Healing of magnetic islands in stellarators by plasma flow

C. C. Hegna
Departments of Engineering Physics and Physics,
University of Wisconsin, Madison, WI 53706

Recent experiments from the Large Helical Device (LHD) indicate that the plasma flow can play a primary role in "healing" vacuum magnetic islands in stellarators. The observed elimination of magnetic islands tends to occur at low collisionality and high plasma β. A model explaining this phenomenon is developed reminiscent of 'mode locking/unlocking' physics of tokamak and reversed field pinch experiments. The theory describes transitions between two asymptotic solutions, a state with a large nonrotating island and a state where rotation shielding suppresses island formation. Transitions between these two states are governed by coupled torque balance and island evolution equations. In conventional stellarators, neoclassical damping physics plays an important role in establishing the flow profiles. The balance of neoclassical damping and cross-field viscosity produces a radial boundary layer for the plasma rotation profile outside the separatrix of a locked magnetic island. The width of this boundary layer decreases as the plasma becomes less collisional. This has the consequence of enhancing the viscous torque at low collisionality making healing magnetic islands occur more readily in high temperature conventional stellarators.

I. INTRODUCTION

Magnetic island physics has been a major topic of interest to the stellarator community. One of the primary reasons for this is that general three-dimensional solutions to the magnetostatic equilibrium equations are not guaranteed to be described by topologically toroidal magnetic flux surfaces. Pressure induced magnetic islands and their subsequent overlap is thought to be a primary mechanism for equilibrium β limits. As such, the elimination of magnetic islands is often used in optimizing stellarators. However, there are occassions where magnetic islands have beneficial effects. In stellarators, island structures can be utilized in diveror design [1] and for creating transport barriers [2, 3]. Additionally, applied resonant magnetic perturbations have been used to improve the performance of tokamaks operating in H-mode [4]. The careful use of applied 3-D magnetic fields may be used to improve plasma peformance [5].

Studies on the Large Helical Device (LHD) have been undertaken to understand the plasma response to a magnetic island introduced into the vacuum configuration [6]. The nonlinear growth or suppression of the magnetic island is systematically studied as a function of plasma parameters. Recently, it has been reported that abrupt changes in the plasma rotation properties are correlated with the suppression of the vacuum magnetic island [7]. These results can be understood by considering the coupled torque balance and island evolution equations. In the following, a theoretical formulation for use in interpreting the experimental results is developed. The theory generalizes previous analytic calculations in cylindrical geometry for tokamak and reversed field pinch applications [8]. In this work, particular emphasis is made on the important role neoclassical physics has in describing the flow profiles of conventional stellarators.

In the LHD experiments, external 3-D coils are intentionally applied to produce a magnetic island chain at a low-order rational surface of the vacuum configuration. Heating via neutral beam injection produces finite beta plasmas. By varying the field strength, density and heating power a dependence of the magnetic island evolution on plasma parameters is observed. Both magnetic island growth and healing is seen with the two disparate plasma responses distinguished by a sharp boundary in a parameter space defined by the plasma β and collisionality at the rational surface. Generally, at low β and high collisionality, the plasma tends to make the island grow in width. However at sufficiently high β and/or low collisionality, the plasma abruptly changes to a configuration with no island. Associated with this sudden loss of the magnetic island is a change in the poloidal rotation profile. In the presence of a locked island, the plasma rotation is inhibited at the rational surface. After the transition to the no island state, rotation is observed at the rational surface. The primary goal of the present calculation is to provide an explanation for this behavior and the experimental dependence on plasma parameters.

The phenomenology reported in the LHD island experiments has similiarities to observations of magnetic island physics in tokamaks [9] and reversed field pinch experiments [10, 11] in the presence of externally produced resonant magnetic fields. Intrinsic or applied 3-D external magnetic perturbations whose harmonic structure corresponds to a resonant surface in the plasma affects the growth and rotation properties of magnetic islands associated with the resonant surface. In tokamaks, generally two classes of problems arise. For tearing stable plasmas, the external resonant magnetic field can provide a source for producing a magnetic island. The penetration of this field is inhibited by plasma rotation. However, at sufficiently large 3-D field amplitude, the resonant field produces a forced reconnection at the rational surface. Associated with mode penetration is a change in the rotation profile with rotation at the rational surface abruptly changing to zero. The second class of problems is associated with the interaction of a rotating magnetic island with an external resonant 3-D field. After a critical field amplitude is breached, the island ceases to rotate and becomes 'locked' to the external 3-D field error. In both cases, an island appears in the plasma that is phase locked to the external 3-D field. Mode locking is to be avoided in tokamaks as it often leads to disruptions.

A theoretical paradigm for understanding the interaction of tearing modes (both stable and unstable) with resonant 3-D magnetic fields produced from an external source has been developed [8]. In this work, the amplitude and phase of the magnetic island is determined by coupled electromagnetic and fluid flow information. The conventional matched asymptotics procedure is used to calculate either a linear layer response or a modified Rutherford theory for the island width evolution. The plasma rotation properties enter through a torque balance equation at the rational surface. This theory predicts 'locking' thresholds when the plasma transitions from high rotation, small island states to small rotation, large island states. The theory also predicts an 'unlocking' threshold for the reverse transition. A hallmark of these theories is hysteresis. Generally, the unlocking threshold differs from the locking threshold.

Finite β healing of vacuum magnetic islands in stellarator configurations has been a topic of prior numerical and analytic study [12]. Plasma pressure effects alter nonlinear island widths through resonant Pfirsch-Schlüter currents [13–17]. If the phase of the resonant magnetic field produced by these currents is 180^o out of phase with respect to the phase of the island producing vacuum magnetic field, pressure effects tend to counteract the vacuum magnetic island. At a critical β, the Pfirsch-Schlüter effect can exactly cancel the external 3-D field and the island is healed. At yet higher β, pressure effects produce an island with a flipped phase relative to the vacuum island. Bootstrap currents can also produce finite β contributions that are favorable to nonlinear island suppression [18]. In tokamaks, these effects produce neoclassical tearing modes. However, in stellarators with the correct choice of bootstrap current direction and rotational transform shear, these effects can be highly stabilizing [19]. Efforts to explain the observed island physics of LHD using finite-β Pfirsch-Schlüter or neoclassical theories have not been successful [6].

In the following calculation, we apply the theory developed by Fitzpatrick for cylindrical plasmas [8] to stellarator geometry. Analytic theories for nonlinear island widths in three-dimensional equilibria closely resembles nonlinear tearing mode theory [12, 14, 15]. Following this prescription the desired island width evolution equation is produced. The effect of the external 3-D field perturbation on the island width enters through the 'cosine' contribution to the asymptotic matching. The 'sine' component of the external 3-D field enters through the torque balance equation. Generally, in a finite β equilibrium, the field perturbation produces eddy currents at the rational surfaces and localized $\mathbf{J} \times \mathbf{B}$ torques. In the conventional cylindrical theory, this torque is balanced by a viscous torque associated with changes in the plasma rotation and a phenomenological cross-field viscosity. However rotation properties are influenced by neoclassical physics. In low collisionality conventional stellarators, large neoclassical transport is predicted with transport coefficients that scale inversely with collision frequency [20]. This large neoclassical transport can be lowered through the presence of sufficiently large radial electric fields. However, for a locked island the radial electric field is small since rotation at the rational surface is inhibited. The rotation profile is determined by a combination of neoclassical transport and cross-field viscosity. In the presence of large $1/\nu$ neoclassical transport, a boundary layer in the rotational profile develops. This has the consequence of enhancing the viscous torque at low collisionality. Hence, suppression of magnetic islands through rotation occurs more readily at lower collisionality, a trend that is in agreement with the experimental observations [6].

In the following section, a brief review of analytic island theories of 3-D equilibrium configurations in the presence of a vacuum magnetic island is presented. In Section III, the properties of the plasma flows in the island region and a derivation of the torque balance equation is presented. In this section, special consideration is given to the role of neoclassical physics. In Section IV, a prediction for the spontaneous suppression of a vacuum magnetic island as a function of the rotation properties is derived using information from Sections II and III. In Section V, a prediction for the penetration of a 3-D resonant field into a rotating plasma is given. These two calculations produce different transition thresholds indicative of the hystersis effect. Finally, this work is summarized in Section VI.

