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The effect of anisotropic heat transport on magnetic islands in 3-D configurations

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An analytic theory of nonlinear pressure-induced magnetic island formation using a boundary layer analysis is presented. This theory extends previous work by including the effects of finite parallel heat transport, and is applicable to general three dimensional magnetic configurations. In this work, particular attention is paid to the role of finite parallel heat conduction in the context of pressure-induced island physics. It is found that localized currents that require self-consistent deformation of the pressure profile, such as resistive interchange and bootstrap currents, are attenuated by finite parallel heat conduction when the magnetic islands are sufficiently small. However, these anisotropic effects do not change saturated island widths caused by Pfirsch-Schlüter current effects. Implications for finite pressure-induced island healing are discussed.

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I. INTRODUCTION

Conventional magnetohydrodynamic (MHD) equilibrium theory requires that the force balance equation $\boldsymbol{J}\times\boldsymbol{B} = \nabla p$ is satisfied. In two-dimensional systems, topologically toroidal flux functions are guaranteed to exist with the pressure constant on each surface. In general three-dimensional configurations, the magnetic topology is described not only by regions with flux surfaces but by magnetic islands and volume-filling stochastic magnetic fields. A rigorous application of the force balance equation implies that pressure is constant in regions of magnetic stochasticity. However, recent evidence from stellarator experiments suggest that pressure gradients can be sustained in regions where the magnetic field is thought to be stochastic.[1,2] These findings hint at the importance of including finite parallel transport effects in a self-consistent theory of flux surface destruction. In this work, the effect of finite parallel heat conduction on the magnetic topology of 3-D equilibria is addressed.

Understanding the plasma physics involved in magnetic surface breakup due to pressure-induced islands has been a topic of both analytic theory[2–8] and computational studies.[9–11] Additionally, analytical island theories have been proven useful to interpreting experimental data of magnetic island dynamics in stellarators.[12–14] In the majority of these studies, however, an assumption of equilibration along magnetic field lines $\boldsymbol{B}\cdot\nabla p = 0$ is employed. In the current work, the analytic theory of pressure-induced magnetic islands in 3-D equilibria is revisited, where the rigid requirement of parallel equilibration along field lines is weakened by allowing for the presence of finite, but large, parallel heat conduction.

Analytic island theories can provide insight into the numerical modeling of 3-D plasmas. In 3-D equilibrium computations, different computational tools employ different assumptions concerning the relaxation of pressure along field lines. Whereas some numerical tools rigorously enforce $\boldsymbol{B} \cdot \nabla p = 0$ as described above, others allow for finite diffusion coefficients which can influence physical mechanisms which affect island size. Furthermore, extended MHD codes have the ability to alter the magnitudes and degree of anisotropy in the various diffusion coefficients used in the modeling. Analytic island theory helps inform the use of these coefficients in modeling of stellarator island physics.

Analytic theories of pressure-induced magnetic islands begin by applying the MHD force balance equation. In general 3-D configurations, this ideal MHD theory predicts singular currents at rational surfaces when the magnetic surfaces are assumed to be topologically toroidal. These singular currents can be separated into two classes. Resonant Pfirsch-Schlüter currents appear in the presence of pressure gradients and resonant components of $1/B^2$. Additionally, the general solution allows for $\delta-$function parallel currents at rational surfaces. Both of these singularities can be resolved by allowing for the presence of magnetic islands at the rational surface.

Analytic island theories for 3-D equilibria are closely related to theoretical approaches used to describe nonlinear tearing mode evolution in tokamaks. In particular the effect of finite parallel heat conduction on neoclassical tearing modes has been addressed explicitly in previous work.[15–17] Additionally, it has been noted that resistive interchange effects[18] which are known to affect magnetic islands width calculations are also altered by finite parallel heat conduction[19]. What is shown in these publications is that for sufficiently small magnetic islands, the pressure profile in the vicinity of the magnetic island does not equilibrate along the magnetic field lines. If the time for diffusion across the magnetic island is short compared to the time to diffuse along the helically-deformed island magnetic surfaces, then the pressure profile is relatively unaffected by the presence of the magnetic island. However, for sufficiently large magnetic islands, equilibration along field lines is the

the dominant process; the MHD condition $\boldsymbol{B} \cdot \nabla p = 0$ is restored, and the conventional picture of island-induced helical flat spots in the pressure profile is recovered. The critical island width that separates these two asymptotic behaviors depends upon the ratio of the perpendicular to parallel heat diffusion coefficients $W_C \sim (\chi_\perp/\chi_\|)^{1/4}$.[15–17]

In the current work a resistive MHD model is used to describe plasma quasineutrality. The effects of neoclassical bootstrap currents are included in a modified Ohm's law, and finite parallel heat conduction is employed in the pressure evolution equation, similar to that in previous analytic calculations.[15] As such, two-fluid, plasma flow and kinetic effects are ignored for simplicity. What is found is that the two classes of singular currents that arise in 3-D equilibria are resolved by allowing for the formation of a magnetic island, but these currents are affected by parallel heat conduction in different ways. As in the tokamak case, island resistive interchange and bootstrap current profiles are sensitive to the detailed transport properties that control the island pressure profile. In order for these physical effects to affect magnetic island physics, the pressure profile must self-consistently deform such that the localized island currents feel the effects of the magnetic island topology. As noted above, for sufficiently small islands, the pressure profile is not helically distorted by the island and the self-consistently produced island currents are small.

In contrast, the island resolved resonant Pfirsch-Schlüter currents are largely insensitive to detailed pressure equilibration physics. The nature of the helical resonance for these currents does not rely on pressure equilibration processes in the island region, but rather the structure of the magnetic field spectrum which is controlled by global properties of the plasma. A distinction between the effects of the 'local' currents described in the previous paragraph relative to the 'global' Pfirsch-Schlüter currents has been previously been pointed out in simulation work by Hayashi[10]. Hayashi's result indicated that is was difficult to see the effects of the local currents on magnetic island physics; the global effects dominated. In light of the results of the present calculation, it is possible to interpret why this result was observed. If the ratio of parallel heat conductivity to perpendicular heat conductivity used in the simulations was not too large, the critical island width (W_C) would be large enough to prevent the local physics from affecting saturated island widths.

This paper will follow a procedure similar to previous work to solve the boundary layer problem of magnetic island dynamics.[20] However, finite parallel transport effects are used in calculating the pressure profile in fully 3-D magnetic topology. This technique will yield an equation for island width which is unencumbered by restrictions on β or geometric shaping. First, in Sect. II the boundary layer problem will be set up; the exterior solution of the quasineutrality equation will be given, which highlights the singular nature of the current sheet at the rational surface. This exterior solution will be matched asymptotically with the interior solution in the final section of this paper. In Sect. III, the quasineutrality equation will be solved in the interior region, where special treatment is given to finding the pressure profile in light of finite parallel heat conduction. Once the pressure profile is obtained, MHD force balance is employed to generate a Grad-Shafranov type expression for magnetic island equilibrium which includes the effects of finite parallel transport. Next, neoclassical bootstrap current effects are introduced through a modified Ohm's law. Finally, in Sect. IV asymptotic matching in the island region is employed to provide an expression for equilibrium island width. The results are discussed in Sect. V.

II. SINGULAR CURRENTS

Finite plasma pressure creates currents within the plasma that alter the structure of the magnetic field. To understand how magnetic flux surface destruction creates magnetic islands, one must understand how these currents alter the magnetic field topology. Much can be learned about these pressure-induced currents by assuming the existence of well-defined toroidal magnetic flux surfaces and applying the ideal MHD equilibrium equation, $\boldsymbol{J} \times \boldsymbol{B} = \nabla p$. However, as will be shown later in this section, this analysis results in singular currents near the rational surface.

Analytically, this current singularity can be resolved by removing the constraint of well-defined toroidal magnetic flux surfaces in the vicinity of the rational surface and applying boundary layer theory. That is, two regions are defined: an exterior region, far from the rational surface, and an interior region, close to the rational surface. First, the current will be found in the exterior region by applying MHD equilibrium. This calculation is performed with the constraint of well-defined toroidal magnetic flux surfaces, and will be carried out below in this section. Then, in Sect. III, the constraint of well-defined toroidal flux surfaces will be lifted in the region near the rational surface. Key to relaxing this constraint is obtaining and applying the correct expression for the pressure profile, as anisotropic transport significantly affects the pressure profile in the vicinity of the rational surface. Once the pressure profile is calculated, the solution of the quasineutrality equation and Ohm's law will yield the current in the interior region. Finally, to complete the boundary layer analysis, the interior and exterior solutions will be matched asymptotically, resulting in an equation which describes the equilibrium magnetic island width.

A. Coordinates and the equilibrium magnetic field

Far from the rational surface of interest, it will be assumed that magnetic field lines lie on topologically toroidal magnetic flux surfaces. These surfaces will be labeled with ψ, where $2\pi\psi$ is the enclosed toroidal flux. This provides the radial-like coordinate. The equilibrium magnetic field, using a contravariant and covariant basis in Boozer coordinates, is then represented by[21]

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\theta + \iota\nabla\zeta \times \nabla\psi \tag{1}$$

$$\boldsymbol{B}_0 = \beta_*(\psi, \theta, \zeta)\nabla\psi + I(\psi)\nabla\theta + g(\psi)\nabla\zeta \tag{2}$$

where ζ is the toriodal coordinate, θ is the poloidal coordinate, and ι is the rotational transform. It can be shown that $g(\psi)$ and $I(\psi)$ are related to the total poloidal and toroidal current respectively. Although this work focuses on 3-D magnetic configurations which include stellarators, there is no restriction on the value of the net toroidal current; $I(\psi)$ may not be 0. Taking the dot product of the covariant and the contravariant forms for the magnetic field produces a convenient expression for the Jacobian:

$$\mathscr{J} = \frac{g + \iota I}{\boldsymbol{B}_0^2} = \frac{1}{\nabla\psi \times \nabla\alpha \cdot \nabla\zeta} \tag{3}$$

B. Singular Plasma Current

To proceed with the boundary layer calculation, a solution for the plasma current must first be found in the region far from the island - the "exterior region." This current is determined from the MHD force balance equation,

$$\boldsymbol{J} \times \boldsymbol{B} = \nabla p \tag{4}$$

Eq.(4) implies that the magnetic flux surfaces are aligned with constant pressure surfaces. That is, p is only a function of ψ. To find the plasma current, the quasineutrality condition is employed,

$$\nabla \cdot \boldsymbol{J} = 0 \tag{5}$$

$$(\boldsymbol{B} \cdot \nabla)Q = -\nabla \cdot \frac{\boldsymbol{B} \times \nabla p}{B^2} \tag{6}$$

where the current is decomposed into components parallel and perpendicular to the magnetic field, $\boldsymbol{J} = Q\boldsymbol{B} + \boldsymbol{J}_\perp$. Here Q is the magnitude of the parallel current. The Jacobian and the parallel current magnitude are now expanded as Fourier series,

$$Q = \sum_{m,n} Q_{mn} e^{im\theta - in\zeta} \tag{7}$$

$$\mathscr{J} = \sum_{m,n} \mathscr{J}_{mn} e^{im\theta - in\zeta} \tag{8}$$

Solving Eq.(5) yields

$$Q_{mn} = -p_0' \mathscr{J}_{mn} \frac{g + \frac{n}{m} I}{(\iota I + g)(\iota - \frac{n}{m})} + \hat{Q}_{mn} \delta(\psi - \psi_s) \tag{9}$$

where $\iota = \iota(\psi)$ is the rotational transform, $\iota(\psi_s) = n/m$, and $\hat{Q}_{mn}$ is the magnitude of the current sheet at the rational surface. Notice that both terms in Eq.(9) are singular at the rational surface, ψ_s. The first term in Eq.(9) represents the inhomogeneous solution to the magnetic differential equation, Eq.(6). This term corresponds to the resonant component of the Pfirsch-Schlüter current which becomes singular at the rational surface, $\iota(\psi_s) = n/m$. It is driven by a pressure gradient and a non-zero contribution of the resonant harmonic of $1/B^2$. The second term of Eq.(9) represents the homogeneous solution to Eq.(6) and describes the localized currents driven near the rational surface. From the quasineutrality condition $\hat{Q}_{mn}$ cannot be determined; the constraint of topologically toroidal flux surfaces must be relaxed.

