents to be zero at the surface of interest.

### 2.2.1 Magnetic field line mapping

The existence of a set of closed, nested flux surfaces and a general straight field line coordinate system \((\theta, \zeta)\) is assumed. The position of a magnetic flux surface in physical space is parametrized as a function of the straight field line coordinates as:

\[
x_0(\theta, \zeta) = \left( R(\theta, \zeta) \hat{R}, \phi(\theta, \zeta) \hat{\phi}, Z(\theta, \zeta) \hat{Z} \right).
\]  

(2.27)

In the vicinity of the surface of interest, this mapping is expanded in the radial flux coordinate \((\psi)\) as:

\[
x(\psi, \theta, \zeta) = x_0(\psi_0, \theta, \zeta) + (\psi - \psi_0) \frac{\partial x}{\partial \psi}(\psi_0, \theta, \zeta) + \ldots
\]  

(2.28)

As second order terms in \((\psi - \psi_0)\) do not enter into the leading order description of any MHD equilibrium quantity, only \(\partial x / \partial \psi\) remains to be determined. The equations of ideal MHD equilibrium are used to determine this term and ensure that this mapping gives a consistent equilibrium. However, with the choice of \(x_0\) a number of properties of the magnetic field are set. The following metric elements depend only on derivatives of \(x_0\) within the flux surface:

\[
g_{\zeta \zeta} = \frac{\partial x_0}{\partial \zeta} \cdot \frac{\partial x_0}{\partial \zeta},
\]  

(2.29)

\[
g_{\zeta \theta} = \frac{\partial x_0}{\partial \zeta} \cdot \frac{\partial x_0}{\partial \theta},
\]  

(2.30)

\[
g_{\theta \theta} = \frac{\partial x_0}{\partial \theta} \cdot \frac{\partial x_0}{\partial \theta}.
\]  

(2.31)
The mapping is completed once the jacobian,
\[ \sqrt{g} = \frac{\partial x_0}{\partial \psi} \cdot \left( \frac{\partial x_0}{\partial \theta} \times \frac{\partial x_0}{\partial \zeta} \right), \] (2.32)
is calculated. As seen in the definition of the jacobian, this is equivalent to calculating \( \partial x / \partial \psi \). Once the jacobian is known, the magnetic field and gradient of magnetic flux (|\nabla \psi|) can be constructed:
\[ B = \frac{1}{\sqrt{g}} \left( \frac{\partial x_0}{\partial \zeta} + i_0 \frac{\partial x_0}{\partial \theta} \right), \] (2.33)
\[ \nabla \psi = \frac{1}{\sqrt{g}} \left( \frac{\partial x_0}{\partial \zeta} + i_0 \frac{\partial x_0}{\partial \theta} \right). \] (2.34)

The choice of flux surface parametrization (R, \( \phi \), Z) explicitly determines the terms in parenthesis, and sets the direction of magnetic field lines within the surface. Calculating the jacobian is equivalent to setting the magnetic field strength on the surface.

### 2.2.2 Geometric properties of magnetic field lines

Equations 2.33 and 2.34 can be used to construct normal and tangent vectors for the magnetic field,
\[ \hat{b} = \frac{\frac{\partial x_0}{\partial \zeta} + i_0 \frac{\partial x_0}{\partial \theta}}{(g_{\zeta \zeta} + 2i_0 g_{\zeta \theta} + i_0^2 g_{\theta \theta})^{1/2}}, \] (2.35)
\[ \hat{n} = \frac{\frac{\partial x_0}{\partial \theta} \times \frac{\partial x_0}{\partial \zeta}}{(g_{\zeta \zeta} g_{\theta \theta} - g_{\zeta \theta}^2)^{1/2}}. \] (2.36)

By taking derivatives of these unit vectors (and the binormal vector \( \hat{b} \times \hat{n} \)) along the magnetic field, a set of equations analogous to Frenet’s formulas can be constructed:
\[ (\hat{b} \cdot \nabla) \hat{b} = \kappa_n \hat{n} + \kappa_g \hat{b} \times \hat{n}, \] (2.37)
\[ (\hat{b} \cdot \nabla) \hat{n} = -\kappa_n \hat{b} + \tau_n \hat{b} \times \hat{n}, \] (2.38)
\[ (\hat{b} \cdot \nabla) \hat{b} \times \hat{n} = -\tau_n \hat{n} - \kappa_g \hat{b}, \] (2.39)

where \( \kappa_n \) is the normal curvature, \( \kappa_g \) is the geodesic curvature, and \( \tau_n \) is the normal torsion for magnetic field lines.

One particularly useful property of the local 3D equilibrium model is that these geometric quantities are explicitly determined by the flux surface parametrization and can be analytically calculated. These quantities play an important role in the stability of highly localized instabilities, and exact knowledge...
of them allows one to choose equilibrium parametrizations that modify them in ways that are relevant to the mitigation of these instabilities.

### 2.2.3 Plasma currents

Once the magnetic field is known the current density can be calculated from Ampere’s law ($\nabla \times B = \mu_0 J$) in a straightforward manner,

$$\mu_0 J \cdot \nabla \Phi^i = \epsilon_{ijk} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \Phi^k} \frac{g_{ij} + i g_{ij}}{\sqrt{g}}, \quad (2.40)$$

where $\Phi = (\psi, \theta, \zeta)$ and $\epsilon_{ijk}$ is the Levi-Civita tensor. From the quasineutrality equation

$$\nabla \cdot J = B \cdot \nabla \left( \frac{J \cdot B}{B^2} \right) + \nabla \cdot \left( \frac{B \times \nabla p}{B^2} \right) = 0, \quad (2.41)$$

one can derive an expression describing the plasma currents parallel to the magnetic field:

$$\mu_0 \frac{J \cdot B}{B^2} = \sigma + \frac{d \rho}{d \psi} \lambda. \quad (2.42)$$

$\sigma$ is the flux surface average of the parallel currents, given by

$$\sigma = \mu_0 \frac{J \cdot B}{B^2}, \quad (2.43)$$

where the $<>$ operator is a flux surface average defined by

$$< Q > = \frac{\oint d\theta \oint d\zeta \sqrt{g} Q}{\oint d\theta \oint d\zeta \sqrt{g}} = \frac{\oint d\theta \oint d\zeta \sqrt{g} Q}{V}. \quad (2.44)$$

The Pfirsch-Schluter coefficient ($\lambda$) is specified by the magnetic differential equation:

$$B \cdot \nabla \lambda = 2 \mu_0 \kappa_g \frac{|\nabla \psi|}{B}. \quad (2.45)$$

While the plasma currents perpendicular to the magnetic field typically do not need to be explicitly calculated for local mode stability analysis, they can be obtained through Equation 2.40 by calculating $\partial x / \partial \psi$ on the surface of interest, given by:

$$\frac{\partial x_0}{\partial \psi} = \frac{1}{|\nabla \psi|} \hat{n} + \hat{n} \cdot \hat{b} \times \hat{n}. \quad (2.46)$$

The quantity $D$ contains the variation of the local shear within the flux surface, and will be examined further later in this section. By taking the MHD force balance equation and projecting it into the $\hat{n}$
direction, a magnetic differential equation for $h$ can be derived:

$$\mathbf{h} = -\frac{dp}{d\psi} \frac{1}{B^2} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \psi} \sqrt{g} - \iota \frac{g_{\theta \zeta} + \iota \theta g_{\theta \theta}}{(B\sqrt{g})^2}$$

$$+ 2 \frac{\kappa_n}{|\nabla \psi|} + D(B \cdot \nabla) \lambda. \quad (2.47)$$

Once $h$ and $D$ have been obtained, one can calculate all of the plasma currents in addition to the magnetic field and its geometric properties. However, the machinery laid out here for calculating the magnetic fields and plasma currents implicitly assumes that the inverse coordinate mapping, $x_\alpha$, has been fully determined. A method of constraining the jacobian and thus completing the coordinate mapping needs to be developed in order to use these expressions.

### 2.2.4 Calculation of the jacobian

Once the parametrization for the shape of the flux surface ($x_\alpha$) and two of the three profile quantities ($\iota', \sigma, p'$) are chosen, the jacobian needs to be constrained using only this information. This can be done by noting that ideal MHD equilibrium theory dictates that plasma currents cannot flow across flux surfaces, i.e. $\hat{n} \cdot J = 0$. Inserting Equation 2.40 for $J$ and Equation 2.36 for $\hat{n}$, an equation for the jacobian is obtained:

$$\frac{\partial}{\partial \theta} \frac{g_{\zeta \zeta} + \iota \theta g_{\zeta \theta}}{\sqrt{g}} = \frac{\partial}{\partial \zeta} \frac{g_{\theta \zeta} + \iota \theta g_{\theta \theta}}{\sqrt{g}}. \quad (2.48)$$

This is a first order partial differential equation for $\sqrt{g}$ which depends only on $g_{\theta \theta}, g_{\theta \zeta}$ and $g_{\zeta \zeta}$, all of which are determined by the choice of $x_\alpha$. While there is no guarantee that a particular choice of flux surface shape will lead to a solution of Equation 2.48, once a solution for the jacobian is obtained all of the MHD equilibrium quantities can be calculated, only requiring the solution of a few magnetic differential equations.

### 2.2.5 Profile quantities

One common method of analysis for local mode stability is the self-consistent variation of profile quantities. For example, ballooning mode stability is typically studied by self consistently varying the pressure gradient and surface-averaged shear for a given flux surface, and obtaining a marginal stability curve as a function of these two quantities. Another choice for a profile quantity is the flux surface averaged
parallel current, $\sigma$. However, for a given choice of $i'$ and $p'$, the surface averaged parallel current is determined. The local 3D equilibrium model requires two of these quantities to be chosen and uniquely determines the third.

Once an equilibrium parametrization has been chosen and the jacobian has been calculated, these profile quantities can be self consistently varied subject to the following profile relationship:

$$i' = \sigma \left< \frac{B^2}{|\nabla \psi|^2} \right> \frac{V'}{4\pi^2} + p' \left< \frac{B^2}{|\nabla \psi|^2} \lambda \right> \frac{V'}{4\pi^2} - 2 \left< \frac{B^2}{|\nabla \psi|^2} \tau_n \right> \frac{V'}{4\pi^2}. \quad (2.49)$$

This also provides a concise description of how the magnetic geometry affects the relationship between shear and plasma currents. The first term represents a shearing of the magnetic field due to the net current flowing on the surface. The second term represents shearing of the magnetic field from parallel Pfirsch-Schluter currents driven by the pressure gradient. The third term shows the shearing of the magnetic field caused purely by geometry. The magnetic field properties and Pfirsch-Schluter coefficient can be calculated once for a given equilibrium, and then any two of these profile quantities can be varied consistent with Equation 2.49. For typical local mode stability analysis, the only quantity which must be recomputed as these profiles are varied is the local shear.

### 2.2.6 The local shear

The local shear of the magnetic field plays an important role in the stability of highly localized instabilities. These instabilities typically align with the structure of the equilibrium magnetic field because the bending magnetic field lines have a strong, stabilizing effect. A shearing of the magnetic field across surfaces can diminish the radial extent of a given mode and decorrelate turbulent fluctuations on different surfaces. One useful property of the Local 3D Equilibrium model is that it provides a number of expressions which clearly demonstrate how different physical mechanisms contribute to the local shear.

The local shear can be defined in terms of the binormal unit vector as

$$s = (\hat{b} \times \hat{n}) \cdot \nabla \times (\hat{b} \times \hat{n}). \quad (2.50)$$

By inserting Equations 2.35 and 2.36 into this expression, the local shear can be written in the form typically used for ballooning stability analysis,

$$s = \frac{|\nabla \psi|^2}{B^2} (\hat{B} \cdot \nabla) [i' \zeta + D]. \quad (2.51)$$
The quantity $D$ represents the local variation of the magnetic shear and is given by

$$
(B \cdot \nabla)D = \sigma \left( \frac{B^2}{|\nabla \psi|^2} \right) \left( \frac{V'}{\sqrt{4\pi}} \right) + p' \left( \frac{B^2}{|\nabla \psi|^2} \right) \left( \frac{V'}{\sqrt{4\pi}} \right) + 2 \left( \frac{B^2}{|\nabla \psi|^2} \right) \left( \frac{V'}{\sqrt{4\pi}} \right) - \frac{B^2}{|\nabla \psi|^2} .
$$

(2.52)

$D$ can be written in terms of $\iota'$ and $p'$ by combining this expression with Equation 2.49:

$$
(B \cdot \nabla)D = \iota' \left( \frac{B^2}{|\nabla \psi|^2} \left( \frac{1}{\sqrt{g}} - \frac{1}{\sqrt{g}} \right) \right) + p' \left( \frac{B^2}{|\nabla \psi|^2} \right) \left( \frac{V'}{\sqrt{4\pi}} \right) + 2 \left( \frac{B^2}{|\nabla \psi|^2} \right) \left( \frac{V'}{\sqrt{4\pi}} \right) - \frac{B^2}{|\nabla \psi|^2} .
$$

(2.53)

For analysis of the ballooning stability properties of 3D local equilibria, the surface averaged shear and pressure gradient can be systematically varied and only the local shear needs to be recalculated for changes in these profile quantities.