II. MAGNETIC FIELDS

In this section, the theory of nonlinear magnetic island formation is reviewed with an emphasis on the role of the externally produced 3-D resonant magnetic fields. An isolated magnetic island chain at the rational surface $\iota = \iota_o \equiv n_o/m_o$ is considered with the island width w assumed small relative to equilibrium lengthscales. Specifically, the ordering

$$\delta = \frac{w}{L_{eq}} \ll 1, \tag{1}$$

is used throughout the calculation. Perturbed quantities associated with island producing fields are assumed to vary rapidly in the "radial" variable (across magnetic surfaces) but have order unity variations in other variables.

The primary problem of interest is of direct relevance to the experiments described in Refs. [6] and [7] where a magnetic island is present in the vacuum configuration. The goal is to calculate the conditions for the plasma flows to 'heal' this island.

A. Island producing fields

The magnetic field is written as the sum of an equilibrium field with robust magnetic surfaces and an island producing magnetic field.

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1. \quad (2)$$

Here $\mathbf{B}_o$ satisfies $\mathbf{B}_0 \cdot \nabla\psi = 0$ where ψ labels topologically toroidal magnetic surfaces. We write this field using Boozer coordinates [21]

$$\mathbf{B}_0 = \nabla\psi \times \nabla\theta + \iota\nabla\phi \times \nabla\psi = g\nabla\phi + I\nabla\theta + h\nabla\psi, \quad (3)$$

where θ and ϕ represent the poloidal and toroidal angles, respectively and the rotational transform ι is a function of the toroidal flux function ψ. The quantities g and I are flux functions and h is a function of all three coordinates in general. The formation of an island at the rational surface $\iota_o = n_o/m_o$ is considered due to a magnetic field of the form $\mathbf{B}_1 \cdot \nabla\psi/\mathbf{B}_0 \cdot \nabla\phi = m_o A_o \sin(n_o\phi - m_o\theta)$. It is convenient to switch coordinates from (ψ, θ, ϕ) to (ρ, α, ζ) given by

$$\alpha = \theta - \iota_o\phi, \quad (4)$$

$$\zeta = \phi, \quad (5)$$

and ρ a radial-like variable derived from $\psi = \psi(\rho) \sim \rho^2$. The Jacobian is given by $\sqrt{g} = 1/\nabla \cdot \rho \times \nabla\alpha \cdot \nabla\zeta = \mathbf{e}_\rho \cdot \mathbf{e}_\alpha \times \mathbf{e}_\zeta = \gamma\psi'/B^2$ where $\gamma = g + \iota I = \mathbf{B}_0 \cdot [\mathbf{e}_\zeta + (\iota - \iota_o)\mathbf{e}_\alpha]$. The perturbed fields due to the magnetic island can be written [22]

$$\mathbf{B}_1 = \nabla A \times \nabla\zeta + \nabla\chi \times \nabla\alpha, \quad (6)$$

where helically resonant components of A produce the symmetry breaking magnetic field responsible for the island. The sum of the two fields can be written using the (ρ, α, ζ) coordinate system in the form

$$\mathbf{B} = \nabla[\psi(\rho) + \chi] \times \nabla\alpha + \psi'\nabla\zeta \times \nabla\Psi^*, \quad (7)$$

where $\psi' \equiv d\psi/d\rho(\rho = \rho_o)$ and the helical flux function Ψ^* is given by the approximate formula in the vicinity of the island

$$\Psi^* = \int d\rho[\iota(\rho) - \iota_o] - \frac{A}{\psi'}, \quad (8)$$

with $\iota_o \equiv \iota(\rho_o)$. In general, the quantity A can have a rich magnetic spectrum. The relevant component for island formation corresponds to the helically resonant part of the spectrum that is independent of ζ at fixed α.

$$\overline{A}(\alpha) = \oint \frac{d\zeta}{2\pi} A(\alpha, \zeta). \quad (9)$$

Using a single harmonic approximation for $\overline{A} = \overline{A}(\rho_o, \alpha) = A_o\cos(m_o\alpha)$, the helical flux function describing the magnetic island flux surfaces is written

$$\overline{\Psi^*} = \frac{1}{2}\iota'_o x^2 - \frac{A_o}{\psi'}\cos(m_o\alpha), \quad (10)$$

where $\iota'_o \equiv d\iota/d\rho(\rho = \rho_o)$ and $x = \rho - \rho_o$. From Eq. (10), the island width can be derived and is given by

$$w = 4\sqrt{|\frac{A_o}{\iota'_o\psi'}|}. \quad (11)$$

In writing Eqs. (7) and (10), a small island width expansion ($\delta = w/L_{eq} \ll 1$) is utilized

In the absence of any plasma response the perturbed fields $\mathbf{B}_1$ are solutions to the vacuum equations ($\nabla \cdot \mathbf{B}_1 = \nabla \times \mathbf{B}_1 = 0$) subject to a boundary condition describing the external source. Following the prescription described above, the vacuum island width can be defined again assuming a dominant single harmonic approximation to the vacuum solution, $\overline{A^V} = A^V_o\cos(m_o\alpha - \phi^V)$ with the vacuum island width given by

$$w^V = 4\sqrt{|\frac{A^V_o}{\iota'_o\psi'}|}. \quad (12)$$

The quantity ϕ^V denotes the phase of the vacuum island. In the definition of the helical flux, Eq. (10), the coordinate α is chosen so that the phase of the island has an O-point at $m_o\alpha = 0$ ($m_o\alpha = \pm\pi$) for $\iota'_o > 0$ (< 0) and an X-point at $m_o\alpha = \pm\pi$ ($m_o\alpha = 0$) for $\iota'_o > 0$ (< 0). While in the vacuum configuration, the difference in the phase between the island and the vacuum island is zero ($\Delta\phi = 0$), in the general finite-β case a non-zero phase shift will be present.

B. Asymptotic matching

To describe the effects of the plasma response, the theoretical machinery of nonlinear tearing modes is utilized. In this calculation, the plasma response is treated differently in two distinct plasma regions and matched asymptotically.

Away from the rational surface, the marginal ideal magnetohydrodynamic (MHD) equations are solved assuming a linear response. These equations are solved subject to boundary conditions and as such, are affected by the presence of the external source for the vacuum magnetic islands. In general, the perturbed vector potential can be written in Fourier series $A = \Sigma A_{mn}(\rho)e^{im\theta - in\zeta}$. Due to the singular nature of the marginal ideal MHD equations, generally a discontinuity in the solutions is present at each harmonic's rational surface. In a cylindrical plasma, each m/n harmonic can be treated independently and the jump discontinuity is quantified by the parameter Δ' defined in the zero β limit as

$$\Delta'_{mn} = \frac{1}{A_{mn}(\rho_o)}\frac{dA_{mn}}{d\rho}|^{\rho_o+}_{\rho_o-}. \quad (13)$$

With finite β, Δ'_{mn} is defined as the jump discontinuity in the ratios of the small to large solutions of the

Frobenius expansion [23]. In the absence of the external source, Δ'_{mn} is a measure of the free energy for tearing mode growth. Low-current stellarators typically satisfy $\Delta'_{mn} < 0$ indicating stability to tearing modes. In toroidal plasmas, rational surfaces with a common toroidal mode number n are coupled together in the exterior region. In this case, the collection of individual Δ'_{mn}'s for each rational surface is replaced by a matrix of matching data which describes how the presence of an island at a particular rational surface affects the island growth properties at coupled rational surfaces. In general three-dimensional systems, the degree of geometric coupling is even more complicated owing to both toroidal and helical shaping generally present in stellarators.