III. ISLAND REGION

To determine the currents near the rational surface, the quasineturality equation is again solved, but with the restriction of topologically toroidal flux surfaces relaxed. To solve this problem, an ordering approach is used. In the following, it is assumed that a thin magnetic island at the rational surface $\iota(\psi_s) = \iota_0$ perturbs the equilibrium quantities, and the quasineutrality equation is solved order by order. This analysis is similar to standard Rutherford analysis of nonlinear tearing mode theory. In the following, we assume that the nonlinear island width exceeds the linear resistive layer.[22–25] Once this inner layer solution is obtained, it will be matched asymtotically with the exterior solution found previously to produce an equation for equilibrium magnetic island width.

A. Perturbations and Ordering

The perturbations to all quantities are described by the form $f = f_0 + \delta f$ where f_0 describes the equilibrium assuming topologically toroidal flux surfaces, and δf is the perturbation due to the island. The basic ordering assumption is that $\delta = W/\Phi \ll 1$, where W is the island width in units of toroidal flux and Φ is the total enclosed toroidal flux in the plasma. As stated above, it is assumed that $W \gg \delta_{\text{res}}$, where δ_{res} is the width in toroidal flux of the resistive layer. All f_0 quantities are order unity, as are the derivatives of f_0 quantities. For perturbed quantities, the radial derivatives $(\partial/\partial\psi)$ are order $1/\delta$, while other derivatives with respect to the angular coordinates $(\partial/\partial\zeta, \partial/\partial\theta)$ are order unity.

Using these orderings, pressure and parallel current will be expanded in powers of δ, and the quasineutrality equation will be solved order by order,

$$p = p^0 + \delta^1 p^1 + \dots \tag{10}$$

$$Q = Q^0 + \delta^1 Q^1 + \dots \tag{11}$$

where superscripts refer to the order in δ.

Before discussing perturbations to the magnetic field, it is convenient to convert the poloidal coordinate to a helical angle-like coordinate, $\alpha = \theta - \iota_0\zeta$, where ι_0 is the rotational transform at the rational surface. The contravariant and covariant forms for the magnetic field become

$$\boldsymbol{B}_0 = \beta_*(\psi, \theta, \zeta)\nabla\psi + I(\psi)\nabla\alpha + [\iota_0 I(\psi) + g(\psi)]\nabla\zeta \tag{12}$$

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\alpha + (\iota - \iota_0)\nabla\zeta \times \nabla\psi \tag{13}$$

Island-producing magnetic perturbations will be described by the vector magnetic potential

$$\delta\boldsymbol{A} = A_\alpha \nabla\alpha + A_\zeta \nabla\zeta$$

$$\delta\boldsymbol{B} = \nabla \times \delta\boldsymbol{A}$$

where the gauge $A_\psi = 0$ is chosen. This allows the full magnetic field (equilibrium plus perturbed) to be written as

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\alpha + \nabla\zeta \times \nabla\Psi^* + \nabla A_\alpha \times \nabla\alpha \tag{14}$$

where $\Psi^* = \iota' x^2/2 - A_\zeta$ is the helical flux function and $x = \psi - \psi_s$. If A_ζ is dominated by a single resonant harmonic, $A_\zeta = A_\zeta(x)\cos(m\alpha)$, then the width of the island is given by $W = 4\sqrt{A_\zeta(W/2)/\iota'}$. See Figure 1. Given the ordering above, near the island region Ψ^*, A_α, and A_ζ are all order δ^2.

The key quantity which will be responsible for introducing finite anisotropic effects is the perturbed pressure, δp. It will be found that $p = p_0 + \delta p_0 + \delta p_1$, where δp_1 contains the anisotropic heat transport effects. Here, the subscript on δp_1 refers to the order of the anisotropic correction.

Finally, the following notation is used to indicate an average over ζ at fixed ψ and α:

$$\bar{f} \equiv \oint \frac{d\zeta}{2\pi} f(\psi, \alpha, \zeta) \tag{15}$$

$$\tilde{f} = f - \bar{f} \tag{16}$$

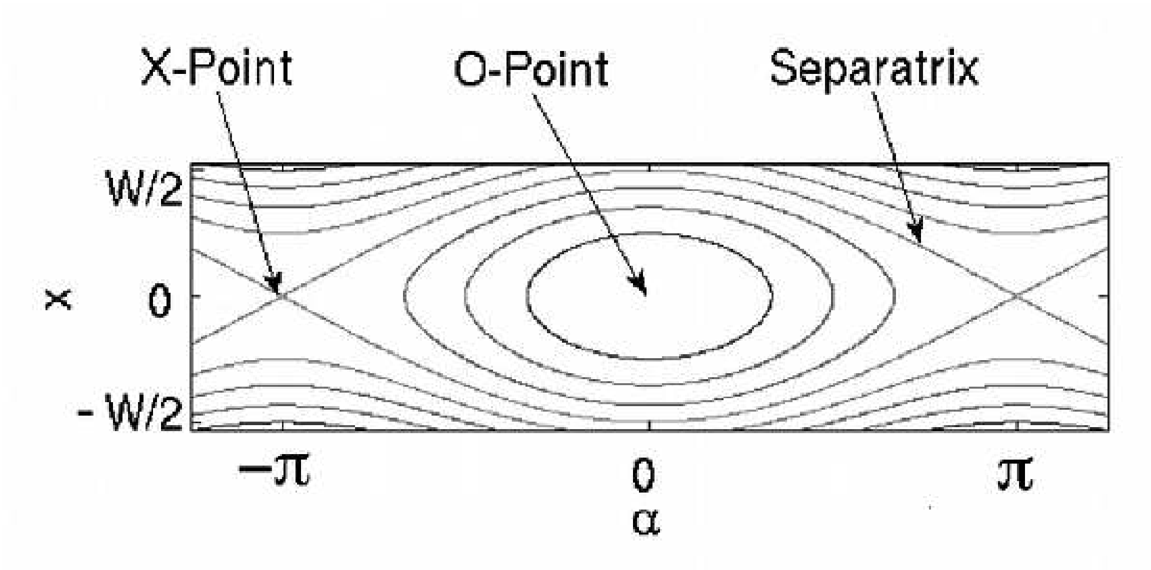

FIG. 1. Contours of Ψ^*, plotted in (x, α).

And, bracket notation represents the derivative operator in the plane perpendicular to $\nabla\zeta$,

$$[f, g] = \frac{\partial f}{\partial \psi}\frac{\partial g}{\partial \alpha} - \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial \psi} \tag{17}$$

B. Resolution of the singular plasma current through island formation

In this subsection, we demonstrate that the singular nature of Eq.(9) is resolved by allowing island formation at the rational surface. We now repeat the calculation of Sect.(II B), but with total magnetic field as described by Eq.(14). The resulting equation for parallel current, after averaging over ζ, is

$$[\overline{\Psi^*}, \overline{Q}] = -[\overline{p}, \overline{\mathscr{J}}] + ... \tag{18}$$

where a number of terms on the RHS are, for the moment, neglected for simplicity (a more complete quasineutrality condition is considered in the following section). Recall from Sect.(II B), the source of the singular Pfirsch-Schlüter current is the resonant component of the Jacobian, $\overline{\mathscr{J}}$. The Jacobian and the parallel current are now expanded as Fourier series in α and ζ. Assuming for the moment that only a single resonant harmonic is important, the Jacobian and the helical flux function can be written,

$$\overline{\mathscr{J}} = \mathscr{J}_{m_s n_s}(x)\cos(m_s\alpha) \tag{19}$$

$$\overline{\Psi^*} = \frac{\iota' x^2}{2} - A_\zeta(x)\cos(m_s\alpha) \tag{20}$$

As will be shown in the following subsection, pressure equilibrates on flux surfaces to lowest order, so that the pressure can be written, $p \approx p_0(\psi_s) + p_0(\psi_s)'x$. With this simplification, Eq.(18) becomes

$$\left(\iota' x - \frac{\partial A_\zeta}{\partial x}\cos(m_s\alpha)\right)\frac{\partial \overline{Q}}{\partial \alpha} - m_s A_\zeta \sin(m_s\alpha)\frac{\partial \overline{Q}}{\partial x} = p_0' m_s \mathscr{J}_{m_s n_s}(0)\sin(m_s\alpha) \tag{21}$$

In the absence of A_ζ, the solution to Eq.(21) produces the singular "1/x" response described in Eq.(9). However with $A_\zeta \neq 0$, the resonant Pfirsch-Schlüter current is not singular. Making the assumption that $\partial A_\zeta/\partial x$ is small in the island region ("constant psi approximation"), the solution to Eq.(21) is

$$\overline{Q} = \frac{-p_0' \mathscr{J}_{m_s n_s}(0) x}{\overline{A_\zeta}} + f(\overline{\Psi^*}) \tag{22}$$

where $f(\overline{\Psi^*})$ is the solution to the homogeneous problem $[\overline{\Psi^*}, \overline{Q}] = 0$. By allowing the formation of a magnetic island at the rational surface, the singularity in the parallel current is resolved. The parallel current is found to depend directly on the resonant Jacobian, and inversely with A_ζ.

In reality, this situation is more complicated than just resolving a single resonant harmonic. More generally, the resonant Jacobian and the helical flux function can be written as a sum of many harmonics with varying phases,

$$\overline{\mathscr{J}} = \sum_k \mathscr{J}_{km_skn_s}(x)\cos(km_s\alpha + \phi_k) \tag{23}$$

$$\overline{\Psi^*} = \frac{t'x^2}{2} - \overline{A_\zeta} \tag{24}$$

$$= \frac{t'x^2}{2} - \sum_k A_{\zeta k}(x)\cos(km_s\alpha + \eta_k) \tag{25}$$

If stellarator symmetry is present, the Jacobian can be written as a cosine series with $\phi_k = 0$.[26] However, this simplification is not required in the following analysis, and a solution with $\phi_k \neq 0$ can be found. In this more general framework, Eq.(18) now becomes

$$\left(t'x - \sum_k \frac{\partial A_{\zeta k}}{\partial x}\cos(km_x\alpha + \phi_k)\right)\frac{\partial \overline{Q}}{\partial \alpha} - \sum_k km_sA_{\zeta k}\sin(km_s\alpha + \eta_k)\frac{\partial \overline{Q}}{\partial x}$$
$$= p_0'\sum_k km_s\mathscr{J}_{km_skn_s}(0)\sin(km_s\alpha + \phi_k) \tag{26}$$

With $\overline{A_{\zeta k}} \neq 0$, the "1/x" singularity can be resolved with special properties of $A_{\zeta k}$. A solution to Eq.(26) is given by

$$\overline{Q} = -p_0'x\frac{\mathscr{J}_{m_sn_s}}{A_{\zeta 1}} + f(\overline{\Psi^*}) \tag{27}$$

with $A_{\zeta k}/A_{\zeta 1} = \mathscr{J}_{km_skn_s}/\mathscr{J}_{m_sn_s}$ and $\eta_k = \phi_k$. The parallel current is found to depend directly on the resonant Jacobian, and inversely with $A_{\zeta 1}$. Furthermore, since $A_{\zeta k}/A_{\zeta 1} = \mathscr{J}_{km_skn_s}/\mathscr{J}_{m_sn_s}$ and $\eta_k = \phi_k$, the helical flux function can be written

$$\overline{\Psi^*} = \frac{t'x^2}{2} - A_{\zeta 1}\left[\cos(m_s\alpha + \phi_1) + \frac{\mathscr{J}_{2m_s2n_s}}{\mathscr{J}_{m_sn_s}}\cos(2m_s\alpha + \phi_2) + \frac{\mathscr{J}_{3m_s3n_s}}{\mathscr{J}_{m_sn_s}}\cos(3m_s\alpha + \phi_3) + ...\right] \tag{28}$$