Another useful expression for the local shear can be derived by examining the parallel component of Equation 2.40,

$$
\frac{\mu_0 J \cdot \hat{b}}{B} = \iota' \frac{|\nabla \psi|^2}{\sqrt{g}} - x_0' \left( \frac{B \times \nabla \psi}{B^2} \cdot \nabla \right) B + \frac{B \times \nabla \psi}{B^2} \cdot (B \cdot \nabla) x_0'.
$$

(2.54)

Using Equations 2.39, 2.46 and 2.51 this can be written simply as

$$
s = \mu_0 \frac{J \cdot \hat{b}}{B} - 2\tau_n.
$$

(2.55)

The normal torsion is a geometric property of the flux surface shape, and the parallel currents are described by the profile quantities and the Pfirsch-Schluter spectrum. This expression highlights the role of geometry and profile effects on the local shear.

### 2.2.7 Comparison to global 3D equilibrium calculations

Local 3D equilibrium theory provides a complement to existing global equilibrium codes. As there is no guarantee that a solution to the local 3D equilibrium equations can be realized within an actual global
equilibrium, it cannot be used for the design of an experiment. It does not solve the problem of infinite
current sheets at rational surfaces, it merely sidesteps the problem by either choosing an irrational
surface, or for a rational surface, constraining the resonant component of the parallel currents at the
surface of interest to be zero. Local equilibrium theory facilitates the study of localized instabilities in
3D configurations, and can provide insights to aid present day Stellarator optimization efforts.

The only partial differential equation which needs to be solved in constructing solutions to the
local equilibrium equations is Equation 2.48. Global equilibrium codes used today require significantly
more computational resources, as they minimize an energy functional over a wide parameter space,
or iteratively solve a large system of nonlinear partial differential equations. While the shape of the
outer boundary can be chosen in global equilibrium calculations, local equilibrium theory allows for
exact control of the geometry at the surface of interest. One is free to choose parametrizations which
highlight a quantity of interest and construct a set of equilibria at insignificant computational cost.

Numerical studies of local mode stability typically require the equilibrium used to be given in straight
field line coordinates, and each individual calculation requires a summation of all Fourier modes for the
MHD equilibrium quantities needed. This can be quite computationally expensive for realistic 3D equi-
libria which require 100,000’s of modes. Global equilibrium codes generally use geometric coordinates,
and their output must be converted to a straight field line coordinate system. This transformation
typically requires a large number of Fourier modes, further increasing the computational cost. The
parametrizations used with local equilibrium theory are written as functions of straight field line angles,
which greatly reduces the number of Fourier modes needed to describe the equilibrium.

2.3 Ideal MHD Stability Theory

Ideal MHD provides an excellent description of the biggest, fastest growing instabilities which typically
determine the operational boundaries for magnetic confinement experiments. These instabilities are
normal modes of the ideal MHD system which are driven to exponential growth by harnessing sources
of free energy within the plasma, i.e. plasma currents and the pressure gradient inherent to any magnetic
confinement experiment. Instabilities that are well described by MHD are at length scales comparable
to that of the device, and on timescales comparable to \( L/v_A \), where L is the device size and \( v_A \) is the
2.3.1 MHD Energy Principle

Linearization of the ideal MHD equations allows one to determine whether or not any mode of the system will be driven to exponential growth, or instability, for a given equilibrium. Each MHD quantity is written as the sum of a large, time-independent background value and a small, fluctuating component. For example, the magnetic field is written as

\[ B = B_0 + \tilde{B} \]  

(2.56)

with \(|\tilde{B}|/|B_0| \ll 1\) for linear stability analysis. Typically, all of the perturbed, fluctuation quantities are described in terms of a fluid displacement, \(\xi(x,t)\), with:

\[ \frac{\partial}{\partial t} \xi = \tilde{v}. \]  

(2.57)

This procedure is used for the ideal MHD equations to derive equations for the evolution of \(\tilde{p}, \tilde{J}, \tilde{B}\) in terms of \(\xi\) and from this MHD force balance can be used to describe the evolution of \(\tilde{\xi}\):

\[ \rho_0 \frac{\partial \tilde{v}(\xi)}{\partial t} = \tilde{J}(\xi) \times B_0 + J_0 \times \tilde{B}(\xi) - \nabla \tilde{p}(\xi). \]  

(2.58)

Assuming solutions with a time dependence like

\[ \xi(x,t) = \xi(x)e^{-i\omega t}, \]  

(2.59)

leads to an eigenvalue problem for \(\xi\) of the form

\[ -\rho_0 \omega^2 \xi = F(\xi). \]  

(2.60)

\(F(\xi)\) is called the ideal MHD force operator, which has some unique properties which facilitate ideal MHD stability analysis. \(F\) is self-Hermitian, i.e.:

\[ \int \eta \cdot F(\xi) d^3x = \int \xi \cdot F(\eta) d^3x. \]  

(2.61)

This implies that all eigenvalues of the system are purely real, i.e. \(\omega\) is either purely real (corresponding to a stable, oscillating mode) or purely imaginary (corresponding to an unstable, exponentially growing
mode). The eigenvectors of the system (different $\xi$ corresponding to different modes of the system) are also orthogonal [19].

By integrating Equation 2.60 over the volume of interest, one can write the eigenvalue in the following form:

$$\omega^2 = -\frac{1}{2} \int d^3x \frac{\xi^* \cdot F(\xi)}{\rho_0|\xi|^2} = \frac{\delta W}{K}. \tag{2.62}$$

From this, it can be seen that the plasma is stable if (and only if) $\delta W \geq 0$ for every $\xi$. $\delta W$ can also be written in an intuitive form which highlights the competing effects which determine ideal MHD stability:

$$\delta W = \frac{1}{2} \int d^3x \left( \frac{\vec{B}_\perp}{\mu_o}^2 + \frac{|B_0|^2}{\mu_o} |\nabla_\perp \cdot \xi + 2\xi \cdot \kappa|^2 + \frac{5}{3} p_0 |\nabla \cdot \xi|^2 
- 2(\xi_\perp \cdot \nabla p_0)(\xi_\perp \cdot \kappa) - J_0 \times \xi \cdot \hat{b} \cdot \vec{B}_\perp \right) \tag{2.63}$$

where $\kappa = (\hat{b} \cdot \nabla)\hat{b}$ is the curvature vector. The first term represents the stabilizing effect of shear Alfven waves, as a perturbation which drives these waves will be losing energy to bend magnetic fields. The second term is the stabilizing effect of compressing the magnetic field, which also draws energy from the perturbation. The third term represents the stabilizing effect of compressing the plasma as a whole. The fourth and fifth terms are the only terms which can be negative, representing the sources of free energy which can drive a mode unstable. The fourth term is negative when the pressure gradient and curvature vector align, representing the free energy available due to the pressure gradient. The fifth term is negative when $J_0$ is nonzero, representing free energy from plasma currents [19].

The process of determining whether or not a plasma is unstable to any ideal MHD modes is thus reduced to finding a perturbation $\xi$ which gives a negative $\delta W$. For various MHD instabilities, the components of $\xi$ can be constrained and a single condition, or eigenvalue equation for one component of $\xi$, only need be evaluated to determine stability. This procedure is used to describe ideal MHD ballooning instabilities, which are the focus of this work.

### 2.3.2 Ballooning Mode Theory

Ballooning instabilities are short wavelength modes which are driven unstable by the coupling of pressure gradients and unfavorable curvature (term 4 on the right hand side of Equation 2.63). Unstable ballooning modes tend to localize within regions of negative normal curvature where the local shear
crosses through zero. There is a competition between a localization of mode structure within regions of negative normal curvature and stabilizing effects which extend the mode structure into regions where favorable curvature can stabilize the mode. The structure of ballooning instabilities contains two distinct scales, a long wavelength parallel to the magnetic field and a short wavelength perpendicular to it. This invites the use of a WKB-like theory of describing the plasma displacement,

$$\tilde{\xi} = \xi(x)e^{\frac{i\varphi(x)}{\epsilon}}$$ (2.64)

where the ratio of the parallel to perpendicular wavevectors,

$$\epsilon = \frac{k_{||}}{k_{\perp}} \ll 1$$ (2.65)

is treated as a small parameter. For axisymmetric configurations, to ensure periodicity within a magnetic surface it is required that $$\epsilon = 1/n$$ where $$n$$ is an integer. The eikonal, $$S$$, is constrained to satisfy

$$B_0 \cdot \nabla S = 0.$$ (2.66)

Both $$S$$ and the function $$\xi(x)$$ vary slowly, while the phase in the exponential term captures the rapid variation perpendicular to the magnetic field. The eigenvalue and eigenmode structure can be determined using an expansion in powers of $$\epsilon$$, with the interesting property that the lowest order theory is sufficient to test for instability. Including terms of higher order in $$\epsilon$$ there is a small, stabilizing correction to the eigenvalue, and the global (i.e. radial) structure of the eigenmode can be determined in terms of the lowest order eigenmodes on each surface [24]. The inherent problem of constructing a global eigenmode that is periodic in both toroidal and poloidal angle in a configuration with a sheared magnetic field will be discussed in the next section.

It is convenient to decompose the perturbation as

$$\tilde{\xi} = \tilde{X} \frac{B_0 \times \nabla S}{B_0^2} + \tilde{\xi}_{||} \frac{B_0}{B_0}.$$ (2.67)

Taking the linearized force balance equation projected along $$B_0$$, an equation for $$\tilde{\xi}_{||}$$ can be derived

$$-\omega^2 \rho_0 B_0 \tilde{\xi}_{||} = \gamma p_0 (B_0 \cdot \nabla) \nabla \cdot \tilde{\xi}.$$ (2.68)

From this it is shown that the stabilizing, compressional component of the displacement ($$\nabla \cdot \tilde{\xi}$$) will go to zero at marginality (where $$\omega^2 \to 0$$). The compressional part can be calculated,

$$\nabla \cdot \tilde{\xi} = \frac{B_0^2}{B_0^2 + \gamma p_0} \left[ (B_0 \cdot \nabla) \frac{\tilde{\xi}_{||}}{B_0} - 2\tilde{X} \frac{\nabla S \times B_0}{B_0^2} \cdot \kappa \right]$$ (2.69)
and these two relations give an equation for the parallel component of the displacement,

\[ -\omega^2 \rho_0 B_0 \ddot{\xi}_\parallel = (B_0 \cdot \nabla) \gamma p_0 B_0^3 \left( B_0 \cdot \nabla \right) \frac{\ddot{\xi}_\parallel}{B_0} + 2 \ddot{X} \nabla \times B_0 \cdot \kappa - 2 \dddot{X} \nabla \times B_0 \cdot \kappa \]  

(2.70)

To get a complete description of the ballooning perturbation, another equation can be derived from the force balance equation projected perpendicular to \( B_0 \),

\[ -\omega^2 \rho_0 \mu_0 \left| \nabla S \right| B_0^2 \ddot{X} = (B_0 \cdot \nabla) \left( \left| \nabla S \right| B_0^2 \right) (B_0 \cdot \nabla) \ddot{X} + 2 \mu_0 \dddot{X} \frac{\nabla \times B_0 \cdot \kappa \nabla \times B_0 \cdot \nabla \dot{\rho}_0}{B_0^2} + 2 \mu_0 \gamma p_0 \frac{\nabla \times B_0 \cdot \kappa \nabla \cdot \ddot{\xi}}{B_0^2}. \]  

(2.71)

These two coupled ordinary differential equations describe the evolution of linear ballooning instabilities. To facilitate the study of the marginal stability properties of ballooning modes, we can note that near marginality Equation 2.70 provides no information, and Equation 2.71 reduces to

\[ -\omega^2 \rho_0 \mu_0 \left| \nabla S \right| B_0^2 \ddot{X} = (B_0 \cdot \nabla) \left( \left| \nabla S \right| B_0^2 \right) (B_0 \cdot \nabla) \ddot{X} + 2 \mu_0 \dddot{X} \frac{\nabla \times B_0 \cdot \kappa \nabla \times B_0 \cdot \nabla \dot{\rho}_0}{B_0^2} + 2 \mu_0 \gamma p_0 \frac{\nabla \times B_0 \cdot \kappa \nabla \cdot \ddot{\xi}}{B_0^2}. \]  

(2.72)

when the \( \omega^2 \to 0 \) limit of Equation 2.70 is used to eliminate \( \dddot{\xi}_\parallel \). This is the marginal ballooning equation which is solved in this work.