To simplify the analysis in the following, we concentrate on the dynamics of a single isolated island chain at the $\iota_o = n_o/m_o$ surface and assume that the responses of coupled magnetic surfaces is negligible. Noting that the exterior region is governed by a linear equation and that the effect of the exterior source enters through a boundary condition, the quantity $\Delta'_{m_o n_o}$ will have a contribution linearly proportional to A^V. In particular

$$\Delta'_{m_o n_o} = \Delta'_o + \Delta'_{BC}, \tag{14}$$

where Δ'_o represents the asymptotic matching data in the absence of the external source and describes the inherent stability properties of the plasma and Δ'_{BC} is proportional to A^V. Additionally, a finite phase shift between the magnetic field associated with the locked island at the rational surface and the external source is allowed. The 'cosine' component corresponds to a contribution that affects the nonlinear island width and the 'sine' component enters into torque balance. Denoting these two different components Δ'_c and Δ'_s, we have

$$\Delta'_c = \Delta'_o[1 - k_v \frac{A^V}{A_o}\cos(\Delta\phi)], \tag{15}$$

$$\Delta'_s = -k_v \Delta'_o \frac{A^V}{A_o}\sin(\Delta\phi), \tag{16}$$

where k_v is an order unity parameter that is a function of the plasma equilibrium. In vacuum, $k_v = 1$. As viewed from the exterior region, the quantities Δ'_c and Δ'_s denote localized current responses at the rational surface. Crudely, this can be written with a δ-function like response

$$-\mu_o \gamma \frac{\mathbf{J}_1 \cdot \mathbf{B}_0}{\mathbf{B}^2} \approx \delta(\rho - \rho_o) g^{\rho\rho} A_o [\Delta'_c \cos(m_o\alpha) + \Delta'_s \sin(m_o\alpha)]. \tag{17}$$

where $g^{\rho\rho} = \nabla\rho\cdot\nabla\rho$. With this, the conventional matching conditions of tearing mode theory are identified.

$$-\int dx \int_0^{2\pi} \frac{d\alpha}{\pi} \int_0^{2\pi} \frac{d\zeta}{2\pi} \cos(m_o\alpha) \frac{\mu_o\gamma}{g^{\rho\rho}} \frac{\mathbf{J}_1 \cdot \mathbf{B}_0}{B^2} = \Delta'_c A_o, \tag{18}$$

$$-\int dx \int_0^{2\pi} \frac{d\alpha}{\pi} \int_0^{2\pi} \frac{d\zeta}{2\pi} \sin(m_o\alpha) \frac{\mu_o\gamma}{g^{\rho\rho}} \frac{\mathbf{J}_1 \cdot \mathbf{B}_0}{B^2} = \Delta'_s A_o, \tag{19}$$

where the island resolved currents used to describe the plasma response are inserted in the integrals. In the vaccum limit ($\mathbf{J}_1 \cdot \mathbf{B}_0 = 0$), $\Delta'_c = \Delta'_s = 0$ and hence the solution $\Delta\phi = 0$ and $A = A^V$ is recovered.

C. Electromagnetic torques

In the island region, the effects of the magnetic island topology are included in a standard boundary layer calculation assuming large ρ derivatives on perturbed quantities. Currents in the island region are calculated and then subsequently matched to the exterior data by the asymptotic matching procedure defined in Eqs. (18) and (19). Since the primary problem of interest here is the supression of vacuum magnetic islands whose width is assumed large compared to the resistive linear layer, the relevant approach is to use Rutherford theory for nonlinear island growth [24]. As noted in the matching conditions, the currents are segregated into 'cosine' components and 'sine' components. The 'cosine' currents affect the nonlinear island growth evolution. A number of physical effects including Pfirsch-Schlüter [13], resistive interchange [12, 14, 15], bootstrap [18, 19] and ion polarization currents [25] can be included in the problem. For simplicity, these effects are neglected for the moment, but can be included in a more general problem. As such, the relevant island evolution equation is obtained from a resistive Ohm's law ($\mathbf{E}\cdot\mathbf{B} = \eta\mathbf{J}\cdot\mathbf{B}$) and results in [24]

$$k_1 \frac{\mu_o}{\eta}\frac{dw}{dt} = \Delta'_c, \tag{20}$$

where η is the plasma resistivity and $k_1 \approx 0.8$. In the island saturation state ($dw/dt = 0$), the island width is simply given from $\Delta'_c = 0$ which yields an equation for the island width given by

$$w = w^V \sqrt{k_v \cos(\Delta\phi)}. \tag{21}$$

To make further progress, an equation for the phase difference $\Delta\phi$ is required. This will be determined from the torque balance relation detailed in the following section.

The presence of a nonzero Δ'_s implies the presence of a localized eddy current in the island region that produces an electromagnetic torque. To make a connection between the asymptotic matching condition Eq. (19) and torque balance, properties of the currents in the island region are considered.

Before proceeding to the torque calculation, the role of the second perturbed field proportional to $\nabla\chi$ is discussed. In the island region, the primary role of this field is to maintain radial force balance. Taking the $\mathbf{e}_\rho$ projection of the steady state momentum balance equation using the small island approximation yields

$$\frac{\partial}{\partial\rho}(\mathbf{B}_0 \cdot \mathbf{B}_1 + \mu_o \delta p) = \mathcal{O}(\delta), \tag{22}$$

where δp denotes the difference in the equilibrium pressure profile with the island present relative to the pressure profile without the island. Using the representation given above, the perturbed radial momentum balance equation is given by

$$\frac{\partial^2 \chi}{\partial \rho^2} - \frac{I}{\gamma}\frac{\partial^2 A}{\partial \rho^2} + \frac{\mu_o \psi'}{B^2}\frac{\partial \delta p}{\partial \rho} = \mathcal{O}(\delta), \qquad (23)$$

where $\gamma = g + \iota_o I + \mathcal{O}(\delta)$. This formula and the use of the magnetic field representations allows one to write the leading order perturbed currents

$$\mu_o \mathbf{J}_1 = -\frac{\partial^2 A}{\partial \rho^2}\frac{g^{\rho\rho}}{\gamma}\mathbf{B}_0 + \frac{\mathbf{B}_0 \times \nabla\rho}{B^2}\frac{\partial \delta p}{\partial \rho} + \mathcal{O}(\delta). \qquad (24)$$

The flux surface averaged torques are defined in the $\mathbf{e}_\alpha$ and $\mathbf{e}_\zeta$ directions by

$$T_{EM\alpha} = \int_0^{2\pi}\int_0^{2\pi} \mathbf{e}_\alpha \cdot \mathbf{J} \times \mathbf{B}\frac{\sqrt{g}}{g^{\rho\rho}} d\zeta d\alpha, \qquad (25)$$

$$T_{EM\zeta} = \int_0^{2\pi}\int_0^{2\pi} \mathbf{e}_\zeta \cdot \mathbf{J} \times \mathbf{B}\frac{\sqrt{g}}{g^{\rho\rho}} d\zeta d\alpha, \qquad (26)$$

where $\sqrt{g} = 1/\nabla\rho \cdot \nabla\alpha \times \nabla\zeta$. In the exterior region where ideal MHD governs the dynamics, these torques are identically zero. Hence, $T_{EM\alpha}$ and $T_{EM\zeta}$ are localized to the island region. Using Eqs. (7) and (24) for the fields appropriate to the island region, one derives

$$T_{EM\alpha} = -\frac{2\pi\psi'}{\mu_o\gamma}\int_0^{2\pi} d\alpha \frac{\partial \overline{A}}{\partial \alpha}\frac{\partial^2 \overline{A}}{\partial \rho^2}. \qquad (27)$$

From the single harmonic approximation of Eq. (10), this becomes

$$T_{EM\alpha} = \frac{2\pi\psi'}{\mu_o\gamma} m_o A_o \int_0^{2\pi} d\alpha \sin(m_o\alpha)\frac{\partial^2 \overline{A}}{\partial \rho^2}. \qquad (28)$$

Integrating this equation in ρ to obtain T_{EM0} one finds

$$T_{EM0} \equiv \int dx\; T_{EM\alpha} = \frac{2\pi^2\psi'}{\mu_o\gamma} m_o A_o^2 \Delta_s', \qquad (29)$$

which is related to the asymptotic matching condition defined in Eq. (19). Using Eqs. (12), (16) and (21), an expression for the electromagnetic torque can be derived

$$T_{EM0} = \frac{\pi^2\psi'^3}{\mu_o\gamma}\frac{m_o k_v^2(-\Delta_o')}{256}(\iota_o')^2 (w^V)^4 \sin(2\Delta\phi). \qquad (30)$$

The above equation describes the integrated torque over the island region produced by the external resonant magnetic perturbation as a function of the phase shift $\Delta\phi$. In order to find a self-consistent solution for $\Delta\phi$, one needs to balance this torque against that from currents flowing in the island region that contribute to the 'sine' component of the asymptotic matching.