The solution given by Eqs. (27) and (28) depends on a special requirement of the magnetic field. In particular it requires the X- and O-points of the island to correspond to the zeroes of the right side of Eq.(26). If this is not the case, at the X- and O-points Eq.(18) assumes the form

$$0 = \frac{\partial p}{\partial x}\frac{\partial \mathscr{J}}{\partial \alpha} \tag{29}$$

To understand how this singularity is resolved, the pressure profile is more carefully treated.[21] As discussed in the introduction, in the case where islands are small, the pressure profile in the island is not flattened as in the large island case. The pressure profile remains largely topologically toroidal, with minor deformation in the vicinity of the island. As a result the radial thermal flux across topologically toroidal surfaces remains constant,

$$\nabla \cdot \mathbf{\Gamma} = 0 \tag{30}$$

Solution of this equation can be realized with a diffusion equation for pressure. Equating the radial particle flux far from the island with that in the vicinity of the island, the pressure gradient is found (see Appendix A)

$$\frac{p_0'}{p_\infty'} = \frac{D_{\rm tot}}{D_{PS}^{NR}\left(1+\frac{\epsilon^2}{x^2}\right) + D_{\rm other}} \tag{31}$$

where p_∞' is the pressure gradient far from the island, $D_{\rm tot}$ is the total diffusion coefficient for all physical drives of radial particle flux, D_{PS}^{NR} is the diffusion coefficient associated with non-resonant Pfirsch-Schlüter drives, $D_{\rm other} = D_{\rm tot} - D_{PS}$ is the total diffusion coefficient with the Pfirsch-Schlüter drives removed, and $\epsilon \ll 1$ is a numerical factor which includes a ratio of resonant and non-resonant Jacobians and is defined in Eqs.(A13) and (A14). Eq.(31) shows that, to maintain constant radial particle flux near the X- and O-points (as $x \to 0$), the transport coefficients become very large, causing the pressure gradient to vanish. Since p_0' vanishes as $x \to 0$, Eq.(18) is not singular at the X- and O-points. However, p_0' obtains its asymptotic value p_∞' at a distance $x \simeq \epsilon\sqrt{D_{PS}^{NR}/D_{\rm tot}} \ll 1$ which is generally a very small distance relative to other scales considered in the calculation. That is, p_0' is modified by this increased transport in a tiny region near the X- and O-points.

Finally, in the calculations throughout the rest of this paper, we do not explicitly make the "constant-ψ " approximation. This approximation will be naturally produced by our analysis below.

Next, finite parallel transport effects will be added by including $\delta p_1(\psi, \alpha)$. The profile δp_1 self-consistently describes the effect of the island on the pressure profile in the island region. These pressure modifications will subsequently produce pressure driven currents that will self-consistently modify the magnetic island width.

C. Specification of the pressure profile via solution of the heat equation

Usually, $\boldsymbol{B} \cdot \nabla p = 0$ is used in island width studies, implying pressure equilibration along field lines. However, in this work we will not be using parallel momentum balance to calculate the pressure profile. Instead, we will use a pressure evolution equation of the form

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} = -\nabla \cdot \boldsymbol{q} \tag{32}$$

to determine the pressure profile. In the following, a magnetostatic equilibrium is assumed in which case Eq.(32) reduces to $-\nabla \cdot \boldsymbol{q} = 0$.

Before proceeding, a comparison of this approach to prior work is warranted. The nonlinear calculation of this paper should reduce to the linear result in the limit of small islands. The key difference between this paper and prior work is the correction resulting from finite parallel heat transport. The pressure closure derived here differs from that used in standard analyses that rely on the resistive MHD model.[18] In Appendix B, the correction for finite parallel heat conduction which is derived in this section is added to the linear layer equations. It will be found that the dispersion relation for this augmented linear analysis exactly matches the small island limit of the nonlinear result of this paper, as expected.

Because of the large anisotropies present in a fusion plasma, the heat equation is separated into parts parallel and perpendicular to the magnetic field

$$-\nabla \cdot \boldsymbol{q} = 0 \tag{33}$$

$$-\nabla \cdot \boldsymbol{q}_{\|} - \nabla \cdot \boldsymbol{q}_{\perp} = 0 \tag{34}$$

Assuming that heat transfer is diffusive, Fourier's law of heat conduction is applied

$$\boldsymbol{q}_{\|} = -\chi_{\|} \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \nabla T \tag{35}$$

$$\boldsymbol{q}_{\perp} = -\chi_{\perp} \nabla_{\perp} T \tag{36}$$

where for simplicity $\chi_{\|}$ and $\chi_{\perp}$ are taken as constant. With these forms for $\boldsymbol{q}$, the pressure evolution equation is written

$$\chi_{\|} \nabla_{\|}^2 p + \chi_{\perp} \nabla_{\perp}^2 p = 0 \tag{37}$$

$$\chi_{\|} \boldsymbol{B} \cdot \nabla \left(\frac{\boldsymbol{B} \cdot \nabla p}{B^2} \right) + \chi_{\perp} \nabla_{\perp}^2 p = 0 \tag{38}$$

where $\nabla_{\perp}^2 p \equiv \nabla^2 p - \nabla_{\|}^2 p$.

For thin islands in the vicinity of the rational surface, crudely the two terms in Eq.(37) scale as

$$\nabla_{\|}^2 p \sim \frac{(\Delta x)^2 p}{L^2} \tag{39}$$

$$\nabla_{\perp}^2 p \sim \frac{p}{(\Delta x)^2} \tag{40}$$

where L is a characteristic length on the order of the equilibrium length scales and $\Delta x = \psi - \psi_s$. When Δx satisfies

$$\Delta x \gg W_C \sim \Phi \left(\frac{\chi_{\perp}}{\chi_{\|}} \right)^{\frac{1}{4}}, \tag{41}$$

the first term in Eq.(37), representing parallel conduction, dominates and the pressure equilibrates along helical magnetic field lines. Here, W_C is a "critical island width" similar to that found by Fitzpatrick.[15] For the current work, it will be found that $W_C \sim$

$\left(\frac{\chi_\perp}{\chi_\parallel}\overline{\mathscr{J} B^2}\,\overline{\mathscr{J} g^{\psi\psi}}\frac{1}{\iota'^2 m_s^2}\right)^{\frac{1}{4}}$, where $g^{\psi\psi} = \nabla\psi\cdot\nabla\psi$. Thus, near the rational surface, cross-field transport effects are important in determining the pressure profile, while far from the rational surface the dominant parallel transport causes isobars to be closely aligned with the magnetic flux surfaces. As previously noted, most analytic island calculations in 3-D equilibria essentially assume $W_C \to 0$, and the solution to the pressure evolution equation becomes $\boldsymbol{B}\cdot\nabla p = 0$. In the following, a solution is developed that assumes W_C to be finite and $W < W_C$.

Now, Eq.(37) is solved order by order for the small island case ($W \ll W_C$), with the ordering parameter $\delta = W/\Phi$. We will find that a secondary ordering will be useful, $\xi = (\delta^2)(\chi_\parallel/\chi_\perp)^{1/2} \sim W^2/W_C^2$. Since $\xi \ll 1$, crossfield transport will be important in determining the pressure profile. The procedure is then to calculate the response order by order in δ. For each order in δ, a secondary perturbative solution is sought which is order by order in ξ. However, it will be seen that the crossfield transport effects will not enter into the solution until higher order in δ. To lowest order, δ^0, Eq.(37) becomes

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{\mathscr{J} B^2}\frac{\partial p^0}{\partial\zeta}\right) = 0 \tag{42}$$

where the superscript on pressure refers to order δ^0. Solving Eq.(42) gives $p^0 = \overline{p^0}$.

To next order, δ^1, Eq.(37) becomes

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{\mathscr{J} B^2}\left[\Psi^*, p^0\right]\right) - \frac{\partial}{\partial\zeta}\left(\frac{1}{\mathscr{J} B^2}\frac{\partial A_\alpha}{\partial\zeta}\frac{\partial p^0}{\partial\psi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{\mathscr{J} B^2}\frac{\partial p^1}{\partial\zeta}\right) = 0 \tag{43}$$

Integrating yields

$$\frac{\partial}{\partial\zeta}\left(-A_\alpha\frac{\partial p^0}{\partial\psi} + p^1\right) = \mathscr{J} B^2 f_1(\psi,\alpha) - \left[\Psi^*, p^0\right] \tag{44}$$

where $f_1(\psi,\alpha)$ is an unknown function of integration. Taking toroidal averages and solving gives

$$\left[\overline{\Psi^*}, p^0\right] = \mathscr{J} B^2\overline{f_1(\psi,\alpha)} \tag{45}$$

Substituting back in to Eq.(44) and solving for p^1 gives

$$p^1 = \widetilde{A}_\alpha \frac{\partial p^0}{\partial \psi} - \int d\zeta \left[\widetilde{\Psi^*}, p^0 \right] + \overline{p^1} \tag{46}$$

At order δ^2, the effect of perpendicular heat conduction competes with equilibration processes along field lines. This equation is given by

$$\left[\overline{\Psi^*}, \frac{1}{\overline{\mathscr{J} B^2}} \left[\overline{\Psi^*}, p^0 \right] \right] = -\frac{\chi_\perp}{\chi_\parallel} \frac{\partial}{\partial \psi} \left(\overline{\mathscr{J} g^{\psi\psi} \frac{\partial p^0}{\partial \psi}} \right) \tag{47}$$

If $\xi \gg 1$ is assumed, we find $p^0 = p(\overline{\Psi^*})$, and the pressure is constant along the helical island magnetic surfaces. However if $\xi \ll 1$, we can make a subsequent expansion in ξ writing the pressure as $p^0(\psi, \alpha) = p_0(\psi_s) + \delta p(\psi, \alpha) = p_0(\psi_s) + \sum_{\nu=0}^{\infty} p_\nu(\psi) \cos(\nu\alpha)$. Now we write $\overline{\Psi^*} = \iota' x^2/2 - \sum_k \overline{A_\zeta}_{km_s,kn_s}(x) \cos(ikm_s\alpha + i\phi_k)$, so that the physics of the full magnetic spectrum are included. Using the small island ordering previously discussed, so that $\overline{\mathscr{J} g^{\psi\psi}}$ is approximately constant across the island region, to lowest order Eq.(47) reduces to:

$$\frac{d^2 p^0}{dx^2} = 0 \tag{48}$$

where $x = \psi - \psi_s$, the distance from the rational surface. Thus, to lowest order, $\delta p = p^0 - p_0 = p_0' x$. To next lowest order, Eq.(47) reduces to the following ODE for the perturbed pressure:

$$\begin{aligned} \frac{d^2}{dx^2} \sum_k p_k \cos(km_s\alpha + \phi_k) - \frac{\chi_\parallel}{\chi_\perp} \frac{1}{\overline{\mathscr{J} B^2}} \frac{1}{\overline{\mathscr{J} g^{\psi\psi}}} \iota'^2 x^2 \sum_k p_k k^2 m_s^2 \cos(km_s\alpha + \phi_k) = \\ \frac{\chi_\parallel}{\chi_\perp} \frac{1}{\overline{\mathscr{J} B^2}} \frac{1}{\overline{\mathscr{J} g^{\psi\psi}}} \iota' p_0' x \sum_k \overline{A_\zeta}_{km_s,kn_s}(x) k^2 m_s^2 \cos(km_s\alpha + \phi_k) \end{aligned} \tag{49}$$

To lowest order in (W^2/W_C^2) this equation is linear in p_k and driven by the resonant component of the magnetic field. Eq.(49) is a form of the parabolic cylinder equation, and can be rewritten, term by term in the index k

$$\frac{d^2 p_k}{dz^2} - \frac{z^2}{4} p_k = \frac{z}{W_{Ck}} \frac{p_0'}{\iota' \sqrt{8}} \overline{A_{\zeta km_s,kn_s}} \left(\frac{z}{\sqrt{2}} W_{Ck} \right) \tag{50}$$

where $W_{Ck} = \left(\frac{\chi_\perp}{\chi_\parallel}\overline{\mathscr{J}B^2}\,\overline{\mathscr{J}g^{\psi\psi}}\frac{1}{\iota'^2k^2m_s^2}\right)^{\frac{1}{4}}$ is the critical island width for each harmonic and $z = \sqrt{2}x/W_{Ck}$. Thus, a Green's function solution can be constructed for this ODE so that the pressure profile can be found

$$p_k(z) = \int_0^\infty \frac{z}{W_{Ck}}\frac{p_0'}{\iota'\sqrt{8}}\overline{A_{\zeta k}}\left(\frac{z}{\sqrt{2}}W_{Ck}\right)G(z,\zeta)d\zeta \tag{51}$$

where $G(z,\zeta)$ is the Green's function solution for the ODE, Eq.(50).