The previous calculation is only accurate to lowest order in \( \epsilon \) and describes perturbations localized to a single flux surface. To construct a global eigenmode, this analysis must be extended to higher order in \( \epsilon \). Typically, the global eigenmodes are peaked at the extrema of \( \omega^2 \), the lowest order local eigenvalue. The global eigenvalue ( \( \Omega^2 \) ) satisfies

\[ \Omega^2 = \omega^2(\psi, \theta_0, \alpha_0) + O(\epsilon). \]  

(2.73)

For axisymmetric configurations the eigenvalue is independent of the field line choice ( \( \alpha_0 \) ). This implies that the local eigenvalues (solutions to Equation 2.72) provide a good approximation of the global eigenvalue and allow for efficient computation of ballooning stability, only requiring the solution of an ordinary differential equation [24].

### 2.3.3 Periodicity

Ensuring that the WKB-like form used to describe the ballooning perturbation ( Equation 2.64 ) satisfies periodicity constraints presents a challenge when there is non-zero shear. Within one flux surface,
perturbations of this form inherently satisfy periodicity in toroidal and poloidal angle. However, when non-zero magnetic shear causes the pitch of magnetic field lines to vary across flux surfaces. Therefore $\xi(x)$ must either vary rapidly across surfaces to accommodate the changing pitch of magnetic field lines (which violates the original assumptions behind the WKB-like analysis) or the periodicity in poloidal angle will not be satisfied [24].

Another way of viewing this problem is to consider the structure of the eikonal, $S$, which has the form:

$$\tilde{\xi} = \tilde{\xi}(x)e^{i\eta[\zeta - q\theta + f(\psi)]}$$

(2.74)

where $f(\psi)$ is constant along flux surfaces. Then, the constraint of periodicity implies

$$\tilde{\xi}(\theta) = \tilde{\xi}(\theta + 2\pi l) = \tilde{\xi}(\theta)e^{in\eta 2\pi l}.$$  

(2.75)

When $dq/d\psi \neq 0$ it is problematic to satisfy this constraint, as

$$\nabla S = \nabla \zeta - q\nabla \theta - \theta \frac{\partial q}{\partial \psi} \nabla \psi + \frac{\partial f}{\partial \psi} \nabla \psi$$

(2.76)

and the $\partial q/\partial \psi$ term is secularly growing in $\theta$. Connor, Hastie, and Taylor developed a clever method of solving this periodicity problem by introducing the Ballooning transform. Utilizing a Fourier decomposition in toroidal angle, a two dimensional linear eigenvalue problem can always be obtained for linear perturbations in an axisymmetric system:

$$L(\theta, \psi)\phi(\theta, \psi) = \lambda \phi(\theta, \psi),$$  

(2.77)

where both the operator $L$ and the perturbation $\phi$ are periodic in $\theta$ and bounded in $\psi$. To obtain a manifestly periodic representation for $\phi$, the following transformation is introduced

$$\phi(\theta, \psi) = \sum_m e^{-im\theta} \int_{-\infty}^{\infty} e^{im\eta} \hat{\phi}(\eta, x) d\eta.$$  

(2.78)

Any $\hat{\phi}$ which is a solution of

$$L(\eta, \psi)\hat{\phi}(\eta, \psi) = \lambda \hat{\phi}(\eta, \psi)$$

(2.79)

on the domain $-\infty < \eta < \infty$ will produce a periodic $\phi(\theta, \psi)$ with the same eigenvalue. Effectively, the eigenvalue problem in the periodic domain is transformed to an artificial but equivalent problem in an infinite domain. Instead of the challenging periodicity requirement, it is only required that $\hat{\phi} \rightarrow 0$.
as \( \eta \to \pm \infty \). The actual, physical eigenmode structure is obtained from a linear superposition of the ‘quasi-modes’ \( \hat{\phi}(\eta) \), but stability can be analysed simply by evaluating the eigenvalues of the system in the fictitious infinite domain [24].

2.3.4 Ballooning instabilities in 3D configurations

There are important differences in the spectrum of ballooning stability for axisymmetric and 3D equilibria. In axisymmetry, level surfaces of the ballooning eigenvalue have a cylindrical shape in \((\psi, \theta_k, \alpha)\) space (where \(\theta_k\) is both a measure of the radial wave-vector and the poloidal angle marking the starting point for integration of the ballooning equation and \(\alpha\) labels different field lines on a given surface). This is because each magnetic field line on a given surface has the same properties in axisymmetry. In three dimensional configurations, different magnetic field lines on the same surface can have different properties. While eigenvalues with cylindrical level surfaces can still exist, there is a new class of ballooning instabilities with spherical level surfaces in \((\psi, \theta_k, \alpha)\) space [22]. These instabilities are more localized and may be stabilized by finite larmour radius (FLR) effects. This possibility is supported by experimental evidence suggesting that ballooning stability predictions may be too pessimistic for Stellarator configurations [25].

This work examines the ballooning stability properties of equilibria generated with the local 3D model, focusing on two problems in particular. The first problem was laid out in Sections 1.3.3 and 1.3.4. Ideal MHD ballooning instabilities are highly correlated with the onset of Kinetic Ballooning Mode (KBM) turbulence, a property which the EPED1 model uses to make predictive calculations of pedestal parameters for high performance Tokamak experiments. The use of externally applied 3D magnetic perturbations to modify plasma profiles and mitigate Edge Localized Modes has yet to be fully explained theoretically, but will be crucial to the success of future Tokamak experiments. The Local 3D equilibrium model is used to gain insight into the effect of 3D magnetic perturbations on an axisymmetric configuration, in particular the geometric and Pfirsch-Schluter modulation of the local shear. Marginal stability analysis of ideal MHD ballooning modes is used to better understand the effect 3D magnetic perturbations may have on KBM turbulence.
The second problem was discussed in Section 1.4.1. Recently attention has turned to further optimizing Stellarator experiments to mitigate turbulent transport. Previous work has demonstrated that optimization of flux surface geometry to mitigate ITG modes has similar effects on ETG and MHD ballooning modes. The similar stabilizing and destabilizing mechanisms at play (described by the local shear and normal curvature) suggest that the behavior of MHD ballooning modes may provide a proxy for ITG and ETG modes which require more computationally intensive gyrokinetic simulations. It is shown how local 3D equilibrium calculations can provide a detailed understanding of how geometry affects ballooning stability properties.
Chapter 3

Code Development

3.1 3DLEQ

3.1.1 Code overview

During the course of this work, the 3DLEQ code has been developed to construct numerical solutions to the Local 3D Equilibrium equations. Given an equilibrium parametrization (R, \(\phi\), Z and their derivatives with respect to \(\theta\) and \(\zeta\)), the rotational transform (\(i_0\)), and two of the three free profile quantities (\(i', p', \sigma\)) the 3DLEQ code can calculate all of the MHD equilibrium quantities, localized about the flux surface of interest.

The equilibrium parametrizations, which must be written into the code, allow for an exact calculation of R, \(\phi\), Z, \(\kappa_n\), \(\kappa_g\) and \(\tau_n\) on a discrete grid in poloidal and toroidal angle. The partial differential equation for the jacobian is solved by projecting these quantities onto cosine and sine basis functions. By inserting the cosine series decompositions into the equation for the jacobian and using orthogonality, a linear system of equations for the coefficients of the cosine series decomposition for the jacobian is produced.

This system is solved for the jacobian, which is then used to calculate the magnetic field and \(\nabla \psi\). The intent of this development is to generate MHD equilibria for use in \(s - \alpha\) analysis of MHD ballooning modes or gyrokinetic turbulence, where the profile quantities are systematically varied. Therefore present implementations of the code leave the calculation of the local shear as a task for stability codes.

While there is no guarantee of the existence of a solution to the local equilibrium equations for a given equilibrium parametrization, in practice it is possible to find a solution for most well behaved (i.e. free of singularities) parametrizations.
3.1.2 Curvature expressions

The set of Equations 2.37, 2.38 and 2.39, analagous to Frenet’s formulas, allow for the exact calculation of curvature quantities as a function of $R, \phi, Z$ and their derivatives with respect to the $\theta$ and $\zeta$. For simplicity, the following notation is used in this derivation:

\[ R_\eta = \partial_\eta R = \frac{\partial}{\partial \eta} R = \left( \frac{\partial}{\partial \zeta} + \iota_0 \frac{\partial}{\partial \theta} \right) R \]  

(3.1)

\[ [A, B] = \partial_\theta A \partial_\zeta B - \partial_\zeta A \partial_\theta B \]  

(3.2)

The unit vectors can be written as

\[ \hat{b} = \frac{R_\eta \hat{R} + R \phi_\eta \hat{\phi} + Z_\eta \hat{Z}}{L}, \]  

(3.3)

\[ \hat{n} = \frac{R[\phi, Z] \hat{R} + [Z, R] \hat{\phi} + R[R, \phi] \hat{Z}}{Q^2}, \]  

(3.4)

where the normalization factors $L$ and $Q$ are given by

\[ L = \sqrt{R^2_\eta + Z^2_\eta + R^2 \phi^2_\eta}, \]  

(3.5)

\[ Q^2 = \sqrt{R^2[\phi, Z]^2 + R^2[R, \phi]^2 + [Z, R]^2}. \]  

(3.6)

From the expressions for $\hat{b}$ and $\hat{n}$ one can construct the binormal unit vector,

\[ \hat{b} \times \hat{n} = \frac{R_\eta \hat{R} + R \phi_\eta \hat{\phi} + Z_\eta \hat{Z}}{LQ^2} + \frac{Z \eta \hat{R} - R_\eta R[R, \phi] \hat{\phi} + R \eta[\phi, Z] - R^2 \phi_\eta \hat{Z}}{LQ^2}. \]  

(3.7)

$\kappa_n$ and $\kappa_g$ can be obtained from

\[ (\hat{b} \cdot \nabla)\hat{b} = \hat{b} \frac{\partial}{\partial \eta} \frac{1}{L} + \frac{1}{L^2} \left( (R_\eta - R \phi^2_\eta) \hat{R} + (R \phi_\eta + 2 R_\eta \phi_\eta) \hat{\phi} + Z_\eta \hat{Z} \right). \]  

(3.8)

The normal curvature is obtained by projecting this vector into the $\hat{n}$ direction,

\[ \kappa_n = \hat{n} \cdot (\hat{b} \cdot \nabla)\hat{b}, \]  

(3.9)

\[ \kappa_n = \frac{[\phi, Z](R R_{\eta \eta} - R^2 \phi^2_\eta) + [Z, R](R \phi_{\eta \eta} + 2 R_\eta \phi_\eta) + Z \eta R[R, \phi]}{Q^2 L^2}. \]  

(3.10)

Similarly, the geodesic curvature is obtained by projecting into the $\hat{b} \times \hat{n}$ direction,

\[ \kappa_g = (\hat{b} \times \hat{n}) \cdot (\hat{b} \cdot \nabla)\hat{b}, \]  

(3.11)

\[ \kappa_g = \frac{1}{L^2 Q^2} \left( [R, \phi](R^2 \phi_\eta R_{\eta \eta} - R^3 \phi^3_\eta - R^2 R \phi_{\eta \eta} - 2 R R^2 \phi_\eta) + [Z, R](-Z \eta R_{\eta \eta} + R Z_\eta \phi^2_\eta + R_\eta Z_{\eta \eta} + [\phi, Z](R^2 Z_\eta \phi_{\eta \eta} + 2 R R \phi_\eta Z_{\eta \eta} - Z_{\eta \eta} R^2 \phi_\eta) \right). \]  