From Eq. (24), one can show

$$\frac{T_{EM\zeta}}{T_{EM\alpha}} \sim \mathcal{O}(\delta). \qquad (31)$$

$T_{EM\zeta}$ to leading order in δ is the parallel component of the electromagnetic torque which is small relative to the term calculated in Eq. (30). In the cylindrical limit, this is expressed by the identity $T_{EM\theta} = -(m_o/n_o)T_{EM\phi}$ [8]. This property can also be derived from the expressions

$$T_{EM\theta} = \int \mathbf{e}_\theta \cdot \mathbf{J} \times \mathbf{B}\frac{\sqrt{g}}{g^{\rho\rho}} d\theta d\phi = -\frac{2\pi\psi'}{\mu_o\gamma}\int d\theta \frac{\partial \overline{A}}{\partial \theta}\frac{\partial^2 \overline{A}}{\partial \rho^2}, \qquad (32)$$

$$T_{EM\phi} = \int \mathbf{e}_\phi \cdot \mathbf{J} \times \mathbf{B}\frac{\sqrt{g}}{g^{\rho\rho}} d\theta d\phi = -\frac{2\pi\psi'}{\mu_o\gamma}\int d\theta \frac{\partial \overline{A}}{\partial \phi}\frac{\partial^2 \overline{A}}{\partial \rho^2}. \qquad (33)$$

Noting $\partial\overline{A}/\partial\theta = -(m_o/n_o)\partial\overline{A}/\partial\phi$, the expected property $T_{EM\theta} = -(m_o/n_o)T_{EM\phi}$ is obtained. The dominant electromagnetic torque is in a direction mutually perpendicular to the normal to the flux surface and the magnetic field direction.

III. PLASMA FLOWS AND VISCOUS TORQUES

To calculate the self-consistent phase shift for the island, the 'sine' component of parallel current in the island region is required. In steady-state, these currents are typically associated with dissipative physics. In the absence of an external 3-D magnetic field source, the torque balance equation establishes the natural frequency of a rotating island [25]. While density, temperature and electrostatic potential profiles are flat within the island separatrix, profile gradients and associated fluid plasma velocities exist outside the plasma. When an island is locked to the external source, the phase velocity at the rational surface is precisely zero and therefore uniquely determines the plasma flow at the rational surface. The requirement that the rotation profile be fixed at $\rho = \rho_o$ is generally in conflict with the solution deduced for the rotation profile from neoclassical and turbulent transport and momentum sources. Recall from the results of the last section, the electromagnetic torque is localized to the island region. Steady state momentum balance is maintained by balancing this torque with a viscous torque in the island region. The viscous torque is characterized by a phenomenological cross-field viscosity coefficient [8]. The torque balance equation produces a prediction for the phase shift $\Delta\phi$ between the island and the external source.

The torque balance relation requires rotation profile information outside the island separatrix. In conventional axisymmetric tokamaks, one typically finds that the localized electromagnetic torque affects the toroidal rotation profile globally. To a large extent this is due to the neoclassical physics operative in a tokamak where toroidal rotation to leading order is undamped while poloidal rotation is strongly damped on the ion-ion collision timescale. (In the presence of three-dimensional fields, toroidal rotation is also damped, but in practice there is generally vast disparity in the poloidal to toroidal damping rates [26]). However, in conventional stellarators the neoclassical damping rates of the poloidal and toroidal rotation are typically comparable. This difference in neoclassical physics produces differences in the amplitude of the viscous torque in conventional stellarators relative to axisymmetric configurations. What is demonstrated in the following is that a boundary layer in the rotational profile in the vicinity of the magnetic island develops due to the balance of the neoclassical physics and the cross-field viscosity. This has the consequence of producing larger viscous torques as the rotation boundary layer becomes smaller. The width of this boundary is determined by the strength of the neoclassical transport. In high temperature stellarators, the flow damping coefficients scale inversely with collision frequency (corresponding to the $1/\nu$ regime of neoclassical cross-field transport). Hence, a smaller collisionality produces a larger viscous force. For the same natural rotational value and external field strength, it is easier to heal magnetic islands in a conventional stellarator than in the corresponding tokamak. Parenthetically, we also note that this physics provides a natural mechanism to produce large sheared $\mathbf{E}\times\mathbf{B}$ velocity profiles in the vicinity of magnetic islands. Hence, this physics can provide an alternative explanation of the observed transport barriers near magnetic islands in stellarators [2, 3] that differs from theories that rely on the presence of the magnetic island from altering the flow damping properties [27].

A. Flows in the absence of islands

As finite β plasmas are produced, the plasma flow generally rises due to both applied torques (e. g., neutral beam injection) or as a result of self-consistently generated radial electric fields needed to obtain ambipolar neoclassical transport. To describe the evolution of the flow profile, a momentum balance equation of the form

$$\rho_M \frac{d\mathbf{v}}{dt} = \mathbf{J}\times\mathbf{B} - \nabla p - \nabla\cdot\vec{\vec{\pi}} + \nabla\cdot(\rho_M \nu_\perp \nabla \mathbf{v}) + \mathbf{S}, \quad (34)$$

is used where ρ_M is the mass density, $\vec{\vec{\pi}}$ the viscous stress tensor and $\mathbf{S}$ external momentum sources. We also include a cross-field viscous force with a phenomenological viscosity coefficient $\nu_\perp$ meant to model turbulent and collisional processes that are not described by $\nabla\cdot\vec{\vec{\pi}}$. In the absence of the island, one can solve for the rotation profiles. In this case $\mathbf{B}\cdot\nabla\rho = 0$ and the bulk flows can be determined from lowest order momentum balance $\mathbf{E} + \mathbf{v}\times\mathbf{B} = \nabla p_i/n_i q_i$:

$$\mathbf{v} = (\frac{d\Phi}{d\rho} + \frac{1}{n_i q_i}\frac{dp_i}{d\rho})\frac{\mathbf{B}\times\nabla\rho}{B^2} + \frac{v_{||}}{B}\mathbf{B}, \quad (35)$$

where Φ is the electrostatic potential and p_i, n_i and q_i are the ion's pressure, density and charge. Alternatively, the flows can be written

$$\mathbf{v} = \Omega^\theta \mathbf{e}_\theta + \Omega^\phi \mathbf{e}_\phi, \quad (36)$$

with $\Omega^\theta = (g/\psi'\gamma)(\Phi' + p_i'/n_i q_i) + \iota v_{||}B/\gamma$, $\Omega^\phi = -(I/\psi'\gamma)(\Phi' + p_i'/n_i q_i) + v_{||}B/\gamma$, $\Phi' = d\Phi/d\rho$ and $p_i' = dp_i/d\rho$. Transforming to the coordinates more convenient to island physics, the flow can also be written

$$\mathbf{v} = \Omega^\alpha \mathbf{e}_\alpha + \Omega^\zeta \mathbf{e}_\zeta. \quad (37)$$

Using with the transformation identities $\mathbf{e}_\theta = \mathbf{e}_\alpha$ and $\mathbf{e}_\phi = \mathbf{e}_\zeta - \iota_o \mathbf{e}_\alpha$, one finds

$$\Omega^\alpha = \Omega^\theta - \iota_o \Omega^\phi = \frac{1}{\psi'}(\Phi' + \frac{p_i'}{n_i q_i}) + \mathcal{O}(\delta), \quad (38)$$

$$\Omega^\zeta = \Omega^\phi = -\frac{I}{\gamma}\Omega^\alpha + \frac{v_{||}B}{\gamma}. \quad (39)$$

While Ω^α is a flux function to leading order Ω^ζ varies within the flux function due to Pfirsch-Schlüter flow effects. Generally, the parallel flow satisfies

$$\frac{v_{||}}{B} = (\Phi' + \frac{p_i'}{n_i q_i})k_{ps} + \frac{< v_{||}B >}{B^2}, \quad (40)$$

where k_{ps} is determined by

$$(\mathbf{B}.\nabla)k_{ps} = -\nabla\cdot(\frac{\mathbf{B}\times\nabla\rho}{B^2}), \quad (41)$$

subject to the constraint $\int\int d\zeta d\alpha \sqrt{g} k_{ps} B^2 = 0$.