The solution for Eq.(50) can be calculated in two asymptotic limits, one where $x \ll W_{Ck}$ and one where $x \gg W_{Ck}$,

$$p^0 = p_0(r_s) + p_0'(r_s)x + \sum_k p_k(\psi,\alpha)$$
$$\text{where } p_k(\psi,\alpha) = \begin{cases} -\frac{p_0'}{\iota'}x\frac{C_0}{W_{Ck}^2}\overline{A_\zeta}_{km_s,kn_s}(W_{Ck})\cos(km_s\alpha+\phi_k) & x \ll W_{Ck} \\ -\frac{p_0'}{\iota' x}\overline{A_\zeta}_{km_s,kn_s}(0)\cos(km_s\alpha+\phi_k) & x \gg W_{Ck} \end{cases} \tag{52}$$

Asymptotic analysis of the Green's function, assuming various forms for $A(x)$, indicates that $C_0 \approx 0.6$.

Eq.(52) contains key information which describes how anisotropic heat transport influences the island dynamics. This expression for pressure will now be used in the solution of the quasineutrality equation.

D. Quasineutrality and plasma current

In Sect.(III B), the singular nature of Eq.(9) was resolved by allowing for the formation of a magnetic island. It was shown in Eq.(27) that the Pfirsh-Schlüter current is driven by the resonant contribution of the Jacobian, and is explicitly not singular when a magnetic island is present at the rational surface. To include finite anisotropic transport effects, we now use the modifications to the pressure profile determined in the last section.

The quasineutrality condition is given by

$$\boldsymbol{B}\cdot\nabla Q = -\boldsymbol{B}\times\nabla p\cdot\nabla(\frac{1}{B^2}) \tag{53}$$

where Q is the magnitude of the parallel current, $\boldsymbol{J}_{\parallel} = Q\boldsymbol{B}$. In the island region, to lowest order the quasineutrality condition takes the form:

$$\frac{\partial Q^0}{\partial\zeta} = -\frac{\partial p^0}{\partial\psi}\left[(\iota_0 I(\psi)+g(\psi))\frac{\partial}{\partial\alpha}\left(\frac{1}{B^2}\right) - I(\psi)\frac{\partial}{\partial\zeta}\left(\frac{1}{B^2}\right)\right]^0 \tag{54}$$

It should be noted here that the Jacobian, $\mathscr{J} = (\iota I+g)/B^2$ has the following ordering

$$\mathscr{J} = \overline{\mathscr{J}} + \widetilde{\mathscr{J}} = \mathscr{J}_{00}(\psi) + \sum_k \mathscr{J}_{km_skn_s}(\psi)\cos(km_s\alpha+\phi_k) + \widetilde{\mathscr{J}} \tag{55}$$

Again we are interested in a single resonant surface defined by multiples of m_s, n_s, where $\iota_0 = n_s/m_s$ at this surface. In the following, $\mathscr{J}_{00}$ and $\widetilde{\mathscr{J}}$ are $\mathcal{O}(1)$, and the second term is $\mathcal{O}(\delta)$. This ordering for the resonant component of the $|B|$ spectrum is needed to be consistent with the small island approximation that permeates the entire analysis. However, as shown in Sect.(III B), it is the presence of the island-producing magnetic field (and not the ordering used) which resolves the previously singular current. Furthermore, $\mathscr{J}$ is Taylor-expanded about the rational surface. The Pfirsh-Schlüter current will be found below to depend linearly on the resonant part of the Jacobian and inversely on the island producing magnetic field amplitude.

Eq.(54) can now be integrated to yield the lowest order Pfirsch-Schlüter current:

$$Q^0 = -\frac{\partial p^0}{\partial\psi}\mathscr{J}^\dagger + f(\psi,\alpha) \tag{56}$$

where $f(\psi,\alpha)$ is an unknown function of integration and $\mathscr{J}^\dagger$ is given by

$$\mathscr{J}^\dagger = \int d\zeta\frac{\partial\mathscr{J}}{\partial\alpha} - \left(\frac{I}{B^2}\right) \tag{57}$$

$$= \sum_{mn}{}' \frac{\mathscr{J}_{mn}e^{im\alpha+i(m\iota_0-n)\zeta+i\phi_k}(g+(n/m)I)}{(\iota_0-n/m)(\iota I+g)} \tag{58}$$

where the prime indicates the exclusion of resonant components where $\iota_0 = n/m$.

At next lowest order, $\mathcal{O}(\delta)$, the quasineutrality condition becomes:

$$
\begin{aligned}
&\frac{\partial Q^1}{\partial \zeta} + [\Psi^*, Q^0] + \frac{\partial A_\alpha}{\partial \psi}\frac{\partial Q^0}{\partial \zeta} - \frac{\partial A_\alpha}{\partial \zeta}\frac{\partial Q^0}{\partial \psi} = x\frac{\partial p^0}{\partial \psi}\left[-(\iota_0 I' + g')\frac{\partial}{\partial \alpha}\left(\frac{1}{B^2}\right) + I'\frac{\partial}{\partial \zeta}\left(\frac{1}{B^2}\right)\right] \\
&+\frac{\partial p^0}{\partial \psi}\left[-\boldsymbol{\delta B}\cdot\boldsymbol{e}_\zeta\frac{\partial}{\partial \alpha}\left(\frac{1}{B^2}\right) + \boldsymbol{\delta B}\cdot\boldsymbol{e}_\alpha\frac{\partial}{\partial \zeta}\left(\frac{1}{B^2}\right)\right] + \frac{\partial p^1}{\partial \psi}\left[-(\iota_0 I + g)\frac{\partial}{\partial \alpha}\left(\frac{1}{B^2}\right) + I\frac{\partial}{\partial \zeta}\left(\frac{1}{B^2}\right)\right] \\
&\qquad + \frac{\partial p^0}{\partial \alpha}\left[\frac{\boldsymbol{B}_0\cdot\boldsymbol{e}_\zeta}{B_0^4}\left(-\frac{\partial B_0^2}{\partial \psi} + 2\frac{\partial p^0}{\partial \psi} - 2p_0'\right) - \boldsymbol{B}_0\cdot\boldsymbol{e}_\psi\frac{\partial}{\partial \zeta}\left(\frac{1}{B^2}\right)\right] \\
&\qquad + \frac{\partial p^0}{\partial \psi}\left[-(\iota_0 I + g)\frac{\partial}{\partial \alpha}\left(\frac{2\delta p}{B_0^4} + x\frac{\partial}{\partial \psi}\left(\frac{1}{B_0^2}\right)\right) + I\frac{\partial}{\partial \zeta}\left(\frac{2\delta p}{B_0^4} + x\frac{\partial}{\partial \psi}\left(\frac{1}{B_0^2}\right)\right)\right] \\
&\qquad\qquad - \left(\frac{\partial \mathscr{J}}{\partial \alpha}\right)\left(\frac{\partial p^0}{\partial \psi}\right) + \left(\frac{\partial \mathscr{J}}{\partial \zeta}\right)\left(\frac{\partial p^0}{\partial \psi}\right)\frac{I}{\iota I + g}
\end{aligned}
\tag{59}
$$

where $x = \psi - \psi_s$ results from Taylor expanding I and g about the rational surface, and defining $I = I(\psi_s)$, $g = g(\psi_s)$, $I' = I'(\psi_s)$, $g' = g'(\psi_s)$, $\boldsymbol{e}_\zeta = \partial\boldsymbol{x}/\partial\zeta = \mathscr{J}\nabla\psi\times\nabla\alpha$, and $\boldsymbol{e}_\alpha = \partial\boldsymbol{x}/\partial\alpha = \mathscr{J}\nabla\zeta\times\nabla\psi$.

Taking the toroidal average of Eq.(59) and simplifying produces

$$
[\overline{\Psi^*}, \overline{Q^0}] - \overline{[\mathscr{J}^\dagger\frac{\partial\widetilde{A_\zeta}}{\partial\psi}, p^0]} + \frac{\partial p^0}{\partial\alpha}\overline{K} + \frac{\partial p^0}{\partial\psi}\delta p\overline{\frac{\partial}{\partial\alpha}\left(\frac{\mathscr{J}}{B_0^2}\right)} + \left(\frac{\partial\overline{\mathscr{J}}}{\partial\alpha}\right)\left(\frac{\partial p^0}{\partial\psi}\right) = 0 \tag{60}
$$

where $\overline{K} = \overline{\frac{\mathscr{J}}{B_0^2}\left(\frac{\partial B_0^2}{\partial\psi} + 2p_0'\right) + \beta_*\frac{\partial}{\partial\zeta}\left(\frac{1}{B_0^2}\right)}$. As will be shown in the following, the middle three terms of Eq.(60) correspond to interchange physics. The last term corresponds to the resonant part of the magnetic field spectrum and is the same as the RHS of Eq.(18). As shown below, and confirming the results of Sect.(III B), the presence of the magnetic island resolves the previously singular resonant Pfirsch-Schlüter current.

This equation will be solved order by order in ξ. However, information from Ampere's law is first required to determine the contribution of the 2nd term.

E. Ampere's law

To obtain the relationship between the parallel current and the magnetic perturbations, Ampere's law is used.

$$\delta\boldsymbol{J} = \nabla \times \nabla \times \delta\boldsymbol{A} \tag{61}$$

To lowest order, the magnitude of the perturbed parallel current is then:

$$\delta Q = Q - Q_0 \approx \frac{\boldsymbol{B}_0 \cdot \delta\boldsymbol{J}}{B_0^2} \approx -\frac{1}{G}\frac{\partial^2 A_\zeta}{\partial\psi^2} \tag{62}$$

where $G = \mathscr{J} B_0^2/g^{\psi\psi}$. The toroidally averaged parallel Ampere's law is then:

$$\overline{\frac{\partial^2 A_\zeta}{\partial\psi^2}} = -\overline{\delta Q}\,\overline{G} - \overline{\widetilde{\delta Q}\widetilde{G}} \tag{63}$$

Subtracting this from the unaveraged equation gives:

$$\frac{\partial^2 \widetilde{A_\zeta}}{\partial\psi^2} = -\widetilde{\delta Q}\,\overline{G} - \overline{\delta Q}\,\widetilde{G} - \widetilde{\delta Q}\widetilde{G} + \overline{\widetilde{\delta Q}\widetilde{G}} \tag{64}$$

Integrating once with respect to ψ yields

$$\frac{\partial\widetilde{A_\zeta}}{\partial\psi} = \widetilde{\mathscr{J}^\dagger\delta p}\,\overline{G} + \widetilde{\mathscr{J}^\dagger\delta p}\,\widetilde{G} - \overline{\widetilde{\mathscr{J}^\dagger}\widetilde{G}}\,\delta p - \frac{\widetilde{G}}{\overline{\overline{G}}}\delta p\,\overline{\widetilde{\mathscr{J}^\dagger}\widetilde{G}} + \frac{\widetilde{G}}{\overline{\overline{G}}}\frac{\partial\overline{A_\zeta}}{\partial\psi} \tag{65}$$

where the boundary condition in ψ-space is far from the island. Note that equilibrium quantities (G, $\mathscr{J}^\dagger$) change slowly with respect to ψ and are considered constant for this integration.