(3.12)
Obtaining the normal torsion requires a calculation of $(\hat{b} \cdot \nabla)(\hat{b} \times \hat{n})$, given by
\[
(\hat{b} \cdot \nabla)(\hat{b} \times \hat{n}) = Q^2(\hat{b} \times \hat{n}) \frac{\partial}{\partial \eta} \frac{1}{LQ^2} + \frac{1}{L^2 Q^2} \left( 2R_{\eta \phi \eta}[R, \phi] + R^2 \phi_{\eta \eta}[R, \phi] + R^2 \phi_{\eta}[R, \phi] - Z_{\eta \phi \eta}[Z, R] - Z_{\eta}[Z, R] - R\phi_{\eta}Z_{\eta}[\phi, Z] + RR_{\eta \phi \eta}[R, \phi] \right) \hat{R} \\
+ \frac{1}{L^2 Q^2} \left( R^2 \phi_{\eta}^2[R, \phi] - Z_{\eta} \phi_{\eta}[Z, R] + R_{\eta}Z_{\eta}[\phi, Z] + RZ_{\eta \eta}[\phi, Z] \\
+ RZ_{\eta \eta \eta}[\phi, Z] - R_{\eta \eta}R[R, \phi] - R_{\eta}^2[R, \phi] - RR_{\eta \eta \eta}[R, \phi] \right) \hat{\phi} \\
+ \frac{1}{L^2 Q^2} \left( R_{\eta \eta}[Z, R] + R_{\eta} \frac{\partial}{\partial \eta} [Z, R] - 2RR_{\eta \phi \eta}[\phi, Z] - R^2 \phi_{\eta \eta}[\phi, Z] - R^2 \phi_{\eta}[\phi, Z] - R^2 \phi_{\eta \eta}[\phi, Z] \right) \hat{Z} \tag{3.14}
\]

The normal torsion is obtained by projecting this vector into the $\hat{n}$ direction,
\[
\tau_n = -\hat{n} \cdot (\hat{b} \cdot \nabla)(\hat{b} \times \hat{n}), \tag{3.15}
\]
\[
\tau_n = \frac{1}{L^2 Q^2} \left( -R^3 \phi_{\eta}[\phi, Z] + RR_{\eta}[Z, R] \right) \frac{\partial}{\partial \eta} [R, \phi] + \frac{1}{L^2 Q^2} (RZ_{\eta}[\phi, Z] - RR_{\eta}[R, \phi]) \frac{\partial}{\partial \eta} [Z, R] \\
+ \frac{1}{L^2 Q^2} (R^3 \phi_{\eta}[R, \phi] - RZ_{\eta}[Z, R]) \frac{\partial}{\partial \eta} [\phi, Z] + \frac{1}{L^2 Q^2} \left( [\phi, Z]^2 R^2 \phi_{\eta}Z_{\eta} + [Z, R]^2 \phi_{\eta}Z_{\eta} \\
- [\phi, Z][R, \phi]R^2 R_{\eta \phi \eta}[\phi, Z] - [R, \phi][Z, R](R^2 \phi_{\eta}^2 - R_{\eta}^2) - [\phi, Z][Z, R]R_{\eta \eta \eta}[Z, R] \right) \tag{3.16}
\]

The expressions used in the 3DLEEQ code can be obtained by expanding the $\eta$ derivatives and bracket notation in Equations 3.11, 3.13, and 3.16. For the normal curvature,
\[
\kappa_n = \frac{1}{Q^2 L^2} \left\{ (\phi \theta Z_{\eta} - \phi_{\theta} Z_{\eta}) (RR_{\zeta \zeta} + \tau_0^2 RR_{\theta \theta} + 2\tau_0 RR_{\theta \zeta} - R^2 (\phi_\zeta + \tau_0 \phi_\theta)^2) \\
+ (Z_{\theta} R_{\zeta} - Z_{\zeta} R_{\theta}) (R \phi_{\zeta \zeta} + \tau_0^2 \phi_{\theta \theta} R + 2\tau_0 R \phi_{\theta \theta} + 2(R_{\zeta} + \tau_0 R_{\theta}) (\phi_\zeta + \tau_0 \phi_\theta)) \\
+ (R_{\theta} \phi_{\zeta} - R_{\zeta} \phi_{\theta}) (RZ_{\zeta \zeta} + \tau_0^2 RZ_{\theta \theta} + 2\tau_0 RZ_{\theta \zeta}) \right\}. \tag{3.17}
\]

For the geodesic curvature,
\[
\kappa_g = \frac{1}{Q^2 L^3} \left\{ (R_\theta \phi_{\zeta} - R_{\zeta} \phi_{\theta}) \left( R^2 (\phi_\zeta + \tau_0 \phi_\theta)(R_{\zeta \zeta} + \tau_0^2 R_{\theta \theta} + 2\tau_0 R_{\theta \zeta}) - R^2 (\phi_\zeta + \tau_0 \phi_\theta)^3 \\
- R^2 (R_{\zeta} + \tau_0 R_{\theta})(\phi_{\zeta \zeta} + \tau_0^2 \phi_{\theta \theta} + 2\tau_0 \phi_{\theta \zeta}) - 2R(R_{\zeta} + \tau_0 R_{\theta})^2 (\phi_\zeta + \tau_0 \phi_\theta) \right) \\
+ \left( Z_{\theta} R_{\zeta} - Z_{\zeta} R_{\theta} \right) \left( (Z_{\zeta} + \tau_0 Z_{\theta}) (R_{\zeta \zeta} + \tau_0^2 R_{\theta \theta} + 2\tau_0 R_{\theta \zeta}) \\
+ (R_{\zeta} + \tau_0 R_{\theta}) (\phi_{\zeta} + \tau_0 \phi_{\theta})^2 + (R_{\zeta} + \tau_0 R_{\theta}) (Z_{\zeta \zeta} + \tau_0^2 Z_{\theta \theta} + 2\tau_0 Z_{\theta \zeta}) \right) \\
+ \left( \phi_{\theta} Z_{\zeta} - \phi_{\zeta} Z_{\theta} \right) \left( (Z_{\zeta} + \tau_0 Z_{\theta}) (\phi_{\zeta \zeta} + \tau_0^2 \phi_{\theta \theta} + 2\tau_0 \phi_{\theta \zeta}) \\
+ 2R(R_{\zeta} + \tau_0 R_{\theta})(Z_{\zeta} + \tau_0 Z_{\theta})(\phi_{\zeta} + \tau_0 \phi_{\theta}) - (Z_{\zeta \zeta} + \tau_0^2 Z_{\theta \theta} + 2\tau_0 Z_{\theta \zeta})(R^2 (\phi_\zeta + \tau_0 \phi_\theta) \right) \right\}. \tag{3.18}
\]
Finally, for the normal torsion,

\[ \tau_n = \frac{1}{L^2 Q_0^4} \left\{ \left( R_{\theta \zeta} \phi_{\zeta} + R_{\theta \phi_{\zeta}} \phi_{\zeta} + i_0 R_{\theta \phi_{\zeta}} - R_{\zeta \zeta} \phi_{\theta} - R_{\zeta \phi_{\zeta}} - i_0 R_{\zeta \phi_{\theta}} - i_0 R_{\zeta \phi_{\zeta}} \right) \right. \\
\left. \cdot \left( - R^3 (\phi_{\zeta} + i_0 \phi_{\theta}) (\phi_{\theta} Z_{\zeta} - \phi_{\zeta} Z_{\theta}) + R (R_{\zeta} + i_0 R_{\theta}) (Z_{\theta} R_{\zeta} - Z_{\zeta} R_{\theta}) \right) + (Z_{\theta} R_{\zeta} + Z_{\zeta} R_{\theta} + i_0 Z_{\theta} R_{\zeta} - i_0 Z_{\zeta} R_{\theta}) \right\} (3.19) \]

All three of these expressions depend only on inputs to the local equilibrium code. The flux surface parametrization ( \( R, \phi, Z \)) and derivatives of these quantities with respect to \( \theta \) and \( \zeta \) are written into the code for each parametrization used. \( i_0 \) is similarly an input quantity, which allows these expressions to be evaluated before the jacobian is solved for. The existence of these exact expressions also allows one to manipulate the curvatures analytically in certain asymptotic limits.

### 3.1.3 Numerical methods

The normal current constraint on the jacobian, Equation 2.48, can be rewritten in the following form, which is used in the 3DLEQ code

\[ \sqrt{g} \frac{\partial}{\partial \zeta} (i_0 g_{\theta \theta} + g_{\zeta \zeta}) - (i_0 g_{\theta \theta} + g_{\zeta \zeta}) \frac{\partial}{\partial \zeta} \sqrt{g} - \sqrt{g} \frac{\partial}{\partial \theta} (g_{\zeta \zeta} + i_0 g_{\zeta \theta}) + (g_{\zeta \zeta} + i_0 g_{\zeta \theta}) \frac{\partial}{\partial \theta} \sqrt{g} = 0. \quad (3.20) \]

For simplicity, the following notation is used

\[ G = i_0 g_{\theta \theta} + g_{\zeta \zeta} \quad (3.21) \]

\[ F = g_{\zeta \zeta} + i_0 g_{\zeta \theta}. \quad (3.22) \]
The quantities F and G are both decomposed into a Fourier cosine series, due to Stellarator symmetry, 

\[ \sum_{m=M,n=-N}^{m=M,n=N} G_{mn} \cos(m\theta + n\zeta) \] (3.23)

\[ F = \sum_{s=0,l=-N}^{s=M,t=N} F_{sl} \cos(s\theta + t\zeta) \] (3.24)

\[ \sqrt{g} = \sum_{k=0,l=-N}^{k=M,l=N} \sqrt{g}_{kl} \cos(k\theta - l\zeta). \] (3.25)

Using these decompositions into Equation 3.20 gives

\[ \sum_{k=0}^{k=M} \sum_{l=-N}^{l=N} \sqrt{g}_{kl} \left[ \sum_{s=0}^{s=M} \sum_{t=-N}^{t=N} \left( sF_{st} \cos(k\theta - l\zeta) \sin(s\theta + t\zeta) - kF_{st} \cos(s\theta + t\zeta) \sin(k\theta - l\zeta) \right) 
\right]
\]

\[ + \sum_{m=0}^{m=M} \sum_{n=-N}^{n=N} \left( -nG_{mn} \cos(k\theta - l\zeta) \sin(m\theta + n\zeta) - nG_{mn} \cos(m\theta + n\zeta) \sin(k\theta - l\zeta) \right) \]
\[ = 0. \] (3.26)

Using trigonometric identities, this can be rewritten with only sine terms,

\[ \sum_{k=0}^{k=M} \sum_{l=-N}^{l=N} \sqrt{g}_{kl} \frac{1}{2} \left[ \sum_{s=0}^{s=M} \sum_{t=-N}^{t=N} \left( sF_{st} \sin((k+s)\theta + (t-l)\zeta) - sF_{st} \sin((k-s)\theta + (l+t)\zeta) 
\right) 
\right]
\]

\[ -kF_{st} \sin((s+k)\theta + (t-l)\zeta) + kF_{st} \sin((s-k)\theta + (l+t)\zeta) 
\]
\[ + \sum_{m=0}^{m=M} \sum_{n=-N}^{n=N} \left( -nG_{mn} \sin((k+m)\theta + (t-l)\zeta) - nG_{mn} \sin((m-k)\theta + (l+n)\zeta) 
\right) 
\]

\[ -lG_{mn} \sin((m+k)\theta + (t-l)\zeta) + lG_{mn} \sin((m-k)\theta + (l+n)\zeta) \]
\[ = 0. \] (3.27)

An equation for the coefficients \( \sqrt{g}_{kl} \) can be obtained by using orthogonality, multiplying this equation by \( \sin(p\theta + h\zeta) \) (with an implicit summation over \( p \) and \( h \) ) and integrating over \( \theta \) and \( \zeta \),

\[ \sum_{k=0}^{k=M} \sum_{l=-N}^{l=N} \sqrt{g}_{kl} \frac{\pi}{2} \left[ \sum_{s=0}^{s=M} \sum_{t=-N}^{t=N} \left( sF_{st} \delta(p-k-s)\delta(h-t+l) - sF_{st} \delta(p-k+s)\delta(h+t+l) 
\right) 
\right]
\]

\[ -kF_{st} \delta(p-s-k)\delta(h-t+l) + kF_{st} \delta(p-s+k)\delta(h-t-l) 
\]
\[ + \sum_{m=0}^{m=M} \sum_{n=-N}^{n=N} \left( -nG_{mn} \delta(p-k-m)\delta(h-n+l) - nG_{mn} \delta(p-m+k)\delta(h-l-n) 
\right) 
\]

\[ -lG_{mn} \delta(p-m-k)\delta(h-n+l) + lG_{mn} \delta(p-m+k)\delta(h-n-l) \]
\[ = 0. \] (3.28)

Evaluating the delta functions and summing over \( m, n, s, \) and \( t \) leads to

\[ \sum_{k=0}^{k=M} \sum_{l=-N}^{l=N} \sqrt{g}_{kl} \left[ (p-2k)F_{s=(p-k),t=(h+l)} + (p+2k)F_{s=(p+k),t=(h-l)} 
\right]
\]

\[ -(h+2l)G_{m=(p-k),n=(h+l)} - (h-2l)G_{m=(p+k),n=(h-l)} \]
\[ = 0. \] (3.29)
This is a system of \((M \times N) + ((M + 1) \times (N + 1))\) equations for each coefficient \(\sqrt{g_{kl}}\), where each unique combination of \(p\) and \(h\) gives one of the equations in the system. This system can be written in the form \(\tilde{A}\mathbf{x} = 0\), where each row of \(A\) corresponds to a unique value of \(p\) and \(h\).