Solutions to the steady state momentum balance equations establish the flow profiles. The flux surface average of the $\mathbf{e}_\alpha$ projection of the steady state momentum balance yields

$$0 = -\psi' < \mathbf{J}\cdot\nabla\rho >_o - < \mathbf{e}_\alpha\cdot\nabla\cdot\vec{\vec{\pi}} >_o + < \mathbf{e}_\alpha\cdot\nabla\cdot(\rho_M \nu_\perp \nabla\mathbf{v}) >_o + < \mathbf{e}_\alpha\cdot\mathbf{S} >_o, \quad (42)$$

where

$$< f >_o = \frac{1}{V'}\int_0^{2\pi}\int_0^{2\pi} d\alpha d\zeta \sqrt{g} f, \quad (43)$$

and $V' = \int_0^{2\pi}\int_0^{2\pi} d\alpha d\zeta\sqrt{g}$. Here, the general function f is expressed as a function of ρ, α and ζ so that $< f >_o$ is purely a function of ρ. By requiring $< \mathbf{J}\cdot\nabla\rho >_o = 0$ in steady state, the above expression can be used to determine the radial electric field. The second term in Eq. (42) describes neoclassical transport in three-dimensional configurations and can be written

$$< \mathbf{e}_\alpha\cdot\nabla\cdot\vec{\vec{\pi}} >_o = -\psi'\sum_s q_s < \vec{\Gamma}_s^{neo}\cdot\nabla\rho >_o . \quad (44)$$

In conventional stellarators, generally the requirement of non-ambipolar neoclassical transport $\vec{\Gamma}_i^{neo} = \vec{\Gamma}_e^{neo}$ produces an equation for the radial electric field or alternatively Ω^α as functions of the plasma density and temperature gradients. In stellarator configurations, due to the lack of intrinsic ambipolar transport, mulitple solutions to the radial electric field profile can be obtained [28–33] which complicate this analysis. Nevertheless, formally a solution to the equilibrium flow profile transport equations can be deduced.

To make analytic progress, a further refinement of the neoclassical transport processes is required. In high temperature conventional stellarators, the operative neoclassical transport regime is the $1/\nu$ regime where cross-field transport coefficients vary inversely with the collision frequency. These high transport rates can be reduced by allowing for radial electric fields of sufficient magnitude. However, since a description of the transition region near the island where the radial electric field is small is desired, these high electric field corrections are ignored for simplicity. As such, the neoclassical transport can be written

$$e < \vec{\Gamma}_s^{neo} >_o = -C_{1/\nu}\rho_M \frac{\omega_{ts}^2}{\nu_s} < \frac{g^{\rho\rho}}{B^2} >_o [\Phi' \frac{q_s}{e} + \frac{p_s'}{n_s e} + k_s \frac{T_s'}{e}]$$
$$= -C_{1/\nu}\rho_M \frac{\omega_{ts}^2}{\nu_s} < \frac{\psi' g^{\rho\rho}}{B^2} >_o \frac{q_s}{e}[\Omega^\alpha - \Omega^\alpha_{amb,s}], \quad (45)$$

where ν_s is the collision frequency, $\omega_{ts} = v_{ts}/R_o$ is the transit frequency with v_{ts} the thermal velocity of species s, R_o the major radius, $p_s' = dp/d\rho, T_s' = dT_s/d\rho$ and $k_s \sim 1$. The dimensionless parameter $C_{1/\nu}$ in the zero radial electric field limit scales as $C_{1/\nu} \sim \epsilon_{eff}^{3/2}$ where ϵ_{eff} is a measure of the effective helical ripple [34]. Using the expression as given in the second form for Γ_s^{neo} demonstrates that Eq. (42) can by viewed as a transport equation for Ω^α as a function of the sources and equilibrium density and temperature profiles.

B. Flow profile in the presence of an island

In the presence of a locked magnetic island of sufficient width, the density, temperature and electrostatic potential profiles equilibrate on the helical magnetic surfaces of the island $\overline{\Psi}^*$ given in Eq. (10). This can be seen by noting that the leading order parallel Ohm's law, density and pressure evolution are governed by $\mathbf{B}\cdot\nabla\Phi + \mathbf{B}\cdot\nabla p_e/n_e e = 0, \mathbf{v}_E\cdot\nabla n = \mathbf{v}_E\cdot\nabla p_s = 0$. In this limit, the Ω^α takes the form

$$\Omega^\alpha = \frac{1}{\psi'}[\frac{d\Phi}{d\overline{\Psi}^*} + \frac{1}{n_i q_i}\frac{dp_i}{d\overline{\Psi}*}]\iota' x = < \Omega^\alpha > \frac{x}{< x >}, \quad (46)$$

where

$$< f > = \frac{\int_0^{2\pi}\int_0^{2\pi} d\alpha d\zeta \sqrt{g} x^{-1} f(\overline{\Psi}^*, \alpha, \zeta)}{\int_0^{2\pi}\int_0^{2\pi} d\alpha d\zeta \sqrt{g} x^{-1}}, \quad (47)$$

with $< f >$ being a function of $\overline{\Psi}^*$ only. In this expression all quantities are expressed as functions of $\overline{\Psi}^*, \alpha$ and ζ. To calculate the flow profile, the same procedure used to construct Eq. (42) can be followed with the result

$$0 = -\frac{\psi'}{\iota'} < \mathbf{J}\cdot\nabla\overline{\Psi}^* > - < x\mathbf{e}_\alpha \cdot \nabla \cdot \vec{\vec{\pi}} >$$
$$+ < x\mathbf{e}_\alpha\cdot\nabla\cdot(\rho_M \nu_\perp \nabla\mathbf{v}) > + < x\mathbf{e}_\alpha\cdot\mathbf{S} > . \quad (48)$$

By setting $< \mathbf{J}\cdot\nabla\overline{\Psi}^* >= 0$, a transport equation for $< \Omega^\alpha >$ is obtained. Following the same procedure suggested in Eq. (45) for the neoclassical transport, Eq. (48) can be written in the vicinity of the magnetic island

$$< \Omega^\alpha > = \Omega_0^\alpha$$
$$+\delta_r^2 \frac{\iota'^2 < x >^2}{c_r < x^2 g^{\rho\rho}/B^2 >}\frac{d}{d\Psi^*}[\frac{< x^4 (g^{\rho\rho})^2/B^2 >}{< x >}\frac{d}{d\Psi^*}\frac{< \Omega^\alpha >}{< x >}], \quad (49)$$

where $c_r = < (g^{\rho\rho})^2/B^2 > / < g^{\rho\rho}/B^2 >$ and Ω_0^α given by

$$\Omega_0^\alpha = < \Omega^\alpha_{amb,i} > + \frac{\nu_i < x\mathbf{e}_\alpha\cdot\mathbf{S} >< x >}{\omega_{ti}^2 \rho_M C_{1/\nu_i} < x^2 \psi'^2 g^{\rho\rho}/B^2 >}, \quad (50)$$

is the rotation profile as determined by sources and neoclassical transport. The effects of cross-field viscosity are described by the last term in Eq. (49) with

$$\delta_r^2 = \frac{c_r \nu_\perp \nu_i}{\omega_{ti}^2 C_{1/\nu_i}}, \quad (51)$$

measuring the relative strength of cross-field viscosity to neoclassical transport. Here, we have taken the 'ion root' which is the pertinent case for solutions for small E_r. In writing the above, it is assumed that the basic functional form for the neoclassical transport is unaffected by the presence of the island. However, the coefficient for the neoclassical transport $C_{1/\nu}$ is modified by the island [27].