To use this form in the second term of Eq.(60), we multiply by $\widetilde{\mathscr{J}^\dagger}$ and take the toroidal average

$$\overline{\widetilde{\mathscr{J}^\dagger}\frac{\partial\widetilde{A_\zeta}}{\partial\psi}} = \delta p\left(\overline{\widetilde{\mathscr{J}^\dagger}\widetilde{\mathscr{J}^\dagger}G} - \frac{1}{\overline{\overline{G}}}\left(\overline{\widetilde{\mathscr{J}^\dagger}\widetilde{G}}\right)^2\right) + \frac{1}{\overline{\overline{G}}}\overline{\widetilde{\mathscr{J}^\dagger}\widetilde{G}}\frac{\partial\overline{A_\zeta}}{\partial\psi} \tag{66}$$

Now, this expression can be related to the functions E, F, and H of resistive interchange theory.[18] For the present case,

$$E = -\frac{p'}{t'}\left(\frac{V''}{t'}\overline{G} + \overline{G\overline{\mathscr{J}}^{\dagger}}\right)$$

$$F = -\frac{p'^2}{t'^2}\left(\frac{1}{(tI+g)}\overline{G\overline{\mathscr{J}}^{2}} + \overline{G}\overline{G\overline{\mathscr{J}}^{\dagger}\overline{\mathscr{J}}^{\dagger}} - \left(\overline{G\overline{\mathscr{J}}^{\dagger}}\right)^2\right)$$

$$H = \frac{p'}{t'}\left(\overline{G\overline{\mathscr{J}}^{\dagger}} - \overline{G\overline{\mathscr{J}}^{\dagger}}\right)$$

where $V'' = \frac{\partial}{\partial\psi}\langle\mathscr{J}\rangle$.

Substituting, Eq.(60) becomes

$$[\overline{\Psi^*}, \overline{Q^0}] + [\frac{t'}{p'\overline{G}}\frac{\partial\overline{A_\zeta}}{\partial\psi}H, p^0] - [\frac{t'^2\delta p}{p'^2\overline{G}}(E+F), p^0] - [\frac{\delta p}{p'}V'', p^0] - [\frac{t'\delta p}{p'}\overline{\mathscr{J}}^{\dagger}, p^0]$$
$$+ [\delta p\overline{\mathscr{J}B^{-2}}, p^0] + \frac{\partial p^0}{\partial\alpha}\overline{K} + \frac{\partial p^0}{\partial\psi}\delta p\overline{\frac{\partial}{\partial\alpha}\left(\frac{\mathscr{J}}{B^2}\right)} + \left(\frac{\partial\overline{\mathscr{J}}}{\partial\alpha}\right)\left(\frac{\partial p^0}{\partial\psi}\right) = 0 \quad (67)$$

Noting that the α-variation of E, F, and H is small and that the ψ-variation of equilibrium quantities is small compared with the ψ-variation of other quantities, Eq.(67) can be simplified to

$$[\overline{\Psi^*}, \overline{Q^0}] + \left[\frac{t'}{p'\overline{G}}\frac{\partial\overline{A_\zeta}}{\partial\psi}H - \frac{t'^2\delta p}{p'^2\overline{G}}(E+F), p^0\right] + \left(\frac{\partial\overline{\mathscr{J}}}{\partial\alpha}\right)\left(\frac{\partial p^0}{\partial\psi}\right) = 0 \quad (68)$$

Before discussing the solution for $\overline{Q^0}$, we note that in the large-x limit we produce the familiar asymptotic solution from tearing mode theory. Using $\overline{Q^0}$ from Eq.(62), and taking the large-x limit of Eq.(68) produces the asymptotic expression, to $\mathcal{O}(W^2/x^2)$, far from the island,

$$\frac{\partial^2\overline{A_\zeta}}{\partial x^2} + \frac{(E+F+H)}{x^2}\overline{A_\zeta} = 0 \quad (69)$$

This linear equation has two solutions of the form

$$\overline{A_\zeta} = A_l|x|^{\alpha_l} + A_s|x|^{\alpha_s} \quad (70)$$

where the large and small Mercier indices are defined as

$$\alpha_{l,s} = \frac{1}{2} \mp \sqrt{-D_I} \tag{71}$$

The Mercier stability criterion, $D_I < 0$ ensures that $\alpha_{l,s}$ are real. If this criterion is violated, ideal interchange instability will result.

Now, at this point the finite anisotropic heat transport effects from Sect. III C are included in the expression for pressure. Subsequently, Eq.(68) is solved order by order in the small parameter $(W/W_C)^2$.

F. Quasineutrality solution: order by order

To solve Eq.(68), the behavior of $\partial\overline{A_\zeta}/\partial\psi$ must be understood. To make further progress, an assertion will be made which is verified in Appendix C after the calculation is complete, that

$$\frac{\partial\overline{A_\zeta}}{\partial\psi} = -\alpha_l \frac{\iota_0'}{p_0'}\delta p\left[1 + \mathcal{O}(\delta^{\sqrt{-4D_I}})\right] \tag{72}$$

where α_l is the large Mercier index. Using the fact that $\mathcal{O}(\delta^{\sqrt{-4D_I}}) \ll 1$, Eq.(68) becomes

$$[\overline{\Psi^*}, \overline{Q^0}] + \left[\left(\frac{-H\alpha_l - E - F}{\overline{G}}\right)\frac{\iota'^2}{p'^2}\delta p, p^0\right] + \left(\frac{\partial\overline{\mathscr{J}}}{\partial\alpha}\right)\left(\frac{\partial p^0}{\partial\psi}\right) = 0 \tag{73}$$

Eq.(73) will now be solved order by order in $\epsilon = (W/W_c)^2$. At order ϵ^0, $p^{0(0)} = p_0 + p_0'x$, where the superscript in parentheses refers to the order in ϵ, and the bare superscript refers to the order in δ. At this order, $\delta p^{(0)} = p_0'x$; finite anisotropy pressure effects do not enter into the calculation, and we will reproduce the result of Sect.(III B).

Ordering on the helical flux function, $\overline{\Psi^*} = \iota' x^2/2 - \overline{A_\zeta}$, must now be specified. $\overline{A_\zeta}(\psi)$ is expanded in orders of ϵ, $\overline{A_\zeta}(\psi) = \sum_n \epsilon^n \overline{A_{\zeta n}} = \overline{A_{\zeta 0}} + \epsilon^1 \overline{A_{\zeta 1}} + ...$ The assertion made in Eq.(72) implies that for order ϵ^0, $\partial\overline{A_\zeta}/\partial\psi \approx 0$. Therefore, at order ϵ^0, the constant-ψ approximation is recovered. For simplicity, we now change coordinates from (ψ, α) to $(\overline{\Psi^*}, \alpha)$. At lowest order, Eq.(73) reduces to

$$\frac{\partial \overline{\Psi^{*(0)}}}{\partial \psi} \frac{\partial \overline{(Q^0)^{(0)}}}{\partial \alpha} = -\frac{\partial \overline{\mathscr{J}^{(0)}}}{\partial \alpha} \frac{\partial p^{0(0)}}{\partial \psi} \tag{74}$$

which is equivalent to Eq.(26).

As in Sect.(III B), expanding $\mathscr{J}$ as a Fourier series at the single resonant surface (multiples of m_s, n_s), $\mathscr{J}^{(0)} = \mathscr{J}_{00}(x) + \sum_{m_s,n_s} \mathscr{J}_{m_s n_s} e^{im_s\alpha + i(m_s \iota_0 + n_s)\zeta + i\phi_k}$, and using the expansion in Eq.(28) for $\overline{\Psi^*}$, Eq.(74) can be solved to give

$$\overline{Q^{0(0)}} = -p_0' \frac{\mathscr{J}_{m_s n_s}(0)}{A_{\zeta 1}} x + \Phi(\overline{\Psi^*}) \tag{75}$$

which reproduces the result of Eq.(27). The first term describes the resonant component of the Pfirsch-Schlüter current and the last term is an undetermined function of $\overline{\Psi^*}$ resulting from the integration. Again, it is noted that in the presence of the magnetic island, the resonant Pfirsch-Schlüter current is no longer singular.

At next order, ϵ^1, Eq.(73) becomes

$$\frac{\partial \overline{\Psi^{*(1)}}}{\partial \psi} \frac{\partial \overline{Q^{0(0)}}}{\partial \alpha} + \frac{\partial \overline{\Psi^{*(0)}}}{\partial \psi} \frac{\partial \overline{Q^{0(1)}}}{\partial \alpha} = \left[\left(\frac{H\alpha_l + E + F}{\overline{G}} \right) \frac{\iota'^2}{p_0'^2} \delta p^{(1)}, p^{0(0)} \right] - \frac{\partial \overline{\mathscr{J}^{(0)}}}{\partial \alpha} \frac{\partial p^{0(1)}}{\partial \psi} \tag{76}$$

Anisotropic pressure effects enter at this order since $\delta p^{(1)} = p_1$ from Eq.(52). Eq.(76) can be integrated to yield, to lowest order

$$\overline{Q^{0(1)}} = \begin{cases} \frac{(H\alpha_l + E + F)}{\overline{G}} C_0 \sum_k \frac{1}{W_{Ck}^2} A(W_{Ck}) \cos(k m_s \alpha + \phi_k) & x \ll W_{Ck} \\ (H\alpha_l + E + F) \frac{\iota'}{2\overline{G}} \ln\left(\frac{\iota' x^2}{2}\right) & x \gg W_{Ck} \end{cases} \tag{77}$$

The toroidally averaged parallel current is now known to within an unknown function of integration, $f(\overline{\Psi^*})$. That is, $\overline{Q^0} = \overline{Q^{0(0)}} + \overline{Q^{0(1)}} + f(\overline{\Psi^*})$ where the first two terms are described by Eqs.(75) and (77). This unknown function can be found from Ohm's law.

G. Ohm's law

Up to this point, we have used the ideal MHD equilibrium equations to produce an expression for the current in the island region. To make further progress, we now apply a modified Ohm's law which includes resistive and neoclassical effects. The use of this modified Ohm's law provides a further constraint on the equilibrium, and produces a solution in the vicinity of the rational surface. This solution will be subsequently asymptotically matched to exterior region.

To solve for the unknown function of integration which arose in the last section, $f(\overline{\Psi^*})$, the projection of Ohm's law along the magnetic field is used.

$$-\boldsymbol{B}\cdot\frac{\partial \boldsymbol{A}}{\partial t}-\boldsymbol{B}\cdot\nabla\phi=\eta Q B^2-\boldsymbol{B}\cdot\nabla\cdot\mathbf{\Pi}\frac{1}{ne} \tag{78}$$

where ϕ is the electrostatic potential and η is the resistivity. Additionally, the viscous stress tensor, $\mathbf{\Pi}$, is added in order to account for electron neoclassical effects. Here, the first two terms represent the $\boldsymbol{B}\cdot\boldsymbol{E}$ component, the third term is the contribution from the parallel current, and the last term is the neoclassical contribution. Note that the resistivity, $\eta=\eta(T)$, can be calculated using an expansion about the rational surface, $T=T_0(r_s)+T_0'(r_s)x+$ higher order terms. So, to lowest order, $\eta=\eta_0(r_s)$. Also, the present work is concerned only with the *equilibrium* island width, so the first term in Eq.(78) is neglected.