To solve this system, one of the coefficients \(\sqrt{g_{kl}}\) must be specified, which is achieved by setting \(\sqrt{g_{00}} = 1\). Stability analysis performed with 3DLEQ equilibria thus far solve a normalized eigenvalue equation, so the \((0,0)\) component is arbitrary. To get a solvable system, the first column of \(\tilde{A}\) (which is multiplied by \(\sqrt{g_{00}}\)) is moved to the right hand side, resulting in the following system

\[
\tilde{A}' \mathbf{x}' = \mathbf{b}
\] (3.30)

where \(b_i = -A_{1,i} \sqrt{g_{00}}\) and \(\mathbf{x}'\) is the array of \(\sqrt{g_{kl}}\) with the \((0,0)\) term removed. The Fourier coefficients of the jacobian are obtained by computing \(\mathbf{x}' = \tilde{A}'^{-1}\mathbf{b}\).

### 3.1.4 Benchmarking

Solutions for the jacobian obtained with the 3DLEQ code can be compared with an approximate analytic solution for a circular cross section, helically modulated equilibrium which rotates with the pitch \((N\zeta - \theta)\) [23]. The parametrization is given by

\[
R(\theta, \zeta) = R_0 + \rho_0 \cos(\theta) + \Delta \cos(N\zeta),
\]

\[
\phi(\theta, \zeta) = -\zeta,
\]

\[
Z(\theta, \zeta) = \rho_0 \sin(\theta) + \Delta \sin(N\zeta).
\]

In the high aspect ratio limit, with \(N\) a large number \((N^2 \gg R_0, N\Delta/R_0 \sim N\rho_0/R_0 \sim 1)\), Equation 3.20 can be written as:

\[
\frac{\partial}{\partial \theta} R_0^2 + N^2 \Delta^2 + \nu_0 N \Delta \rho_0 \cos(N\zeta - \theta) \frac{\sqrt{g}}{\sqrt{g}} = \frac{\partial}{\partial \zeta} N \Delta \rho_0 \cos(N\zeta - \theta) + \nu_0 \rho_0^2
\] (3.34)

Expanding in \(N\), this reduces to an ordinary differential equation along \(\eta = N\zeta - \theta\),

\[
\frac{\partial}{\partial \eta} R_0^2 + N^2 \Delta^2 + N^2 \Delta \rho_0 \cos(\eta) \frac{\sqrt{g}}{\sqrt{g}} = 0.
\] (3.35)

After integration, the solution for the jacobian is given by

\[
\sqrt{g} = \frac{V'}{4\pi^2} (1 + \frac{N^2 \Delta \rho_0}{R_0^2 + N^2 \Delta^2 \cos(\eta)}).
\] (3.36)
Figure 5: Log of the $L_2$ norm of the local truncation error between the 3DLEQ and analytic solution.

The analytic solution was benchmarked against numerical solutions produced with the 3DLEQ code, with the parameters $N = 100$, $R_0 = 10$, $\rho = \Delta = 1$. Figure 5 shows the $L_2$ norm of the disparity between the two solutions at each point on the grid (2000 points poloidal, 2000 points toroidal) as more Fourier harmonics are included in the calculation. The dominant harmonics in the solution are at multiples of 100, and the error drops significantly as the maximum toroidal Fourier mode number reaches multiples of 100.

### 3.2 Ballooning Stability Code

#### 3.2.1 Code overview

A code which calculates the marginal stability boundaries for ideal MHD ballooning modes was also developed within the course of this work. This code was developed to carry out these calculations for equilibria generated with 3DLEQ, by calculating the local shear and Pfirsh-Schluter coefficient consistent
with the profile relations from local equilibrium theory. Once the quantities needed to perform ideal MHD infinite-n ballooning eigenvalue calculations are calculated, the numerical methods used follow those used in the COBRA code [26].

3DLEQ provides the Fourier coefficients for the following MHD equilibrium quantities used for ballooning stability calculations: $R$, $B^2$, $|\nabla \psi|^2$, $\sqrt{g}$, $\kappa_g$, $\kappa_n$, $\tau_n$. The ballooning stability code uses local equilibrium theory to calculate the Pfirsh-Schluter coefficient ($\lambda$) and then uses the profile relation to calculate the local shear once $p'$ and $\iota'$ are chosen. These profile quantities are systematically varied, with the most unstable eigenvalue calculated at each value, to generate a marginal stability curve.

### 3.2.2 Normalized Ballooning Equation

The angular variables are transformed such that $\eta = \zeta$ and $\alpha_0 = \theta - \iota \zeta$, with $\eta$ denoting the position along a magnetic field line and $\alpha_0$ denoting different magnetic field lines. The ballooning equation is normalized using $R_0$ and $\dot{V}' = R_0/B_0$, where each can be set equal to one. With this coordinate system and normalization, the ballooning equation is solved in the following normalized form:

$$\frac{\partial}{\partial \eta} \left[ \frac{\dot{V}'}{\sqrt{g}} \frac{R_0^4}{|\nabla \psi|^2 V^2} (1 + \Lambda^2) \frac{\partial \xi}{\partial \eta} \right] + \frac{\sqrt{g}}{V'} \frac{R_0^2}{V'|\nabla \psi|} 2\mu_0 \rho \dot{V}' R_0 (R_0 \kappa_n + \Lambda R_0 \kappa_g) \xi = -\rho \left[ \frac{\sqrt{g}}{V'} \frac{R_0^4}{|\nabla \psi|^2 V^2} (1 + \Lambda^2) \right] \omega^2 \xi. \tag{3.37}$$

The local shear is calculated using Equation 2.53, in the following normalized form:

$$\left( \frac{\dot{V}'}{\sqrt{g}} \right) \left( \frac{\partial}{\partial \zeta} + \iota_0 \frac{\partial}{\partial \theta} \right) \left( DR_0^3 \frac{R_0^4}{V^2} \right) = \iota' \frac{R_0^3}{V'} \left[ \frac{B^2 R_0^2}{|\nabla \psi|^2} \frac{1}{\langle B^2 R_0^2 \rangle} - \dot{V}' \right] \nonumber$$

$$+ \mu_0 p' \dot{V}' R_0 \left[ \frac{B^2 R_0^2}{|\nabla \psi|^2 \dot{V}^2} - \frac{\langle B^2 R_0^2 \rangle}{\langle |\nabla \psi|^2 \dot{V}^2 \rangle} \right] + 2 \left( \frac{B^2 R_0^2}{|\nabla \psi|^2} \right) \left( \frac{B^2 R_0^2}{|\nabla \psi|^2} \right) \tau_n R_0 + 2 \left( \frac{B^2 R_0^2}{|\nabla \psi|^2} \right) \tau_n R_0, \tag{3.38}$$

and the integrated local shear ($\Lambda$) is given by

$$\Lambda = \frac{|\nabla \psi|^2}{R_0^2} \frac{\dot{V}'}{V^2} \int_{\eta_k}^{\eta} d\eta \left[ \mu_0 \frac{R_0^3}{V'} + \frac{\partial}{\partial \eta} \frac{DR_0^3}{V^2} \right]. \tag{3.39}$$

The ballooning eigenvalue is a function of $\eta_k$, the starting point for the integration along the magnetic field line. For 3D configurations, the ballooning eigenvalue will also depend on the choice of field line.
(\(\alpha\)). In principle this calculation should be carried out over an infinite domain along the magnetic field line of choice, but in practice a large value \(\eta_\infty\) is used as the endpoints for integration. Typical values used are on the order of \(\pm 20 \cdot 2\pi\) (in poloidal angle), but in general \(\eta_\infty\) is increased until there is no further change to the marginal stability boundary.

### 3.2.3 Calculation of Shear and Pfirsch Schluter Current

Most of the information required to solve Equation 3.37 is provided by the 3DLEQ code, but two magnetic differential equations must be solved in order to obtain the Pfirsch-Schluter coefficient (\(\lambda\)) and the local shear (\(D\)). The Pfirsch-Schluter coefficient is determined by Equation 2.45, and noting that in straight field line coordinates

\[
(B \cdot \nabla) = \frac{1}{\sqrt{g}} (\frac{\partial}{\partial \zeta} + \iota_0 \frac{\partial}{\partial \theta}),
\]

the right hand side of Equation 2.45 is Fourier decomposed in a sine series

\[
\sqrt{g} 2 \mu_0 \kappa_g \frac{|\nabla \psi|}{B} = \sum_{m=0}^{M} \sum_{n=-N}^{N} (RHS)_{mn} \sin(m\theta + n\zeta)
\]

which allows the Fourier cosine coefficients for \(\lambda\) to be computed simply as

\[
\lambda_{mn} = -\frac{(RHS)_{mn}}{n + \iota m}.
\]

The same procedure can be used to solve for \(D\) in equation 3.38. However, marginal stability analysis for ballooning modes involves varying the profile quantities \(\alpha\) (here \(\alpha\) is the normalized pressure gradient as opposed to field line label) and \(\iota\) across a range of values. It is convenient to decompose the quantity \(D\) into three component parts,

\[
D_{mn} = \dot{i} D_{1mn} + \dot{p} D_{2mn} + D_{3mn}
\]

with \(\dot{i} = i R_0^3 \dot{V}\) and \(\dot{p} = \mu_0 \dot{p} R_0 \dot{V}\) acting as normalized profile quantities. Each individual component of \(D\) is calculated using the same method as for \(\lambda\), with the magnetic differential equations given
by

\[
\left( \frac{\hat{V}'}{\sqrt{g}} \right) \left( \frac{\partial}{\partial \zeta} + i_0 \frac{\partial}{\partial \theta} \right) \left( D_1 R_0^3 \frac{\hat{V}'}{V'} \right) = \left[ \frac{B^2 R_0^2}{|\psi|^2} \frac{1}{\sqrt{|\psi|^2}} - \hat{V}' \right]
\] (3.44)

\[
\left( \frac{\hat{V}'}{\sqrt{g}} \right) \left( \frac{\partial}{\partial \zeta} + i_0 \frac{\partial}{\partial \theta} \right) \left( D_2 R_0^3 \frac{\hat{V}'}{V'} \right) = \left[ \frac{B^2 R_0^2}{|\psi|^2} \frac{\lambda}{V'} - \frac{B^2 R_0^2}{|\psi|^2} \left( \frac{\hat{V}'}{V'} \right) \right]
\] (3.45)

\[
\left( \frac{\hat{V}'}{\sqrt{g}} \right) \left( \frac{\partial}{\partial \zeta} + i_0 \frac{\partial}{\partial \theta} \right) \left( D_3 R_0^3 \frac{\hat{V}'}{V'} \right) = 2 \left[ \frac{B^2 R_0^2}{|\psi|^2} \left( \frac{\hat{V}'}{V'} \right) \frac{\tau_n R_0}{\tau_n R_0} \right] - \frac{B^2 R_0^2}{|\psi|^2} \tau_n R_0 \right] .
\] (3.46)

The Fourier cosine coefficients for the right hand side of each of these 3 equations are computed and then used to calculate the Fourier sine coefficients for each component of D. As \( \iota' \) and \( \hat{\rho}' \) are varied, the shear can be quickly calculated by summing these three components.