For high temperature stellarators with sufficiently large ϵ_{eff}, the characteristic distance δ_r satisfies $\delta_r \ll L_{eq}$. In this case, the cross field viscosity plays a weak role, and the solution $< \Omega^\alpha >= \Omega_0^\alpha$ is expected. However, in the presence of a locked island, the rotation at the island sepatrix $< \Omega^\alpha(\overline{\Psi}^* = \overline{\Psi}^*_{sx}) >$ is determined by the no-slip condition assumed for the island. This condition provides an additional internal boundary condition for the flow profile. Noting that this internal boundary condition is generally inconsistent with the solution $< \Omega^\alpha >= \Omega_0^\alpha$, a boundary layer solution of radial width δ_r outside the island separatrix is required. In the asymptotic limit $\delta_r \sim X_* \gg w$, the transport equation Eq. (49) simplifies to

$$< \Omega^\alpha > = \Omega_0^\alpha + \delta_r^2 \frac{d^2 < \Omega^\alpha >}{dX_*^2}, \quad (52)$$

where the radial-like coordinate X_* is defined by $\overline{\Psi}^* - \overline{\Psi}^*_{sx} = \iota' X_*^2/2$. The solution to Eq. (52) subject to the boundary conditions previously discussed is given by

$$< \Omega^\alpha > = \Omega_0^\alpha - \Delta\Omega^\alpha e^{-|X_*|/\delta_r}, \quad (53)$$

where $\Delta\Omega^\alpha = \Omega_0^\alpha - < \Omega^\alpha(\Psi^* = \Psi^*_{sx}) >$ denotes the difference in the value of the flow at the rational surface from Ω_0^α. For a locked island, $\Delta\Omega^\alpha = \Omega_0^\alpha$. For radial extents comparable to the island width, there is no simple analytic expression from the solution of the transport equation. Nonetheless, it is clear that δ_r sets the radial scale for the boundary layer solution for the flow profile. Hence, we can formally write the solution to Eq. (49) as

$$<\Omega^\alpha>=\Omega_0^\alpha-\Delta\Omega^\alpha f_r(\frac{X_*}{\delta_r},\frac{w}{\delta_r}), \tag{54}$$

with the asymptotic solution $f_r = e^{-|X_*|/\delta_r}$ in the small island limit. The flow profile described in Eq. (54) is similar to the equivalent locked island calculation for the toroidal flow velocity in tokamaks when neoclassical toroidal viscosity is present [35].

C. Viscous torques

To see how the physics of the flow profile affects the island, the current response associated with these flows is included in the asymptotic matching. For torque balance, the condition of interest is Eq. (19) which can be written in a slightly different form

$$\Delta_s' A_o=\int_{-A_o}^{\infty} d\overline{\Psi^*}\int\frac{d\alpha}{\pi}\int\frac{d\zeta}{\pi}\sqrt{g}\frac{\gamma\mu_o}{\iota' A_o m_o g^{\rho\rho}}\nabla\cdot\mathbf{J}_\perp, \tag{55}$$

where the quasineutrality equation $\mathbf{B}\cdot\nabla(\mathbf{J}_1\cdot\mathbf{B}_0/B^2) = -\nabla\cdot\mathbf{J}_\perp$ to leading order is used. [In writing the integral over $\overline{\Psi^*}$, $\iota_o' > 0$ is assumed. If $\iota_o' < 0$, the integration range is from A_o to $-\infty$.] From Eq. (29), the torque balance can be written

$$T_{EM0}=\frac{2\psi'}{\iota'}\int_{-A_o}^{\infty} d\overline{\Psi^*}\int d\alpha\int d\zeta\frac{\sqrt{g}}{g^{\rho\rho}}\nabla\cdot\mathbf{J}_\perp. \tag{56}$$

From symmetry arguments, the only components to $\mathbf{J}_\perp$ that contribute to steady state torque balance are dissipative terms corresponding to quantities that enter into the transport equation for Ω^α described above. Using the momentum balance equation to determine $\mathbf{J}_\perp$, the torque balance equation can be written

$$T_{EM0}=\int_{-A_o}^{\infty} d\overline{\Psi^*}\frac{2V'}{\iota'<x>}(<\frac{\mathbf{e}_\alpha\cdot\nabla\cdot\vec{\vec{\pi}}}{g^{\rho\rho}}>-<\frac{\mathbf{e}_\alpha\cdot\mathbf{S}}{g^{\rho\rho}}>). \tag{57}$$

Using the previously discussed form for the neoclassical transport and the solution to the transport equation for Ω^α, Eq. (54), one obtains

$$T_{EM0}=\int_{A_o}^{\infty} d\overline{\Psi^*}[-\frac{2V'C_{1/\nu}\rho_M\omega_{ti}^2<\psi'^2x/B^2>}{\nu_i\iota'<x>^2}\Delta\Omega^\alpha f_r -\frac{2V'}{\iota'<x>}<\frac{\mathbf{e}_\alpha\cdot\mathbf{S}}{g^{\rho\rho}}(1-\frac{xg^{\rho\rho}<x/B^2>}{<x^2g^{\rho\rho}>})>]. \tag{58}$$

If we assume that the source function $\mathbf{S}$ is small in the island region (so that the last term can be neglected) and that the resonant component of the magnetic spectrum is sufficiently small (so that $<x/B^2>/<x>\approx<1/B^2>$), the important $\overline{\Psi^*}$ dependence of the integrand is contained by the function f_r defining the boundary layer solution for the flow profile described in Eq. (54). This solution effectively limits the radial extent of the integration to the characteristic scale δ_r. Utilizing this, the torque balance condition $T_{EM0}+T_{V0}=0$ can be derived with the viscous torque described by

$$T_{V0}=2k_fV'\rho_M\nu_\perp c_r<\frac{\psi'^2}{B^2}>\frac{\Delta\Omega^\alpha}{\delta_r}, \tag{59}$$

where the dimensionless coefficient k_f is defined as a function of the quantity $z = w/\delta_r$ by

$$k_f(z)=\int_0^\infty dy f_r(y,z)\frac{2K(k)}{\pi}\frac{y}{\sqrt{y^2+z^2/4}}, \tag{60}$$

where $K(k)$ is the complete elliptic integral of the first kind, $k=\sqrt{z^2/(4y^2+z^2)}$ and f_r is defined with respect to the variable $y = X_*/\delta_r$. In the limit $z = 0, k_f = 1$.

An easier way to understand the derivation of T_{V0} in the small island limit is to directly compute the viscous torque on the island region. Taking the $\mathbf{e}_\alpha$ projection and flux surface averaging in accordance with the calculation of the electromagnetic torques of Eq. (25) produces the expression

$$\frac{T_{EM\alpha}}{V'}=-<(\vec{\vec{P}})^T:\nabla\frac{\mathbf{e}_\alpha}{g^{\rho\rho}}>_o+<\rho_M\nu_\perp(\nabla\mathbf{v})^T:\nabla\frac{\mathbf{e}_\alpha}{g^{\rho\rho}}>_o +\sum_i\frac{d}{d\rho}[<\frac{g^{\rho i}}{g^{\rho\rho}}\mathbf{e}_i\cdot\vec{\vec{P}}\cdot\mathbf{e}_\alpha>_o-<\rho_M\nu_\perp\frac{g^{\rho i}}{g^{\rho\rho}}\mathbf{e}_i\cdot\nabla\mathbf{v}\cdot\mathbf{e}_\alpha>_o] -<\frac{\mathbf{e}_\alpha}{g^{\rho\rho}}\cdot\mathbf{S}>_o\,. \tag{61}$$

where $\vec{\vec{P}} = p\vec{\vec{I}}+\vec{\vec{\pi}}$. An equivalent expression can be derived for the $\mathbf{e}_\zeta$ component of the momentum balance. Integrating the above equations across the island region produces the static torque balance relations $T_{EM0} + T_{V0} = 0$ and $T_{V1} = \int dxT_{EM\zeta} = \mathcal{O}(\delta)$ where

$$T_{V0}=V'\rho_M\nu_\perp[<g_{\alpha\alpha}>_o\frac{d\Omega^\alpha}{d\rho}|_{\rho_o-}^{\rho_o+}+<g_{\zeta\alpha}\frac{\partial\Omega^\zeta}{\partial\rho}>_o|_{\rho_o-}^{\rho_o+}], \tag{62}$$

$$T_{V1}=V'\nu_\perp[<g_{\alpha\zeta}>_o\frac{d\Omega^\alpha}{d\rho}|_{\rho_o-}^{\rho_o+}+<g_{\zeta\zeta}\frac{\partial\Omega^\zeta}{\partial\rho}>_o|_{\rho_o-}^{\rho_o+}]. \tag{63}$$

Here, the ρ derivatives on $\mathbf{v}$ are assumed to produce the largest contributions owing to the radially localized boundary layer solution on the flow profile described previously. In deriving this equation, $\vec{\vec{P}}$ and $\mathbf{v}$ are assumed to be continuous across the island region. Using $T_{V1}/T_{V0} = \mathcal{O}(\delta)$ and the property $g_{\zeta\alpha} = (I/\gamma)g_{\zeta\zeta}[1 + \mathcal{O}(\delta)]$ allows one to eliminate $\Delta\Omega^\zeta$ from T_{V0} so that

$$T_{V0}=V'\rho_M\nu_\perp<\frac{\psi'^2g^{\rho\rho}}{B^2}>_o\frac{d\Omega^\alpha}{d\rho}|_{\rho_o-}^{\rho_o+}, \tag{64}$$

which agrees with the scaling predicted in the zero w/δ_r limit of Eq. (59) with $d\Omega/d\rho|_{\rho_o-}^{\rho_o+} = 2\Delta\Omega^\alpha/\delta_r$.