Taking the toroidal average of Eq.(78) gives

$$-\overline{[\Psi^*,\phi]}=\eta(\iota I+g)\overline{Q}-\overline{\mathscr{J}\boldsymbol{B}\cdot\nabla\cdot\mathbf{\Pi}\frac{1}{ne}} \tag{79}$$

The first term in Eq.(79) is now annihilated by taking an average over the helical flux surface at fixed Ψ^*,

$$\langle\overline{f}\rangle_*=\frac{\oint d\alpha\frac{\overline{f}}{\frac{\partial\Psi^*}{\partial x}}}{\oint d\alpha\frac{1}{\frac{\partial\Psi^*}{\partial x}}} \tag{80}$$

Applying this operation to Eq.(79) gives

$$\eta(\iota I + g)\langle\overline{Q}\rangle_* - \langle\overline{\mathscr{J}\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi}\frac{1}{ne}}\rangle_* = 0 \tag{81}$$

We can write $\langle\overline{\mathscr{J}\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi}\frac{1}{ne}}\rangle_*$ as[27–29]

$$\langle\overline{\mathscr{J}\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi}\frac{1}{ne}}\rangle_* = (\eta - \eta_{NC})(\iota I + g)\langle\overline{Q}\rangle_* + \eta_{NC}(\iota I + g)\langle\overline{Q_{BC}}\rangle_* \tag{82}$$

where η_{NC} is the neoclassical resistivity and $\overline{Q_{BC}}$ is the bootstrap current.[6] Combining Eqs.(81) and (82), $\langle\overline{Q}\rangle_* = \langle\overline{Q_{BC}}\rangle_*$. From an island physics standpoint, the important property of $\overline{Q_{BC}}$ is its dependence on the pressure profile; with no island present $\overline{Q_{BC}} \sim p_0'$. With the island, the pressure profile is perturbed and modifies the bootstrap current. Accounting for the difference in the pressure profile, we can write

$$\langle\overline{Q}\rangle_* = \langle\overline{Q_{BC}}\rangle_* \tag{83}$$

$$= \overline{Q_{BC}}\left[1 + \frac{1}{p_0'}\langle\frac{\partial\delta p}{\partial\psi}\rangle_*\right] \tag{84}$$

$$= \iota' D_{NC}\left[1 + \frac{1}{p_0'}\langle\frac{\partial\delta p}{\partial\psi}\rangle_*\right] \tag{85}$$

Here, the stability of bootstrap current effects are defined by the sign of D_{NC}, which is a measure of the neoclassical bootstrap current effect.[20] Tokamak-like bootstrap currents are stabilizing to island growth when $\iota' > 0$, since, for quasi-symmetric stellarators, $D_{NC} \sim p_0'/\iota'$.

Substituting for $\overline{Q}$ solves for the function of integration

$$f(\overline{\Psi^*}) = \langle\overline{Q_{BC}}\rangle_* - \langle\overline{Q^{0(0)}}\rangle_* - \langle\overline{Q^{0(1)}}\rangle_* \tag{86}$$

Taking the appropriate averages, the parallel current is, for $x \ll W_{Ck}$

$$\overline{Q} = -\frac{p_0'\mathscr{J}_{m_s n_s}(0)(x - \langle x\rangle_*)}{A_{\zeta 1}} + \frac{(H\alpha_l + E + F)}{\overline{G}}\left(\sum_k \frac{C_0}{W_{Ck}^2}A(W_{Ck})\cos(km_s\alpha + \phi_k) - \left\langle\sum_k \frac{C_0}{W_{Ck}^2}A(W_{Ck})\cos(km_s\alpha + \phi_k)\right\rangle_*\right) + \left\langle\overline{Q_{BC}}\right\rangle_* \tag{87}$$

and for $x \gg W_{Ck}$

$$\overline{Q} = -\frac{p_0' \mathscr{I}_{m_s n_s}(0)(x - \langle x \rangle_*)}{A_{\zeta 1}} + \frac{(H\alpha_l + E + F)}{\overline{G}} \frac{t'}{2} \left(\ln\left(\frac{t' x^2}{2}\right) - \left\langle \ln\left(\frac{t' x^2}{2}\right) \right\rangle_* \right) + \langle \overline{Q_{BC}} \rangle_* \quad (88)$$

Introducing the ψ-variation of the perturbed pressure into these expressions results in a convenient form for the parallel current. For $x \ll W_C$

$$\overline{Q} = -\frac{p_0' \mathscr{I}_{m_s n_s}(0)(x - \langle x \rangle_*)}{A_{\zeta 1}} + \frac{-\alpha_s D_R}{\alpha_s - H} \frac{t'}{p_0' \overline{G}} \left(\frac{\partial \delta p}{\partial \psi} - \left\langle \frac{\partial \delta p}{\partial \psi} \right\rangle_* \right) + \langle \overline{Q_{BC}} \rangle_* \quad (89)$$

and for $x \gg W_C$

$$\overline{Q} = -\frac{p_0' \mathscr{I}_{m_s n_s}(0)(x - \langle x \rangle_*)}{A_{\zeta 1}} + \frac{D_R}{\alpha_s - H} \frac{t'}{p_0' \overline{G}} \left(\frac{\partial \delta p}{\partial \psi} - \left\langle \frac{\partial \delta p}{\partial \psi} \right\rangle_* \right) + \langle \overline{Q_{BC}} \rangle_* \quad (90)$$

Note that there is a sign difference when comparing the second terms on the right hand sides of Eqs.(89) and (90). This results from the fact that the perturbed pressure, δp has positive slope for $W \ll W_C$ and has negative slope for $W \gg W_C$.

The results in Eqs.(89) and (90) make it possible to proceed with asymptotic matching; the solution interior to the island region will be matched with the solution exterior to the island region.

IV. ASYMPTOTIC MATCHING AND EQUILIBRIUM ISLAND WIDTH

The equilibrium island width is found by asymptotically matching the inner region solution with the exterior region solution, given by Eq.(70). Formally, these exterior solutions can be expanded in the resonant harmonics as $\overline{A_\zeta} = \sum_k \overline{A}_k(\psi) \cos(k m_s \alpha)$. The key information for matching is contained in the ratios of the resonant harmonics of the exterior solution on either side of the narrow island region. The matching parameter is[18]

$$\Delta' = \frac{A_{k,s+}}{A_{k,l+}} - \frac{A_{k,s-}}{A_{k,l-}} \quad (91)$$

Employing the assumption that the solution is dominated by a single harmonic, $\overline{A_\zeta} \approx \overline{A_\zeta}(\psi)\cos(m_s\alpha)$, results in the use of a single matching parameter, Δ' at the rational surface. This asymptotic matching procedure utilizes Δ' to relate the helically resonant currents in the layer to the external data. In a sense, this procedure resolves the δ-function-like currents described in Eq.(9) by allowing for island formation as noted in the discussion following Eq.(75). The singular Pfirsch-Schlüter current contribution is also resolved by the allowance of an island. Finally, under normal operation, most stellarators are not expected to be tearing unstable. Hence, $\Delta' < 0$ is typically satisfied.

To make further progress with the asymptotic matching, it will prove to be convenient to work with the function T rather than $\overline{A_\zeta}$

$$\frac{\partial T}{\partial x} = \frac{\partial \overline{A_\zeta}}{\partial \psi} + \alpha_l \frac{\iota_s'}{p_0'}\delta p \tag{92}$$

See Appendix C for a discussion of the relation between T and $\overline{A_\zeta}$. Using Eq.(C2) from Appendix C, the asymptotic matching between the interior and exterior regions can now be carried out

$$\Delta^* \overline{A_\zeta} = \Delta' A_l \sqrt{-4D_I}\left|\frac{W}{2}\right|^{-\alpha_l} = \int_{-\infty}^{\infty} dx \oint \frac{d\alpha}{2\pi}\cos(m_s\alpha)\frac{\partial^2 T}{\partial \psi^2} \tag{93}$$

where D_I is the Mercier stability criterion, and α-integration is carried out with a cosine to pick out the correct harmonic. It can be shown that

$$\frac{\partial^2 T}{\partial x^2} = -\overline{\delta Q G} + \frac{D_R}{\alpha_s - H}\frac{\iota'}{p_0'}\frac{\partial \delta p}{\partial \psi} \tag{94}$$

where $D_R = E + F + H^2$ from resistive interchange theory,[18] and $\partial^2 \overline{A_\zeta}/\partial\psi^2$ is found from the toroidally averaged Ampere's law, Eq.(63), where Q is defined by Eqs.(89) and (90). Carrying out the integration in Eq.(93) yields an equation for equilibrium island width,

$$\Delta^* + \frac{1}{2}\left(\frac{1+\alpha_s}{\alpha_s - H}\right) D_R \left(\frac{W}{W_C}\right)\frac{C_0}{W} + 0.81 D_{nc}\left(\frac{W^2}{W_C^2}\right)\frac{C_0}{W} + \frac{C_{PS}}{W^2} = 0 \tag{95}$$

where $\Delta^* = \Delta' \left|W/2\right|^{-2\alpha_l} \sqrt{-4D_I}$ and C_{PS} describes the contribution from Pfirsch-Schlüter currents. Recall from the calculations given in Eq.(56) and Eq.(75) that both the non-resonant and resonant components of the Pfirsch-Schlüter current in the island region are unaffected by finite parallel transport physics. Hence the contribution of the last term to the equilibrium island width does not depend on W_C.

The contribution to equilibrium island width from the Pfirsch-Schlüter current can in principle be determined by solving for the perturbed magnetic potential in Ampere's law,

$$\nabla \times \nabla \times \delta \boldsymbol{A} = \delta \boldsymbol{J} \tag{96}$$

where here $\delta \boldsymbol{A}$ and $\delta \boldsymbol{J}$ arise from the perturbed magnetic field. Within the current fully three-dimensional geometry, currents (and vector potentials) are produced with every helicity. Formally $\delta \boldsymbol{A}$ will take the form

$$\delta \boldsymbol{A} = \sum_{mn} [A_\theta(\psi, m, n)\nabla\theta + A_\zeta(\psi, m, n)\nabla\zeta] e^{i(m\theta - n\zeta)} \tag{97}$$

where the gauge $A_\psi = 0$ is chosen. Since every helicity is represented in $\delta \boldsymbol{A}$, C_{PS} will have both resonant and non-resonant components, $C_{PS} = C_{PS-RES} + C_{PS-NR}$. However, for the current geometry, to obtain C_{PS} through solution of Ampere's law is not tractable. To understand the scaling of the resonant coefficients of Eq.(97), one can make the simplifying assumption of nearly circular flux surfaces and solve Ampere's law by using a Green's function integration, as Cary and Kotschenreuther did.[4] This simplification decouples the helicities, and yields to lowest order

$$C_{PS-RES} \simeq \frac{\beta}{m^2 |\iota'|} \frac{R_0^2}{a^2} \left(\frac{\mathscr{J}_{m_s n_s}}{\mathscr{J}_{00}} \right) \tag{98}$$

To determine the non-resonant coupling coefficients, details of the three-dimensional geometry are required to properly invert the $\nabla \times (\nabla \times)$ operator. As such we write

$$C_{PS} \simeq \frac{\beta}{m^2|\iota'|}\frac{R_0^2}{a^2}\sum_{mn} C_{mn}\frac{\mathscr{I}_{mn}}{\mathscr{I}_{00}} \tag{99}$$

where C_{mn} denotes the order unity (at most) coupling coefficients.

So, incorporating the above into the Pfirsch-Schlüter term, and substituting $C_0 \approx 0.6$ the island equation becomes

$$\Delta^* + 0.3\left(\frac{1+\alpha_s}{\alpha_s - H}\right)\frac{D_R}{W}\left(\frac{W}{W_C}\right) + 0.5\frac{D_{nc}}{W}\left(\frac{W^2}{W_C^2}\right) + \frac{1}{W^2}\frac{\beta}{m^2|\iota'|}\frac{R_0^2}{a^2}\sum_{mn} C_{mn}\frac{\mathscr{I}_{mn}}{\mathscr{I}_{00}} = 0 \tag{100}$$

When $W \ll W_C$, the neoclassical contribution is negligible. In this limit the dominant finite-β drive for the island comes from the Pfirsch-Schlüter contribution. In this case, the island width is approximated by

$$W^2 = \frac{C_{PS}}{-\Delta^* - 0.3\left(\frac{1+\alpha_s}{\alpha_s - H}\right)\frac{D_R}{W_C}} \tag{101}$$

assuming Δ^* is negative. Also, note that for $W \ll W_C$, the resistive interchange term produces a finite contribution, whereas the neoclassical term asymptotes to zero.

Now, the above calculations are applicable for the case where perpendicular heat transfer is not negligible relative to parallel heat transfer ($W \ll W_C$). Conversely, when islands are large, $W \gg W_C$, the parallel heat transfer equilibrates temperature on the island perimeter, flattening the temperature profile within the island. Previous island width calculations are characterized by the $W \gg W_C$ case, where these studies have essentially assumed that $W_C \to 0$. For this large-island case, the corresponding island width equation is

$$\Delta^* + k_0\frac{D_R}{(\alpha_s - H)W} + k_1\frac{D_{nc}}{W} + \frac{1}{W^2}\frac{\beta}{m^2|\iota'|}\frac{R_0^2}{a^2}\sum_{mn} C_{mn}\frac{\mathscr{I}_{mn}}{\mathscr{I}_{00}} = 0 \tag{102}$$

where k_0 and k_1 are constants of order unity. It is instructive to compare Eqs.(100) and (102), the equilibrium island width equations in the small- and large-island limits, respectively.