### 3.2.4 Numerical methods

Solution of the magnetic differential equation for D requires several flux surface averaged quantities, requiring numerical integration. The CUBA library for multidimensional integration is used here, with the Cuhre routine being used for these integrations. Cuhre uses a deterministic method based off of cubature rules [27]. In practice, only a small number of numerical integrations need to be performed for each equilibrium, the values can then be recorded and reused.

The ideal MHD ballooning equation (Equation 3.37) is a second order, linear ordinary differential equation. This can be rewritten using the notation

\[
\frac{\partial}{\partial \eta} \left[ P(\eta) \frac{\partial}{\partial \eta} \right] \xi + Q(\eta) \xi + \Lambda R(\eta) \xi = 0
\] (3.47)

where \( \Lambda = \omega^2 \). The equilibrium is unstable if there exists a negative \( \Lambda \) for some value of \( \eta_k \) and \( \alpha_0 \).

The magnetic field line of interest is descretized along a domain ranging from \( \eta = [-\eta_\infty, \eta_\infty] \). With an odd number of points \( N_{pts} \), the full and half grids are defined as

\[
\eta_j = -\eta_\infty + h(j - 1), \quad j = 1, N_{pts}
\] (3.48)

\[
\eta_{j+1/2} = -\eta_\infty + h(j - 1/2), \quad j = 1, N_{pts} - 1
\] (3.49)

with \( h = 2 \eta_\infty/(N_{pts} - 1) \) being the step size. The boundary conditions imposed are that

\[
\xi_1 = \xi_{N_{pts}} = 0.
\] (3.50)
The ballooning equation is then written in the form

\[ \tilde{A} \cdot \xi = \Lambda \xi \quad (3.51) \]

where the values of \( A \) are given by

\[
h^2 A_{ij} = \delta(i - j + 1) \left( \frac{P_{j+3/2}}{R_j} \right) + \delta(i - j - 1) \left( \frac{P_{j+1/2}}{R_{j+1}} \right) - \delta(i - j) \frac{P_{j+3/2} + P_{j+1/2} - h^2 Q_{j+1}}{R_{j+1}}
\]

\[ i, j = 1, ..., N_{pts} - 2 \quad (3.52) \]

The problem is then reduced to calculating the most negative eigenvalue of \( \tilde{A} \) and the corresponding eigenvector, with \( \tilde{A} \) being a tridiagonal matrix with non-negative off-diagonal components. This property allows for a very efficient computation of the most negative eigenvalue.

### 3.2.5 Benchmarking

The typical benchmark for a ballooning stability code is a circular cross section Tokamak equilibrium (with \( A \) the aspect ratio \( R_0/\rho_0 \))

\[
R = R_0(1 + \cos(\theta)/A) \quad (3.53)
\]

\[
Z = \sin(\theta)/A \quad (3.54)
\]

In the high aspect ratio limit, the components of the ballooning equation can be analytically calculated. A marginal stability diagram is then generated for a numerically calculated, high aspect ratio equilibrium and compared to a marginal stability diagram generated where the coefficients in the ballooning equation are directly computed using the analytic solution [19].

The components of the ballooning equation analytically simplify to the following for this benchmark,

\[
\Lambda = s(\theta - \theta_0) - \alpha(sin(\theta) - sin(\theta_0)), \quad (3.55)
\]

\[
P = R = (1 + \Lambda^2), \quad (3.56)
\]

\[
Q = \alpha(A sin(\theta) + \cos(\theta)). \quad (3.57)
\]

Figure 6 shows the resulting marginal stability diagrams for the analytic solution and for a numerical solution with \( A = 100 \).
Figure 6: Marginal stability curves for the $A = 100$ equilibria (red) and analytic ballooning profile, $A = \infty$ (blue).
Chapter 4

Tokamak Equilibria with Resonant Magnetic Perturbations

4.1 Equilibrium Calculations

4.1.1 Motivation

As discussed in Section 1.3.4, understanding how externally imposed 3-D magnetic fields (RMPs) modify the temperature and density profiles, allowing mitigation of Edge Localized Modes, is one of the most important areas of research in the Tokamak community. After energizing the coils which produce the RMP fields, significant changes to the density profile occur featuring a large pump-out of particles through the pedestal. The decreased density lowers the height of the pressure pedestal, reducing the drive for ELM instabilities. The mechanism by which this density pump-out occurs is not presently understood.

The EPED1 model has been highly successful in predicting the height and width of the pedestal in standard, ELMing H-mode discharges. One of the central assumptions used in this model is that the pressure gradient in the H-mode pedestal is limited by the onset of strong Kinetic Ballooning Mode (KBM) turbulence, which is closely related to ideal MHD ballooning stability. Equation 1.3, based off of an analysis of typical marginal ballooning stability curves, is used to predict the pedestal width. The success of the EPED1 model provides strong evidence that the onset of KBM turbulence sets a limit on the pedestal pressure gradient in ELMing H-mode discharges.

In this section, the 3DLEQ code is used to model the effect of RMP fields on an axisymmetric Tokamak equilibrium. The physics of how axisymmetric equilibria respond to helical deformation of
the flux surface shape is examined. Marginal stability diagrams are generated for these equilibria to study the effect of RMP fields on MHD ballooning modes with the goal of understanding how these perturbations could affect the onset of KBM turbulence.

### 4.1.2 Equilibrium parametrization

The following parametrization is used to model the use of RMP fields,

\[
R = R_0 + \rho_0 \cos(\theta + \arcsin(\delta)\sin(\theta)) + \sum_i \gamma_i \cos(M_i \theta - N_i \zeta),
\]

\[
\phi = -\zeta,
\]

\[
Z = \kappa \rho \sin(\theta) + \sum_i \gamma_i \sin(M_i \theta - N_i \zeta).
\]

The terms leading with \( \gamma \) correspond to a radial modulation of the flux surface shape, consistent with the effect of RMP coils. A full spectrum of different M and N values can be included, and the magnitude of the perturbation for each harmonic can be independently set. \( \delta \) and \( \kappa \) are the triangularity and elongation of the axisymmetric equilibrium, respectively. The stability of marginal ballooning instabilities is quite sensitive to 2D shaping, so comparisons with experiments must use carefully chosen equilibria. In this work the following values are used:

- \( A = R_0/\rho_0 = 3.17 \)
- \( \kappa = 2.0 \)
- \( \delta = 0.416 \)
- \( \iota = 1/3.03 \sim 0.33033 \) and \( \iota = \pi/3 \sim 1.047 \)

121 poloidal modes (\( M = 0 \rightarrow 120 \)) are used in all cases, and 61 toroidal modes (\( N = -30 \rightarrow 30 \)) are used for the nonaxisymmetric equilibria. These parameters are representative of the geometry of typical Tokamak equilibria, though obtaining a realistic distribution of magnetic field strength across the surface requires using a more detailed parametrization, as laid out in Reference [21] (though the parametrization there is in terms of the geometric angle as opposed to a straight field line angle). A detailed comparison with experimental equilibria would require adding the 3D perturbations to Miller’s parametrization, though this work focuses on this simpler parametrization as a starting point.
4.1.3 Radial magnetic field perturbation

The key quantity used in experimental studies of RMPs is $\delta b_r/B_0$, the radial component of the perturbation divided by the background toroidal field. While $\gamma$ in the parametrization used here crudely corresponds to an equal magnitude perturbation of $|B|$, determining a radial $\delta b$ requires a more detailed calculation. Equations 4.1 through 4.3 give a magnetic field which can be written as

$$B = \frac{1}{\sqrt{g}} \left[ \frac{\partial x}{\partial \zeta} + \iota \frac{\partial x}{\partial \theta} \right] = B^{axi} + \tilde{B} \quad (4.4)$$

$$B = B^{axi} + \frac{1}{\sqrt{g}} \sum_i (M_i \iota - N_i) \gamma_i \left[ -\sin(M_i \theta - N_i \zeta) \tilde{R} + \cos(M_i \theta - N_i \zeta) \tilde{Z} \right] \quad (4.5)$$
In the coordinate system of the axisymmetric field, there is a component of $\tilde{B}$ which is in the radial ($\rho$) direction

$$\tilde{B}_\rho = \tilde{B} \cdot \nabla \psi_{axi}$$

$$\nabla \psi_{axi} = \frac{1}{\sqrt{g}} \left[ \frac{\partial Z_{axi}}{\partial \theta} \hat{R} - \frac{\partial P_{axi}}{\partial \zeta} \hat{Z} \right]$$

Using the axisymmetric component of the equilibrium, Equation 4.6 can be constructed as a series solution

$$\nabla \psi_{axi} \sqrt{g} \frac{d\psi_{axi}}{d\rho} = \frac{B_0}{R_0 \rho} \sum_k \left[ r_k \cos(k\theta) \hat{R} + z_k \sin(k\theta) \hat{Z} \right] \, .$$

In the high aspect ratio, circular cross section limit this reduces to

$$\nabla \psi_{axi} \sqrt{g} \frac{d\psi_{axi}}{d\rho} \approx \frac{B_0}{R_0} \left[ \cos(\theta) \hat{R} + \sin(\theta) \hat{Z} \right] \, .$$

The radial component of the perturbed magnetic field is given by

$$\frac{\tilde{B}_\rho}{B_0} = \sum_{i,k} (M_i \bar{\iota} - N_i \gamma_i) \frac{\gamma_i}{R_0} \left[ \frac{z_k - r_k \sin((M_i + 1)\theta - N_i\zeta)}{2\rho} - \frac{z_k + r_k \sin((M_i - 1)\theta - N_i\zeta)}{2\rho} \right] \, .$$

In the high aspect ratio, circular cross section limit this expression reduces to

$$\frac{\tilde{B}_\rho}{B_0} \approx - \sum_i (M_i \bar{\iota} - N_i) \frac{\gamma_i}{R_0} \sin((M_i - 1)\theta - N_i\zeta) \, .$$

The relationship between $\gamma_i$ and $\tilde{B}_\rho$ for a given Fourier harmonic can be estimated as

$$\frac{\tilde{B}_\rho}{B_0} (M = M_i - 1, N = N_i) \approx (N_i - M_i \bar{\iota}) \frac{\gamma_i}{R_0} \, .$$

This work focuses on the stability properties of two equilibria with $\gamma_i = 0.001$, $N_i = 3$, and $M_i = [4 \rightarrow 14]$. One equilibrium has a rotational transform near a low order resonant surface, $\iota = 1/3.03 \sim 0.33033$, and the other at a value which does not resonant with any component of the perturbation, $\iota = \pi/3 \sim 1.047$. Figure 8 shows the value of the radial magnetic perturbation for each equilibrium, evaluated using Equation 4.12. For the $\iota \sim 0.33$ equilibrium, the value of $\delta b_r/B_0$ ranges from $8 \times 10^{-4}$ to $-8 \times 10^{-4}$, on the same order as the vacuum levels of RMP fields. For the $\iota \sim 1.047$ equilibrium the radial perturbation is larger, reaching almost $6 \times 10^{-3}$, much larger than even the vacuum level of RMP fields typically used.
Figure 8: Radial magnetic perturbation amplitude with $N=3$ for the two non-axisymmetric equilibria.
4.1.4 Modifications to the MHD Equilibrium Quantities

Most of the equilibrium quantities display only a weak helical modulation due to the 3D perturbation. Figures 9 through 12 compare the normal curvature and magnetic field strength ($B^2$) for each of the three equilibria. While local mode stability is quite sensitive to the normal curvature, stability analysis suggests that the weak perturbation to the normal curvature shown here has little impact.

The geodesic curvature and normal torsion display a stronger modulation. Figures 13 and 14 compare the geodesic curvature for each equilibrium. The helical perturbation is more noticeable here than for the normal curvature of magnetic field strength. Figures 15 and 16 display the normal torsion, where there is an even stronger effect. The axisymmetric structure of the normal torsion is significantly altered. Both the geodesic curvature and normal torsion play an important role in setting the local shear, and the stability analysis which follows suggests that the perturbation to the geodesic curvature can become very important depending on the value of the rotational transform.