These calculations also imply

$$< \frac{\partial}{\partial \rho}(\frac{v_{||}}{B}) >_o |_{\rho_o -}^{\rho_o +} = 0, \tag{65}$$

to leading order. To make contact with the conventional cylindrical theory, the condition $T_{V1}/T_{V0} = \mathcal{O}(\delta)$ can also be written to express a relationship between $\Delta\Omega^\phi$ and $\Delta\Omega^\theta$. This yields

$$< \sqrt{g}\frac{\partial \Delta\Omega^\phi}{\partial \rho} >_o |_{\rho_o -}^{\rho_o +} = -\frac{I}{g} < \sqrt{g}\frac{\partial \Delta\Omega^\theta}{\partial \rho} >_o |_{\rho_o -}^{\rho_o +}. \tag{66}$$

Since $I/g \sim \epsilon^2$ is typically quite small (ϵ is the inverse aspect ratio), the change in the jump in the poloidal flow gradient is generally much larger than the change in the toroidal flow gradient.

As is clear from the calculation, in order to properly account for torque balance at the rational surface, there is a competition between neoclassical physics, cross-field viscosity and electromagnetic effects. This point has been made in a recent publication from Nishimura et al [36] . The treatment of the flow profile given here is predicated on the assumption $\delta_r \ll L_{eq}$. At higher collisionality, this approximation breaks down. In this limit, the radial extent of the flow profile "boundary layer" extends to macroscopic scales.

IV. HEALING A LOCKED ISLAND

From the information in Sections II and III, a prediction for the phase shift between the island and the external resonant field perturbation can be derived. From torque balance $T_{EM0} + T_{V0} = 0$ and Eqs. (30) and (59), one finds

$$D_A \sin(2\Delta\phi) = D_\Omega, \tag{67}$$

where

$$D_A = \omega_A^2 \frac{m_o^2 k_v^2}{k_f c_r} \frac{(-\rho_o \Delta_o')}{2m_o} \frac{(\iota_o')^2 (w^V)^4}{512}, \tag{68}$$

$$D_\Omega = -\nu_\perp \rho_o \frac{d\Omega^\alpha}{d\rho}|_{\rho_o -}^{\rho_o +} = -2\nu_\perp \frac{\rho_o}{\delta_r}\Omega_0^\alpha, \tag{69}$$

and $\omega_A^2 = 4\pi^2 \psi'/V'\gamma\rho_M \mu_o < 1/B^2 > \approx v_A^2/R_o^2$ is the Alfven frequency.

As noted previously, in vacuum $D_\Omega = 0$ and the island is locked to the external source with $\Delta\phi = 0$. At finite β, D_Ω obtains a finite value that produces a finite phase shift $\sin(2\Delta\phi) = D_\Omega/D_w$. With larger $|D_\Omega|$, the amplitude of the phase shift becomes larger until $|\sin(2\Delta\phi)|$ reaches its maximal value at unity corresponding to $|\Delta\phi| = \pi/4$. At this point $|D_\Omega| = D_w$ and $w^2 = k_v (w^v)^2/\sqrt{2}$. Beyond this point, $|D_\Omega|$ exceeds D_w which indicates steady state momentum balance is no longer satisfied. At this point, the momentum balance equation now includes the effect of finite inertia

$$\int dx \int_0^{2\pi} \int_0^{2\pi} \sqrt{g} d\alpha d\zeta \; \rho_M \frac{\partial \Omega^\alpha}{\partial t} \frac{\psi'^2}{B^2} = T_{EM0} + T_{V0}. \tag{70}$$

When $|D_\Omega| > D_w$, the viscous torque overwhelms the electromagnetic torque and the plasma at $\rho = \rho_o$ starts to rotate. The island is no longer locked to the wall and the eddy currents at the rational surface become sufficiently large to inhibit the penetration of the vacuum magnetic fields into the plasma; the large magnetic island disappears. The condition $|D_\Omega| = D_w$ is the criteria for healing a magnetic island with plasma flow.

To make firmer contact with the predictions of the theory and the observations of island suppression on LHD, the healing criteria can be used to predict a scaling of the critical β for island suppression as a function of collisionality. As noted in the previous section, the effect of the collisionality naturally enters through the dependence of the flow boundary layer solution described in Eq. (54). To make further progress, a model for the cross-field viscosity is required. Assuming a gyro-Bohm scaling for $\nu_\perp \sim (\rho_i/L_{eq})(T/eB)$ where ρ_i is the ion gyro-radius and diamagnetic level flows, the healing criteria $|D_\Omega| = D_w$ produces a scaling for the critical β as a function of collisionality given by

$$\beta_{crit} \sim (\nu^*)^{1/4} (\frac{w^V}{L_{eq}})^2 \frac{L_{eq}\omega_{pi}}{C_{1/\nu}^{1/4} c}, \tag{71}$$

where $\nu^* = \epsilon^{3/2}\nu_i/\omega_{ti}$ is the normalized collisionality and $\omega_{pi}/c = \sqrt{ne^2\mu_o/m_i}$ is the ion skin depth. For fixed vacuum island width, the above scaling indicates that the critical β for island healing scales weakly but monotonically with collisionality. This result is qualitatively consistent with the results from LHD [6, 7].

V. MODE PENETRATION INTO A ROTATING PLASMA

While the previous calculations were concerned with the conditions under which a magnetic island can be healed with plasma rotation, the issue of interest in this section is the conditions for when an external resonant perturbation penetrates into a rotating plasma. As will be shown, this criteria will differ from the healing criteria derived in the previous section.

In this section, the plasma is posited to be rotating at the rational surface. The external resonant perturbation attempts to force reconnection at the rational surface. However, this reconnection is inhibited at the rational surface due to a localized eddy current response.

Since rotation has such a dramatic effect on the amplitude of the reconnected flux at the rational surface, it is common to use linear tearing theory to describe the plasma response rather than the nonlinear model used previously. A distinct difference between the linear theory and theory used previously is a relaxation of the noslip constraint on the plasma flow. For a nonlinear island, the island and the plasma rotate together. In the linear theory, the bulk plasma flow is decoupled from the resistive layer solution. As such a linear layer response driven by the external source can move relative to the bulk plasma motion at the rational surface.

The response to an external resonant magnetic field source is taken to be described by linear layer physics (Δ_L) which is used in the asymptotic matching procedure $\Delta'_{m_o n_o} A_o = \Delta_L A_o$. From Eqs. (14)-(16), A_o is determined by

$$A_o = \frac{k_v A^V}{1 + \Delta_L/(-\Delta'_o)}. \qquad (72)$$

The quantity Δ_L is a measure of the eddy current response at the rational surface. The layer response depends upon the sophistication of the layer model equations employed [8, 37], but generally is described by an expression of the form $\Delta_L/(-\Delta'_o) = [(\omega - \omega_i^*)^p(\omega - \omega_e^*)^q \omega^{1-p-q}\tau_{rec}]^r e^{i\chi_L}$ where τ_{rec} denotes a characteristic linear layer response time, $\omega = m_o\Omega^\alpha(\rho_s)$ is the frequency of the tearing mode in a rotating plasma, ω_s^* the diamagnetic frequency of species s and p, q and r are exponents of order unity. Here, complex notation is employed so that $\sin(\chi_L)$ denotes the reactive layer response. For specificity in the following, a simplified visco-resistive limit is used where $\Delta_L/(-\Delta'_o) = i\omega\tau_L$, $\tau_L = S^{2/3}P_r^{1/6}/\omega_A$, $S = \tau_R\omega_A$ is the Lundquist number and $P_r = \nu_\perp\mu_o/\eta$. For many flows of characteristic interest $\omega\tau_L \gg 1$. Plasma flows produce a large layer response that dramatically reduces the magnetic island width from its value in vacuum. From Eq. (72), the value of the resonant component of the magnetic field at the rational surface is given by

$$A_o = \frac{kA^V}{\sqrt{1 + \omega^2\tau_L^2}} \qquad (73)$$

and the differential phase between the external source and the island producing field is given by

$$\sin(\Delta\phi) = -\frac{\omega\tau_L}{\sqrt{1 + \omega^2\tau_L^2}}. \qquad (74)$$