First, note that the resistive interchange term is attenuated by a factor of W/W_C in the small-island case compared with the large-island case. Also, the neoclassical term is attenuated by a factor of W^2/W_C^2 in the small-island case compared with the large-island case. That is, resistive interchange and neoclassical physics have smaller effects when islands are small. However, the destabilizing Pfirsch-Schlüter currents have the same effect whether islands are large or small.

V. DISCUSSION

In this work, we extend the theory of nonlinear pressure-induced magnetic islands to include cases where finite parallel transport effects are important. This work was completed for general three dimensional non-axisymmetric magnetic geometry, with no limits on aspect ratio or β. An expression for equilibrium magnetic island size is derived for the case where finite parallel transport effects are important; that is, for the cases where there is competition between crossfield transport across the island, and parallel transport which maps the perimeter of the island. The result of the current work, Eq.(100), applies to magnetic islands which are sufficiently small. As discussed above, prior workers have generated island equations for the case where $\kappa_\parallel/\kappa_\perp \sim \infty$; that is, the large-island case where parallel transport dominates and temperature equilibrates on the island perimeter, flattening the temperature profile within the island. The current work can be bridged to this prior work via a Padé approximation to produce an equation for equilibrium island width which spans both small and large island regimes.

$$\Delta^* + \frac{k_0 D_R}{(\alpha_s - H)} \frac{1}{W + \frac{k_0}{0.3(1+\alpha_s)} W_C} + k_1 D_{nc} \frac{W}{W^2 + \frac{k_1}{0.5} W_C^2} + \frac{C_{PS}}{\iota' W^2} = 0 \tag{103}$$

where k_0 and k_1 are positive dimensionless coefficients[20] and C_{PS} represents both the resonant and non-resonant contribution of the Pfirsch-Schlüter current as discussed in the last section.

Eq.(103) provides insight into how anisotropic transport effects influence island dynamics, as small island cases are compared with large island cases. First, note that for small islands, island growth driven by resistive interchange physics is attenuated by a factor of W/W_C compared with the same large island case. Next, observe that for small islands, the neoclassical drive for island growth is attenuated even more, by a factor of W^2/W_C^2, when compared with the large island case. This can be explained by the fact that, in the small-island case, the pressure profile is relatively unaffected by the presence of the magnetic island. As discussed above, island growth caused by resistive interchange effects and neoclassical effects depends on localized island currents caused by self-consistent deformation of the pressure profile in the vicinity of the island. Since the pressure profile is minimally affected by the small magnetic island, resistive interchange physics and neoclassical physics therefore play a small role in determining island width.

Equally important is the fact that the inclusion of finite parallel transport effects does not change island growth caused by Pfirsch-Schlüter currents, whether the island is small or large. This finding is consistent with the fact that Pfirsch-Schlüter currents depend very weakly on pressure equilibration physics. Instead, these currents depend on the magnetic spectrum, which is determined by the global plasma properties.

The results of this work depend upon the closure used to describe the pressure response. Whereas standard approaches employ a strict resistive model, we use a model that explicitly allows for anisotropic transport as the determining physics in describing the pressure response. An important result is that even in the limit of large (but not infinite) parallel heat conduction, perpendicular transport can affect pressure-induced island physics in stellarators. This finding has implications for extended MHD simulation codes which have a variety of pressure and heat transport schemes that can be employed. The choice of closure scheme may affect the simulation results, since proper understanding of island dynamics is critical to predicting quality of confinement.

Furthermore, in practice, the scale size W_C is very small (fractions of a centimeter) for high temperature stellarators, owing to the vast differences in scale between perpendicular and parallel transport. This scale disparity is difficult to realize in full MHD numerical simulations. In particular, previous simulations using HINT did not produce islands caused by local currents. However, for these simulations, W_C was a significant fraction of the radius. Conversely, taking $\boldsymbol{B} \cdot \nabla p = 0$ overestimates the local current effects. These findings indicate that efforts to design reactors which minimize island size via stabilizing neoclassical or interchange physics may not be successful for small islands. Conversely, success in controlling growth of small islands is more likely in reactors which attempt to control island size by limiting the drives from Pfirsch-Schlüter effects.

This work assumes that transport is purely diffusive in nature, as evidenced by Eq.(37), with local transport coefficients specifying the transport. In more general treatments, integral models for parallel transport may be operative.[30,31] These effects are not considered in the present calculation.

Finally, we note that in addition to the finite pressure effects considered here, plasma flow effects can also influence island formation in stellarators.[32,33] In particular, flow effects are suspected of healing large islands present in the vacuum configuration of the Large Helical Device.[34,35]

While this work is concerned with the formation of a single island chain, a practical issue in stellarator operation is the presence or lack thereof of stochastic regions which result from overlapping magnetic islands. While the present calculation cannot be easily extended to this case, the current work does point out the need to properly account for self-consistent anisotropic heat conduction.

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Appendix A: Calculation of the pressure profile at X- and O-points.

For the case where islands are small, the pressure profile in the island is not flattened as in the large island case. The pressure profile remains largely topologically toroidal, with minor deformation in the vicinity of the island. As a result the radial thermal flux across topologically toroidal surfaces remains constant,

$$\nabla \cdot \mathbf{\Gamma} = 0 \tag{A1}$$

Solution of this equation can be realized with a diffusion equation for pressure. If the source for plasma pressure is assumed to be far from the island region, the diffusion equation for pressure can be written

$$\nabla \cdot \mathbf{\Gamma} = 0 \tag{A2}$$

$$-\nabla \cdot D \nabla p = 0 \tag{A3}$$

The solution of this equation far from the island can be matched with the solution at $x = \psi - \psi_s = 0$ to yield the pressure gradient at the X- and O-points.

Far from the island, isobars are aligned with the topologically toroidal flux surfaces, so we write $dp/d\psi|_\infty \equiv p'_\infty$. Therefore, in the region far from the island, the radial thermal flux is

$$\langle \boldsymbol{\Gamma} \cdot \nabla\psi \rangle = -\langle D\nabla p \cdot \nabla\psi \rangle \tag{A4}$$

$$= -\left\langle D p'_\infty |\nabla\psi|^2 \right\rangle \tag{A5}$$

$$= -D_{\text{eff}} p'_\infty \tag{A6}$$

For the region in the vicinity of the X- and O-points, the radial thermal flux can be decomposed based on the physical drive.[36] In this region we are interested in the drive from the large Pfirsch-Schlüter currents. Since finite parallel transport effects can be ignored to lowest order, in the vicinity of the island $p(\psi_s) \approx p_0(\psi_s) + p'_0(\psi_s)x$, and we have

$$\langle \boldsymbol{\Gamma} \cdot \nabla\psi \rangle = \Gamma^R_{PS} + \Gamma^{NR}_{PS} + ... \tag{A7}$$

$$= -\left(D^R_{PS} + D^{NR}_{PS} + D_{\text{other}}\right) p'_0 \tag{A8}$$

Here, the diffusion coefficient for pressure is decomposed into parts associated with resonant and non-resonant Pfirsch-Schlüter currents and all other physical drives. Also, the geometric factor $|\nabla\psi|^2$ has been absorbed into the diffusion coefficients in the last expression, as was done in Eq.(A6).

For simplicity, in the rest of the derivation we will employ expressions for radial particle fluxes. For the present stellarator-like magnetic configuration, in Boozer coordinates, it can be shown that the radial particle flux driven by Pfirsch-Schlüter currents is

$$\Gamma_{PS} = -n\eta_\| p'_0 \left\langle \frac{Q^2 B^2}{p'^2_0} \right\rangle \tag{A9}$$

where n is the number density, $\eta_\|$ is the Spitzer resistivity, and Q is the parallel current, which has both resonant and non-resonant components.

We now define the resonant and non-resonant Pfirsch-Schlüter particle fluxes

$$\Gamma_{PS}^{R} \equiv -n\eta_{\|}p_0' \left(\frac{g+n_s/m_s I}{\iota I+g}\right)^2 \left\langle \frac{\mathscr{J}_{m_s n_s}^2}{(\iota' x)^2} B^2 \right\rangle \tag{A10}$$

$$\Gamma_{PS}^{NR} \equiv -n\eta_{\|}p_0' \left\langle \left(\sum_{mn}{}' \frac{\mathscr{J}_{mn} e^{im\alpha+i(m\iota_0-n)\zeta}(g+n/mI)}{(\iota_0-n/m)(\iota I+g)}\right)^2 B^2 \right\rangle \tag{A11}$$

where $\Gamma_{PS} = \Gamma_{PS}^{R} + \Gamma_{PS}^{NR}$, and in Eq.(A11) the prime on the sum denotes a sum where $n/m \neq \iota_0$. Additionally, the resonant parallel current, $Q = -p_0' \mathscr{J}_{m_s n_s}/\iota' x$ in Eq.(A10) is obtained from the external solution to $[\overline{\Psi^*}, \overline{Q}] = -[\overline{p}, \overline{\mathscr{J}}]$. Eq.(A9) can now be written

$$\Gamma_{PS} = \Gamma_{PS}^{NR}\left(1+\frac{\Gamma_{PS}^{R}}{\Gamma_{PS}^{NR}}\right) \tag{A12}$$

$$= \Gamma_{PS}^{NR}\left(1+\left\langle \frac{\mathscr{J}_{m_s n_s}^2 (g+n_s/m_s I)^2}{(\iota' x)^2 \left(\sum_{mn}' \frac{\mathscr{J} e^{im\alpha+i(m\iota_0-n)\zeta}(g+n/mI)}{\iota_0-n/m}\right)^2}\right\rangle\right) \tag{A13}$$

$$= \Gamma_{PS}^{NR}\left(1+\frac{\epsilon^2}{x^2}\right) \tag{A14}$$

The expressions for the flux far from the island and in the vicinity of the island can now be matched, giving an expression for the pressure gradient near the island. Since $\Gamma_{PS}^{NR} = D_{PS}^{NR} p_0'$,

$$\frac{p_0'}{p_\infty'} = \frac{D_{\text{tot}}}{\left(D_{PS}^{NR}\left(1+\frac{\epsilon^2}{x^2}\right) + D_{\text{other}}\right)} \tag{A15}$$

where D_{tot} is the total diffusion coefficient for all physical drives of radial flux. Eq.(A15) shows that, to maintain constant radial flux near the X- and O-points, the transport coefficients change, causing the pressure gradient to vanish in a very small region of order $\epsilon\sqrt{D_{PS}^{NR}/D_{\text{tot}}} \ll 1$. In practice, this is a very small radial extent and is assumed small for the bulk of this calculation.

Appendix B: Comparison with linear theory

In this appendix, we present an outline of the corresponding linear theory associated with the nonlinear calculation discussed in the main part of this paper. As mentioned in

Sect.(III C), a key difference between the calculation which is the subject of this paper and the linear calculation of Ref. (18) is the closure relation used for pressure effects. Here, a linear calculation is outlined which includes the pressure closure from Eq.(52). In the following, we assume the length scale W_C is large compared to the linear resistive layer scale length.