4.2 Ballooning Stability Analysis

4.2.1 Marginal Stability Diagrams

Using the ballooning stability code developed in this work, the marginal stability properties of the axisymmetric and two $\gamma = 0.001$ equilibria were analyzed. At each point in s-\(\alpha\) space, the starting point for integration of the ballooning equation ($\theta_0$) was varied along the magnetic field line to determine the most negative eigenvalue. For the two perturbed equilibria, this procedure was repeated for 4 different magnetic field lines equally spaced across the flux surface, with no difference in the marginal stability properties for each field line. This suggests that the weak 3D perturbations do not introduce a field line dependence for the ballooning eigenvalues, though a large number of field lines should be checked to confirm this.

Figure 17 shows the resulting marginal stability diagrams. Surprisingly, the marginal stability properties of the $\iota \sim 1.047$ equilibrium, with a larger magnitude radial magnetic perturbation, are nearly identical to the axisymmetric equilibrium. At low average shear and high pressure gradient, the stability boundary expands for the $\iota \sim 0.33$ equilibrium, while the stability boundary is less restrictive
Figure 9: Normal curvature for the axisymmetric (top) and $\gamma = 0.001$, $\iota = 1.047$ (bottom) equilibria.
Figure 10: Normal curvature for equilibria with $\gamma = 0.001$, $\iota = 1.047$ (top) and $\iota = 0.33$ (bottom).
Figure 11: $B^2$ for the axisymmetric (top) and $\gamma = 0.001, \nu = 1.047$ (bottom) equilibria.
Figure 12: $B^2$ for equilibria with $\gamma = 0.001$, $\iota = 1.047$ (top) and $\iota = 0.33$ (bottom).
Figure 13: Geodesic curvature for the axisymmetric (top) and $\gamma = 0.001$, $\iota = 1.047$ (bottom) equilibria.
Figure 14: Geodesic curvature for equilibria with $\gamma = 0.001, \iota = 1.047$ (top) and $\iota = 0.33$ (bottom).
Figure 15: Normal torsion for the axisymmetric (top) and $\gamma = 0.001, \iota = 1.047$ (bottom) equilibria.
Figure 16: Normal torsion for equilibria with $\gamma = 0.001$, $\iota = 1.047$ (top) and $\iota = 0.33$ (bottom).
at high average shear. Figure 18 shows a computation of the values of $s$ and $\alpha$ in the edge of the plasma for a H-mode experiment in DIII-D. $\alpha$ decreases from a peak value of 8 near $\Psi_n = 0.95$ down to roughly 1.5 at the separatrix. Much of the pedestal region is at $\alpha > 3$ and $s < 5$, suggesting that the relevant portion of the marginal stability diagram is the lower shear branch. The mechanism which destabilizes ballooning modes for the $\iota \sim 0.33$ can be examined by looking at changes to the local shear at $s = 1$, $\alpha = 3$.

### 4.2.2 Comparison at fixed average shear, pressure gradient

At roughly $\alpha = 3$, the marginal stability boundary for the $\iota \sim 0.33$ equilibrium begins to expand to lower shear than for the axisymmetric equilibrium. The disparity grows larger at higher $\alpha$, suggesting that the destabilizing mechanism scales with the pressure gradient. At $s = 1$, $\alpha = 3$ all of the equilibria are close to marginality but the $\iota \sim 0.33$ equilibrium is unstable. This is a useful point in s-$\alpha$ space to examine ballooning eigenvectors and the local shear to understand the destabilizing mechanism.

Figure 19 shows the eigenvector, local shear, and normal curvature (each normalized to one in this plot) for the unstable, $\iota \sim 0.33$ equilibrium. As expected, the eigenvector is largely localized to the
outboard midplane (θ ∼ 0) where the normal curvature is negative and the local shear is relatively weak. Figure 20 compares the normal curvature and local shear for each of the three equilibria at these parameters. At first glance the difference between the $\ell \sim 0.33$ equilibrium and the other two appears negligible, apart from different peak magnitudes due to the dependence on the rotational transform (as these plots are not normalized).

A comparison of the local shear for each equilibrium across the entire surface, shown in Figures 21 and 22, leads to an important insight. For the $\ell \sim 0.33$ equilibrium, there is a resonant helical structure present which aligns with the path of magnetic field lines. These helical structures have a significant impact on the local shear, significantly lowering it over parts of the inboard midplane. Lowering the magnitude of the local shear has a destabilizing effect in general, and may be responsible for the onset of ballooning instability at these parameters.

The local shear can be decomposed into separate components corresponding to different physical effects, as shown in Equations 3.44 through 3.46. This is performed for each of the three equilibria in Figures 23, 24, and 25. The contribution to the local shear from the $\partial D_2/\partial \eta$ term scales with the pressure gradient and represents modulation of the local shear by spectrum of Pfirsch-Schluter currents. Figure 25 shows that this term is responsible for the strong, resonant helical modulation.

This can be explained by noting that the Fourier coefficients of the Pfirsch-Schluter spectrum are

![Graph](image-url)
Figure 19: Normal curvature, local shear, and eigenvector for the $\iota = 0.33$ equilibria, unstable at $s = 1, \alpha = 3$. Each quantity is normalized to one here.

Figure 20: Comparison of the normal curvature and local shear for each of the three equilibria.
determined by a magnetic differential equation, such that

\[ \lambda_{mn} \sim -\frac{\kappa_{gmn}}{n + \iota_0 m}. \]  

(4.13)

This suggests that components of the geodesic curvature with \( n + \iota m \leq 1 \) will play a dominant role in the Pfirsch-Schluter spectrum. This is demonstrated in Figures 26 and 27, where the Pfirsch-Schluter spectrum is shown for each equilibrium. While the perturbation to the geodesic curvature was of comparable magnitude for each of the \( \gamma = 0.001 \) equilibria, the resonant denominator effect leads to a much stronger perturbation of the Pfirsch-Schluter spectrum for the \( \iota \sim 0.33 \) equilibrium. Figure 28 shows the contribution to the local shear from the Pfirsch-Schluter modulation for each of the perturbed equilibria, highlighting the role of the resonance.

### 4.2.3 Discussion

It is interesting to note that the magnitude of \( \delta b_r/B_0 \) is significantly larger for the \( \iota \sim 1.047 \) equilibrium yet there is little change in the ballooning stability properties. For the \( \iota \sim 0.33 \) equilibrium, the resonant component of the geodesic curvature \((M = 9, N = 3)\) dominates the Pfirsch-Schluter spectrum and results in a stronger modulation of the local shear. This has a stabilizing effect at large \( s \) and also at low \( \alpha \), but is destabilizing at low \( s \) when \( \alpha > 2 \). As \( \alpha \) increases the destabilizing effect becomes more prominent, which is consistent with the hypothesis that Pfirsch-Schluter modulation of the local shear is causing the destabilization (as the \( D_2 \) term scales with the pressure gradient).

These calculations show that 3D deformations of magnetic flux surfaces, of comparable magnitude to those imposed by RMP fields in vacuum, can lead to surprisingly large modulation of the local shear. Resonant 3D perturbations of the geodesic curvature can have a drastic impact on the Pfirsch-Schluter spectrum and play an important role in modulation of the local shear. There is a destabilizing effect on ideal MHD ballooning modes particularly at large pressure gradients, comparable to those found in the pedestal of typical H-mode experiments. However, there is also a stabilizing effect at larger values of the surface averaged shear. There was no change in stability for 4 different, equally spaced magnetic field lines, but a more detailed analysis is needed to confirm that the ballooning eigenvalues are still independent of magnetic field line choice.
Figure 21: Local shear for the (top) axisymmetric and (bottom) $\gamma = 0.001$, $\iota = 1.047$ equilibria at $s = 1, \alpha = 3$. 
Figure 22: Local shear for equilibria with $\gamma = 0.001$, $\iota = 1.047$ (top) and $\iota = 0.33$ (bottom), at $s = 1, \alpha = 3$. 
Figure 23: Decomposition of the local shear into its three components for the $\gamma = 0$ equilibrium.
Figure 24: Decomposition of the local shear into its three components for the $\gamma = 0.001$, $\iota = 1.047$ equilibrium.
Figure 25: Decomposition of the local shear into its three components for the $\gamma = 0.001$, $\iota = 0.33$ equilibrium.
Figure 26: Pfirsch-Schluter coefficient for the (top) axisymmetric and (bottom) γ = 0.001 equilibria.
Figure 27: Pfirsch-Schluter coefficient for the $\gamma = 0.001$ equilibrium with (top) $\iota_0 = 1.047$ and (bottom) $\iota_0 = 0.33$ at $s = 1$ and $\alpha = 3$. 
Figure 28: Comparison of the $D_2$ contribution to the local shear for the $\iota = 1.047$ (top) and $\iota = 0.33$ (bottom) equilibria at $s = 1$ and $\alpha = 3$. 
Chapter 5

Shaping of Stellarator Equilibria

5.1 Equilibrium Calculations

5.1.1 Motivation

As discussed in Section 1.4.1, efforts to further optimize Stellarator configurations today focus on mitigating turbulent transport driven by microinstabilities such as Ion Temperature Gradient modes. The STELLOPT code performs global 3D MHD equilibrium calculations with VMEC to optimize a cost function \( C^2 = \sum_i w_i C_i^2 \) which is determined by a set of weighted constraints. For example, to minimize turbulent heat flux one would ideally choose \( C_{\text{turb}} = \langle Q_{\text{GK}} \rangle \), the flux surface averaged heat flux from a nonlinear gyrokinetic simulation. Due to the computationally intensive nature of gyrokinetic simulations, a simple proxy function for \( Q_{\text{GK}} \) is instead used in STELLOPT. The resulting configurations can then be examined using gyrokinetic simulations. Using a relatively simple analytic estimate of the heat flux due to ITG turbulence, STELLOPT has been able to lower turbulent heat transport by up to a factor of 2−2.5 relative to the starting equilibrium.

As shown in Chapter 4, local equilibrium theory is a powerful tool for analysis of local mode stability in 3D configurations. It allows for direct control of the geometric properties of magnetic field lines and provides insight into how geometry affects the local shear. While future work could incorporate gyrokinetic simulations using equilibria produced with the 3DLEQ code, ballooning stability calculations are used here. The computational cost of ballooning stability calculations is significantly less than gyrokinetic simulation, and a better understanding of the relationship between ITG and ballooning modes may even allow for the use of ballooning stability calculations as a proxy for ITG stability during optimization.

In this work, local equilibrium theory is used to isolate one particular property of Quasi-Helically
Symmetric (QHS) Stellarators: suppression of the 'toroidal' component of the normal curvature. A simple circular cross section, helically modulated equilibrium is used as a starting point. An additional term in the flux surface parametrization is chosen to cancel out the toroidal component of the normal curvature. The impact on the ballooning stability properties of these equilibria is examined.

5.1.2 Equilibrium Parametrization

An additional term is added to the circular cross section, helically modulated equilibrium used earlier,

\[
R = R_0 + \rho_0 \cos(\theta) + \Delta \cos(N \zeta) + C \frac{2R_0 \rho_0}{N^2 \Delta} \sin(N \zeta) \sin(\theta),
\]

(5.1)

\[
\phi = -\zeta,
\]

(5.2)

\[
Z = \rho_0 \sin(\theta) + \Delta \sin(N \zeta) - C \frac{2R_0 \rho_0}{N^2 \Delta} \sin(N \zeta) \cos(\theta).
\]

(5.3)

In the limit that \(N^2 \Delta/R_0 \gg 1 > N \Delta/R_0\) the normal and geodesic curvatures can be written

\[
\kappa_n = -\frac{N^2 \Delta}{R_0^2} \cos(N \zeta - \theta) - C \frac{\rho_0}{R_0} \cos(2N \zeta - \theta) + C - \frac{1}{R_0} \cos(\theta) + O \left( \frac{1}{N^2 \Delta} \right),
\]

(5.4)

\[
\kappa_g = -\frac{N^2 \Delta}{R_0^2} \sin(N \zeta - \theta) \left[ 1 + \frac{N^2 \Delta^2}{R_0^2} \right] + 2C \frac{\rho_0}{R_0} \sin(\theta) - C \frac{\rho_0}{R_0} \sin(2N \zeta - \theta) + C - \frac{1}{R_0} \sin(\theta) + O \left( \frac{1}{N^2 \Delta} \right).
\]

(5.5)

The terms leading with \((C-1)\) correspond to the contributions to each quantity from toroidal curvature. As \(C \to 1\) the toroidal component of the curvatures goes to zero.