The layer equations result from the quasineutrality equation and Ohm's law. Keeping the pressure term and the inertia term in the linear quasineutrality equation, $\nabla \cdot \boldsymbol{J} = 0$, one obtains

$$(\boldsymbol{B}_0 \cdot \nabla)\widetilde{Q} + (\widetilde{\boldsymbol{B}} \cdot \nabla)Q_0 + \nabla \cdot \left(\frac{\boldsymbol{B}_0 \times \rho \frac{\partial \widetilde{\boldsymbol{v}}}{\partial t}}{B^2} \right) + \nabla \cdot \left(\frac{\boldsymbol{B}_0 \times \nabla \widetilde{p}}{B^2} \right) = 0 \tag{B1}$$

Following standard linear tearing mode theory, a linear expression can be derived using the $x \ll W_C$ expression for $\widetilde{p_1}$ in Eq.(52). Eq.(B1) can be written

$$\hat{\gamma} \frac{\partial^2 \hat{\phi}}{\partial X^2} = -X \frac{\partial^2 \hat{A}_\zeta}{\partial X^2} - \hat{A}_\zeta \lambda' - X \frac{\alpha_s D_R}{\alpha_s - H} \frac{C_0}{\hat{W}_C^2} \hat{A}_\zeta \tag{B2}$$

where $x = \psi - \psi_s$, $X = \iota' x$, $\hat{\phi} = -\phi i m \iota' \tau_A$, $\hat{\gamma} = \gamma \tau_A$, $\hat{A}_\zeta = A_\zeta \iota'$, $\hat{W}_C = W_C \iota'$, and $\lambda' = \mu_0 (dQ_0/d\psi)\overline{G}/\iota'^2$, and the Alfv́en time is given by

$$\tau_A = \frac{\sqrt{\mu_0 \rho}}{m} \left(\overline{\frac{\mathcal{J} g^{\psi\psi}}{B^2}} \, \overline{\frac{\mathcal{J} B^2}{g^{\psi\psi}}} \right)^{1/2} \tag{B3}$$

The last term in Eq.(B2) describes the interchange contributions and differs from the conventional resistive MHD prediction due to the different pressure closure.

Next, the parallel projection of Ohm's law is treated,

$$-\boldsymbol{B}_0 \cdot \nabla \widetilde{\phi} - \boldsymbol{B}_0 \cdot \frac{\partial \widetilde{A_\zeta}}{\partial t} = \eta \boldsymbol{B} \cdot \widetilde{\boldsymbol{J}} \tag{B4}$$

which can be written

$$\hat{\gamma} \hat{A}_\zeta = X \hat{\phi} + \frac{1}{S} \frac{d^2 \hat{A}_\zeta}{dX^2} \tag{B5}$$

using the same normalizations defined above with the Lundqusit number is given by

$$S=\frac{\mu_0}{\eta t'^2\tau_A}\frac{\overline{\overline{G}}}{\mathscr{J}B^2} \tag{B6}$$

A dispersion relation is now derived for the linear layer equations, Eqs.(B2) and (B5), by matching the inner solution to the outer solution of these equations. The result is

$$\Delta^*+\frac{1+\alpha_s}{2}\frac{D_R}{\alpha_s-H}\frac{C_0}{W_C}=2\pi\gamma^{5/4}S^{3/4}\frac{\Gamma(3/4)}{\Gamma(1/4)}\left[1+\mathcal{O}\left(\frac{L_W}{W_C}\right)\right] \tag{B7}$$

where L_W is the layer width and small terms of order L_W/W_C are neglected in the dispersion relation. Note that the interchange contribution in the nonlinear theory of Eq.(95) is exactly the same as its corresponding contribution in Eq.(B7).

Appendix C: Verification of Eq.(72)

Following the procedure of Ref. (20) , we assume that Δ' is an order unity quantity, and introduce the function T,

$$\frac{\partial T}{\partial x}=\frac{\partial\overline{A_\zeta}}{\partial\psi}+\alpha_l\frac{\iota_s'}{p_0'}\delta p \tag{C1}$$

Far away from the magnetic island (at large x), $\boldsymbol{B}\cdot\nabla p=0$; it is easily seen that $\delta p\approx -p_0'\overline{A_\zeta}/\iota_0'/x$. Using this fact and Eq.(91),

$$\begin{aligned}\lim_{x\to\infty}\frac{\partial T}{\partial x}\Big|_{-x}^{+x}&=\lim_{x\to\infty}\left[A_{s\pm}(\alpha_s-\alpha_l)|x|^{-\alpha_l}\right]_{-x}^{+x}\\&=(\alpha_s-\alpha_l)\left|\frac{W}{2}\right|^{-\alpha_l}\sum_k\Delta_k'A_{k,l}\cos(km_s\alpha)\end{aligned} \tag{C2}$$

where continuity is satisfied by $A_{l+}=A_{l-}$. Using $\partial\overline{A_\zeta}/\partial x\sim\alpha_l\overline{A_\zeta}/x$ we find that

$$\frac{\frac{\partial T}{\partial x}}{\frac{\partial\overline{A_\zeta}}{\partial x}}\sim\Delta'\frac{\alpha_s-\alpha_l}{\alpha_l}|x|^{\sqrt{-4D_I}}\sim\delta^{\sqrt{-4D_I}}\ll 1 \tag{C3}$$

which implies that

$$\frac{\partial\overline{A_\zeta}}{\partial x}=-\alpha_l\frac{\iota_0'}{p_0'}\delta p+\mathcal{O}\left(\delta^{\sqrt{-4D_I}}\right) \tag{C4}$$

which is used in Eq.(72).

REFERENCES

[1]A. Reiman, M. C. Zarnstorff, D. Monticello, A. Weller, and J. Geiger, Nuclear Fusion **47**, 572 (2007).

[2]J. A. Krommes and A. H. Reiman, Physics of Plasmas **16**, 072308 (2009).

[3]A. Reiman and A. H. Boozer, Phys.Fluids **27**, 2446 (1984).

[4]J. R. Cary and M. Kotschenreuther, Phys.Fluids **28**, 1392 (1985).

[5]C. C. Hegna and A. Bhattacharjee, Phys.Fluids B **1**, 392 (1989).

[6]C. C. Hegna and J. D. Callen, Physics of Plasmas **1**, 3135 (1994).

[7]A. Bhattacharjee, T. Hayashi, C. C. Hegna, N. Nakajima, and T. Sato, Phys.Plasmas **2**, 883 (1995).

[8]C. C. Hegna, Plasma Physics and Controlled Fusion **53**, 024003 (2011).

[9]A. Reiman and H. Greenside, Computer Physics Communications **43**, 157 (1986).

[10]T. Hayashi, A. Takei, and T. Sato, Physics of Fluids B: Plasma Physics **4**, 1539 (1992).

[11]T. Hayashi, T. Sato, P. Merkel, J. Nuhrenberg, and U. Schwenn, Physics of Plasmas **1**, 3262 (1994).

[12]K. Narihara, K. Y. Watanabe, I. Yamada, T. Morisaki, K. Tanaka, S. Sakakibara, K. Ida, R. Sakamoto, N. Ohyabu, N. Ashikawa, M. Emoto, H. Funaba, M. Goto, H. Hayashi, H. Idei, K. Ikeda, S. Inagaki, N. Inoue, O. Kaneko, K. Kawahata, T. Kobuchi, A. Komori, S. Kubo, R. Kumazawa, S. Masuzaki, J. Miyazawa, S. Morita, O. Motojima, S. Murakami, S. Muto, T. Mutoh, Y. Nagayama, Y. Nakamura, H. Nakanishi, K. Nishimura, N. Noda, T. Notake, S. Ohdachi, Y. Oka, K. Ohkubo, M. Osakabe, S. Ozaki, B. J. Peterson, A. Sagara, K. Saito, H. Sasao, M. Sasao, K. Sato, M. Sato, T. Seki, T. Shimozuma, C. Shoji, S. Sudo, H. Suzuki, A. Takayama, M. Takechi, Y. Takeiri, N. Tamura, K. Toi,

N. Tokuzawa, Y. Torii, K. Tsumori, T. Watari, H. Yamada, S. Yamaguchi, S. Yamamoto, K. Yamazaki, and Y. Yoshimura, Phys.Rev.Lett. **87**, 135002 (2001).

[13]N. Ohyabu, K. Ida, T. Morisaki, K. Narihara, A. Komori, K. Watanabe, Y. Narushima, Y. Nagayama, M. Shoji, N. Ashikawa, M. Emoto, H. Funaba, M. Goto, H. Idei, K. Ikeda, S. Inagaki, N. Inoue, M. Isobe, K. Khlopenkov, T. Kobuchi, A. Kostrioukov, S. Kubo, R. Kumazawa, Y. Liang, S. Masuzaki, T. Minami, J. Miyazawa, S. Morita, S. Muto, T. Mutoh, S. Murakami, Y. Nakamura, H. Nakanishi, K. Nishimura, N. Noda, T. Notake, K. Ohkubo, S. Ohdachi, Y. Oka, M. Osakabe, T. Ozaki, B. J. Peterson, R. Sakamoto, S. Sakakibara, A. Sagara, K. Saito, M. Sasao, K. Sato, M. Sato, T. Seki, T. Shimozuma, S. Sudo, H. Suzuki, Y. Takeiri, K. Tanaka, N. Tamura, K. Toi, T. Tokuzawa, Y. Torii, K. Tsumori, T. Watanabe, K. Yamazaki, I. Yamada, S. Yamamoto, M. Yokoyama, Y. Yoshimura, T. Watari, Y. Xu, O. Kaneko, K. Kawahata, H. Yamada, and O. Motojima, Phys.Rev.Lett. **88**, 055005 (2002).

[14]N. Ohyabu, Y. Narushima, Y. Nagayama, K. Narihara, T. Morisaki, A. Komori, and E. G. LHD, Plasma Physics and Controlled Fusion **47**, 1431 (2005).

[15]R. Fitzpatrick, Phys.Plasmas **2**, 825 (1995).

[16]N. N. Gorelenkov, R. V. Budny, Z. Chang, M. V. Gorelenkova, and L. E. Zakharov, Phys.Plasmas **3**, 3379 (1996).

[17]T. A. Gianakon, C. C. Hegna, and J. D. Callen, Physics of Plasmas **3**, 4637 (1996).

[18]A. H. Glasser, J. M. Greene, and J. L. Johnson, Phys.Fluids **18**, 875 (1975).

[19]H. Lutjens, J.-F. Luciani, and X. Garbet, Phys.Plasmas **8**, 4267 (2001).

[20]C. C. Hegna, Phys.Plasmas **6**, 3980 (1999).

[21]A. H. Boozer, Phys.Fluids **24**, 1999 (1981).

[22]P. H. Rutherford, Phys.Fluids **16**, 1903 (1973).

[23]R. B. White, D. A. Monticello, M. N. Rosenbluth, and B. V. Waddell, Physics of Fluids **20**, 800 (1977).

[24]B. Carreras, B. V. Waddell, and H. R. Hicks, Nuclear Fusion **19**, 1423 (1979).

[25]A. Thyagaraja, Physics of Fluids **24**, 1716 (1981).

[26]R. L. Dewar and S. R. Hudson, Physica D: Nonlinear Phenomena **112**, 275 (1998).

[27]H. R. Wilson, J. W. Connor, R. J. Hastie, and C. C. Hegna, Physics of Plasmas **3**, 248 (1996).

[28]R. Carrera, R. D. Hazeltine, and M. Kotschenreuther, Physics of Fluids **29**, 899 (1986).

[29]C. C. Hegna and J. D. Callen, Physics of Fluids B **4**, 1855 (1992).

[30]E. D. Held, J. D. Callen, C. C. Hegna, C. R. Sovinec, T. A. Gianakon, and S. E. Kruger (AIP, 2004) Chap. 5, pp. 2419–2426.

[31]E. D. Held, J. D. Callen, C. C. Hegna, and C. R. Sovinec, Physics of Plasmas **8**, 1171 (2001).

[32]C. Hegna, Nuclear Fusion **51**, 113017 (2011).

[33]C. C. Hegna, Physics of Plasmas **19**, 056101 (2012).

[34]Y. Nakamura, the LHD Experimental Group, Y.Narushima, K.Y.Watanabe, S.Sakakibara, K.Narihara, I.Yamada, Y.Suzuki, S.Ohdachi, N.Ohyabu, and H.Yamada., Nuclear Fusion **48**, 075010 (2008).

[35]S. Nishimura, the LHD Experiment Group, TJ-II Experiment Group, Y.Narushima, F.Castejón, S.Sakakibara, K.Y.Watanabe, S.Ohdachi, Y.Suzuki, T.Estrada, F.Medina, D.López-Bruna, M.Yokoyama, M.Yoshinuma, and K.Ida, Nuclear Fusion **51**, 083030 (2011).

[36]K. C. Shaing and J. D. Callen, Physics of Fluids **26**, 3315 (1983).