This work examines a series of equilibria with increasing \(C\) (\(C = 0, 0.2, 0.5, 0.7\)), with the following parameters:

- \(N = 4\)
- \(R_0 = 4\)
- \(\rho_0 = \Delta = 1\)
- \(\iota_0 = \pi/3 \approx 1.047\)

with a total of 101 poloidal \((M = 0 \to 100)\) and 61 toroidal \((N = -30 \to 30)\) Fourier modes used in the decomposition.
5.1.3 Modifications to the MHD Equilibrium Quantities

The normal curvature is shown in Figure 29 for $C = 0, 0.7$ equilibria. With $C = 0$ the normal curvature is predominantly toroidal (negative near the outboard midplane, $\theta \sim 0, 2\pi$ and positive near the inboard midplane, $\theta \sim \pi$) except for narrow regions corresponding to the 'corners' of the device (where separate field periods meet). With $C = 0.7$ the structure is helically rotated, though there is still a toroidal structure present.

The geodesic curvature is shown in Figure 30 for the same two equilibria. The effect is similar, with the $C = 0$ structure being helically rotated slightly when $C = 0.7$. The normal torsion is shown in Figure 31. In each case the normal torsion has its largest magnitude near the corners of the device, which is also where the strongest modulation occurs as $C$ increases. With $C = 0.7$ some of the fine scale structure in the normal torsion near the device corners is smoothed out.

Surprisingly, the magnetic field is strongly affected by increasing $C$. Figure 32 shows the change in magnetic field strength from $C = 0$ to $C = 0.7$. The areas of strongest magnetic field start at the outboard midplane and are moved to the inboard side. Figure 33 shows the change from $C = 0$ to $C = 0.2$, where the region of strong magnetic field is just beginning to migrate to the inboard side. Figures 34 and 35 show this effect more clearly by showing the magnetic field strength on 3D flux surface plots.

The net effect of increasing $C$ is relatively modest for the normal and geodesic curvatures. The structure of each is helically rotated somewhat as the toroidal components are cancelled out. The normal torsion and magnetic field strength are modulated significantly near the corners of the device on the inboard side. The most visible effect of increasing $C$ is that the region of strongest magnetic field migrates to the inboard side of the device, but the following stability analysis shows that the modulation of the normal curvature is quite important.
Figure 29: Normal curvature with $C = 0$ (top) and $C = 0.7$ (bottom).
Figure 30: Geodesic curvature with $C = 0$ (top) and $C = 0.7$ (bottom).
Figure 31: Normal torsion with $C = 0$ (top) and $C = 0.7$ (bottom).
Figure 32: Magnetic field strength ($B^2$) with $C = 0$ (top) and $C = 0.7$ (bottom).
Figure 33: Magnetic field strength \( (B^2) \) with \( C = 0 \) (top) and \( C = 0.2 \) (bottom).
Figure 34: Contours of magnetic field strength ($B^2$) with $C = 0$. 
Figure 35: Contours of magnetic field strength \( (B^2) \) with \( C = 0.7 \).
5.2 Ballooning Stability Analysis

5.2.1 Marginal Stability Diagrams

Using the ballooning stability code developed in this work, the marginal stability properties of the equilibria with $C = 0, 0.2, 0.5, 0.7$ were examined for one magnetic field line (passing through $(\theta, \zeta) = (0.0)$). At each point in $s-\alpha$ space, the starting point for integration of the ballooning equation ($\theta_0$) was varied along the magnetic field line to determine the most negative eigenvalue. It is important to note that the ballooning eigenvalue also depends on choice of magnetic field line for fully 3D configurations. In order to capture the full effect of 3D shaping, information from the entire flux surface should be incorporated (either by averaging the growth rate over the entire surface, or by picking out the most unstable eigenvalue by varying $\theta_0$ and the field line label at each value of $s$ and $\alpha$). This work solely examines one field line, as a demonstration of how the tools developed here can facilitate the study of local mode stability in Stellarator configurations.

The marginal stability diagram for this field line is shown in Figure 36 for each equilibrium. The net effect of increasing $C$ is to tilt the marginal stability diagram in $s, \alpha$ space. At high average shear the unstable region is expanded, but at low shear there is a stabilizing effect.

5.2.2 Comparison at fixed average shear, pressure gradient

In order to examine the role of increasing $C$ on stability, the different equilibria are examined with $s = 1.2, \alpha = 1.97$. At these parameters, the $C = 0, 0.2$ equilibria are unstable while the $C = 0.5, 0.7$ equilibria remain stable. The eigenvector, local shear, and normal curvature are shown for the unstable $C = 0$ equilibrium in Figure 37. As one would expect, the eigenvector is localized in a region of negative curvature and low shear.

The normal curvature and local shear for all four equilibria are shown in Figures 38 and 39 respectively. In part of the region where the $C = 0$ eigenvector was localized, the normal curvature is flipped from negative to positive for the $C = 0.5$ and $C = 0.7$ equilibria.

The modulation of the local shear with increasing $C$ also appears to have a stabilizing effect. The structure of the local shear does not change significantly with increasing $C$, but its magnitude increases with $C$. This effect is surprisingly similar to the study performed in Reference [15], where the turbulent
Figure 36: Marginal stability diagrams for equilibria with $C = 0, 0.2, 0.5, 0.7$. 
heat flux driven by ITG modes was decreased by artificially doubling the local shear. In this case, increasing the magnitude of the local shear (by increasing C) also has a stabilizing effect.

The structure of the local shear across the entire flux surface, for $C = 0$ and $C = 0.7$, is shown in Figure 40. Again it can be seen that the dominant effect of increasing C is to uniformly increase the magnitude of the local shear. The decomposition technique used in the previous section is repeated here for the $C = 0$ and $C = 0.7$ equilibria in Figures 41 and 44 respectively.

It appears that the contribution from $D_1$ is several orders of magnitude larger than the others, but this is largely cancelled out by the contribution from the flux surface average term (i.e. the first term on the right hand side of Equation 2.51). Figure 43 shows the cancellation which occurs by combining the $D_1$ and flux surface average terms for each equilibrium. Figure 44 shows just $\partial D_1 / \partial \eta$. The local

Figure 37: Eigenvector, local shear, and normal curvature (each normalized) for the unstable $C = 0$ equilibrium at $s = 1.2, \alpha = 1.97$. 
Figure 38: Normal curvature for equilibria with $C = 0, 0.2, 0.5, 0.7$. 
Figure 39: Local shear for equilibria with $C = 0, 0.2, 0.5, 0.7$. 
shear is sensitive to small changes in $\partial D_1 / \partial \eta$ due to the large magnitude of $B^2 / |\nabla \psi|^2$ (again noting that these plots of the local shear are unnormalized.)

5.2.3 Discussion

Modulating the normal curvature by increasing C has a destabilizing effect at large average shear, but a stabilizing effect at the lower average shear where Stellarators typically operate. While the toroidal structure of the normal curvature is helically modulated, there are a number of other interesting effects. The distribution of magnetic field strength on the surface is significantly altered.

The shift in the normal curvature plays a stabilizing role at low average shear. Part of the region in which the eigenmode localizes for the $C = 0$ equilibrium flips from negative to positive normal curvature for the $C = 0.5$ and $C = 0.7$ equilibria. There is also a roughly uniform increase in the magnitude of the local shear as C increases, which is another stabilizing effect.

While the stability analysis is incomplete due to its consideration of only a single field line, the usefulness of the Local 3D Equilibrium has been demonstrated. One can directly manipulate the geometric properties of magnetic field lines, perform a stability analysis, and study the changes to the normal curvature and local shear in detail.
Figure 40: Comparison of the local shear for the $C = 0$ (top) and $C = 0.7$ (bottom) equilibria.
Figure 41: Decomposition of the local shear for the $C = 0$ equilibrium. The large magnitude of the $D_1$ term is largely cancelled out by the contribution to the local shear from the flux surface averaged shear as shown in Figure 43.
Figure 42: Decomposition of the local shear for the $C = 0.7$ equilibrium. The large magnitude of the $D_1$ term is largely cancelled out by the contribution to the local shear from the flux surface averaged shear as shown in Figure 43.
Figure 43: Comparison of the contribution to the local shear from $D_1$ AND the flux surface average term for the $C = 0$ (top) and $C = 0.7$ (bottom) equilibria.
Figure 44: Comparison of $\partial D_1/\partial \eta$ for the $C = 0$ (top) and $C = 0.7$ (bottom) equilibria.
Chapter 6

Conclusions

6.1 Development of computational tools

The 3DLEQ code, which is a numerical implementation of local 3D equilibrium theory, was developed over the course of this work. Once an equilibrium parametrization is chosen it must be hard written into the code once, but it is straightforward to generate families of equilibria for stability analysis, where one systematically varies certain equilibrium parameters. A first order, linear partial differential equation which sets the jacobian is solved using a spectral method with Fourier sine and cosine series as the basis functions. The code generates Fourier coefficients for the MHD equilibrium quantities as output. Straight field line coordinates are used throughout, so the resulting equilibrium information typically requires fewer Fourier modes than an equilibrium solution which is transformed to straight field line coordinates from geometric coordinates.

A ballooning stability code was also developed for usage with 3DLEQ. The algorithms used in the finite difference solution of the ballooning eigenvalue equation are those from the widely used COBRA code. As profile quantities are systematically varied during the generation of marginal stability curves, the local shear must be recomputed each time consistent with the profile relationship from local equilibrium theory.

6.2 The physics of resonant magnetic perturbations

The use of resonant magnetic perturbations to mitigate ELM instabilities during H-mode Tokamak operation is both poorly understood and crucially important to the future success of the ITER device. The 3DLEQ code was used to examine the equilibrium physics associated with a 3D perturbation to the
shape of a typical Tokamak flux surface. The impact on ideal MHD ballooning stability was examined, due to its role as a proxy for the onset of KBM turbulence which is thought to set the pressure gradient in the pedestal region during H-mode operation.

It is shown that perturbations to the geodesic curvature can be particularly important due to its role in setting the Pfirsch-Schluter spectrum. A resonant component of the geodesic curvature dominates the Pfirsch-Schluter spectrum and leads to substantial modulation of the local shear (an effect which is more prominent as the pressure gradient increases). This can have both a stabilizing and destabilizing effect depending on the value of the surface averaged shear and pressure gradient, but at high pressure gradient and low average shear the effect is destabilizing.

This analysis has shed some light on the physics of local mode stability for axisymmetric equilibria with small 3D perturbations. The rotational transform and 3D structure of the geodesic curvature are of particular importance, and the magnitude of the radial magnetic perturbation is surprisingly unimportant for the equilibria examined here. This provides a framework for a detailed analysis of ballooning stability in the presence of RMPs, where future work should incorporate Miller’s parametrization. It is also possible to modify the spectrum of 3D perturbations used (by varying $\gamma$ for each of the 3D terms in the parametrization) as knowledge of the structure and magnitude of RMP fields within the plasma improves.

### 6.3 Optimization of stellarator equilibria

The usefulness of local 3D equilibria as a complement to global equilibrium calculations for Stellarator configurations has been demonstrated. A parametrization which allows one to manipulate the normal curvature and cancel out the 'toroidal' component was introduced, reproducing one property of quasi-helically symmetric Stellarators. While only one field line was examined during this analysis, the modulation of the normal curvature was stabilizing at parameters relevant to Stellarator optimization. Cancelling the toroidal component of the normal curvature also lead to a uniform increase in the magnitude of the local shear, another stabilizing effect.

The tools developed here complement existing efforts to optimize the turbulent transport properties of Stellarator equilibria. There is an enticing prospect of using local 3D equilibria to develop better
proxy functions for use in optimization loops. The ubiquitous nature of the role played by the local shear and normal curvature for a range of highly localized instabilities suggests it may be possible to develop a cost function based on these quantities alone. While further comparisons between ideal MHD ballooning modes and ITG modes are needed, it may even be useful to use ballooning stability calculations as a proxy function during optimization. In general, the tools developed here allow for a highly detailed description of the physics of local mode stability in 3D configurations at very little computational cost. The ability to explicitly control the geometric properties of magnetic field lines is invaluable for these studies.
Chapter 7

Bibliography