Associated with the localized eddy current and the small rotationally shielded resonant magnetic field is a localized electromagnetic torque. Inserting the above expressions into Eq. (30), one obtains

$$T_{EM0} = -\frac{\pi^2\psi'^3}{\mu_o\gamma}\frac{m_o k_v^2(-\Delta'_o)}{256}(\iota'_o)^2(w^V)^4\frac{2\omega\tau_L}{1 + \omega^2\tau_L^2}. \qquad (75)$$

As in the calculation of the previous section, this electromagnetic torque is balanced by a viscous torque which depends upon the properties of the flow profile. From Eq. (64), note that the viscous torque depends upon the change in the rotation value away from its 'natural' rotation rate $T_{V0} \sim \Delta\Omega^\alpha = \Omega_o^\alpha - <\Omega^\alpha(\rho = \rho_s)>$. Denoting $\omega_o = m_o\Omega_o^\alpha$ as the natural frequency, the viscous torque is given by

$$T_{V0} = V'\rho_M\nu_\perp < \frac{\psi'^2 g^{\rho\rho}}{B^2} >_o \frac{2\rho_o}{m_o\delta_r}(\omega_o - \omega). \qquad (76)$$

The torque balance condition $T_{EM0} + T_{V0} = 0$ can then be written

$$C_A\frac{\omega\tau_L}{1 + \omega^2\tau_L^2} = \omega_o - \omega, \qquad (77)$$

where $C_A = \delta_r m_o D_A/\nu_\perp\rho_o$ with D_A given in Eq. (68).

Torque balance provides a self-consistent prediction for ω. Generally, there are three solution, only two of which are dynamically stable. When $\omega_o\tau_L$ is very large, only one of the solutions is realized. In this limit, the operative solution is $\omega \approx \omega_o$ which corresponds to a state with large rotation at the rational surface and a small island. In the small $\omega_o\tau_L$ limit, the only solution is $\omega \approx 0$ corresponding to small rotation at the rational surface and a nearly fully penetrated magnetic island. In the general case, both solutions may be present.

For a plasma rotating with sufficient amplitude, the small island state is realized. However, if the natural rotation rate of the plasma drops to a critical value, the high rotation/small island state disappears. In this case, the plasma abruptly changes from the high rotation/small island state to a low rotation/large island state [8, 9]. An approximate criteria for this transition can be obtained in the $\omega_o\tau \ll 1$ limit of Eq. (77). In this limit, only the large rotation root is described and given by

$$\omega = \omega_o[\frac{1}{2} + \sqrt{1 - \frac{4C_A}{\omega_o^2\tau_L}}], \qquad (78)$$

As ω_o decreases at fixed C_A, the quantity ω/ω_o drops from unity at high ω_o until it reaches the value one half. As ω_o reaches the critical condition $\omega_o = \omega_{crit}$ given by

$$\omega_{crit}^2 = \frac{4C_A}{\tau_L}, \qquad (79)$$

the plasma abruptly jumps to a state where the magnetic island produced from the external magnetic perturbation appears and rotation at the rational surface ceases. This critical transition is referred to as mode penetration.

The threshold condition for mode penetration is different than the threshold condition for healing a vacuum island. This is a distinctive feature of mode locking theory [8]. Another way to write the island penetration threshold is to use the parameters D_Ω and D_A defined in the Eqs. (68) and (69). At $\omega_o = \omega_{crit}$, $D_\Omega = \nu_\perp\rho_o\Omega_0^\alpha/\delta_r$ and the penetration threshold is given by

$$|D_\Omega| \approx \frac{2D_A}{\omega_o\tau_L}, \qquad (80)$$

While the healing condition occurs at $|D_\Omega| = D_A$, the condition for field penetration satisfies $|D_\Omega| \ll D_A$ if

$\omega_o \tau_L \gg 1$. For a given external resonant field strength, the value of the rotation at the rational surface is higher after the threshold for healing is reached than the strength of the rotation at the rational surface just before field error penetration occurs. The equivalent critical β for mode penetration β_{crit}^{pen} relative to the critical β for island healing β_{crit} from Eq. (71) is given by

$$\beta_{crit}^{pen} \approx \beta_{crit}\sqrt{\frac{2}{\omega_o \tau_L}}. \quad (81)$$

VI. SUMMARY AND DISCUSSION

In this work, a theory describing the interaction of a tearing stable conventional stellarator with an externally produced resonant magnetic field perturbation is given. The calculation highlights the important role plasma rotation can have in healing vacuum magnetic islands. The model put forth parallels earlier theoretical efforts to describe the interaction of tearing modes with resonant magnetic perturbations in cylindrical geometry [8]. In this work, transitions between different asymptotic states of the plasma are described by coupled electromagnetic and fluid flow physics.

An important distinction between conventional stellarators and tokamaks lies in the treatment of neoclassical physics. In high temperature conventional stellarators, the viscous force associated with non-ambipolar neoclassical transport plays an important role in establishing the flow profile in the vicinity of a locked mode. A model for the rotation profile is provided that accounts for a balance between cross-field viscosity and neoclassical flow damping. The presence of $1/\nu$ neoclassical transport in high temperature conventional stellarators produces a radial boundary layer in the flow profile outside the separatrix of a locked island. This has the consequence of effectively raising the viscous torque as the plasma becomes less collisional. Therefore, island suppression by plasma flow becomes easier at lower collisionality.

The healing criteria is given by $|D_\Omega| = D_A$ where D_Ω and D_A are measures of the viscous and electromagnetic torques, respectively as described in Eqs. (68) and (69). This theory produces a critical β for island healing as a function of collisionality and external 3-D field resonant field amplitude described in Eq. (71). The scaling indicates a weak but monotonic dependence of the critical β for healing with plasma collisionality, a prediction which is in qualitative agreement with the experimental results from LHD [6, 7].

Additionally, a theory for the penetration of a resonant 3-D magnetic field into a rotating conventional stellarator is also given. The threshold condition for mode penetration, Eqs. (80), is different than the healing criteria indicating that once a vacuum island is healed the plasma is typically required to drop well below the critical β for healing before the island reappears.

The neoclassical physics of conventional stellarators have been emphasized. For these devices, the flow damping rates in the toroidal and poloidal directions can formally have comparable values. The flow properties of quasisymmetric configurations are different than those considered here [38, 39]. These devices have weak neoclassical flow damping in the symmetry direction. The equivalent calculation for the healing of magnetic islands by plasma flows requires a procedure more akin to the theories used for axisymmetric (or nearly axisymmetric) tokamak plasmas [35]. One of the purported advantages of quasisymmetric stellarators is the presence of undamped flows in the symmetric directions which are known to have a variety of beneficial effects on stability and transport in tokamaks. However, for the conventional stellarator case discussed here, it is the presence of the $1/\nu$ transport and the associated neoclassical viscous force that enhances the viscous torque that heals the magnetic island.

For simplicity the presence of the nonlinear dependence of the neoclassical viscosities on radial electric field (or Ω^α) is ignored. A discussion of this effect is delayed for future work.

An important implication of this work is that plasma rotation physics can play a crucial role on magnetic island physics and surface fragility in stellarator configurations. The requirement of ambipolar neoclassical transport or external momentum sources can produce self-consistent plasma flows of sufficient magnitude to eliminate vacuum magnetic islands. The physics of flow suppression is not incorporated in 3-D MHD equilibrium codes [16, 17, 40, 41]. These tools are unduly pessimistic in predicting the topological breakup of magnetic surfaces. Initial value extended MHD codes are capable of describing magnetic island healing due to plasma flows. However, in order to obtain correct quantitative solution, extended MHD modeling would need to include 3-D field induced non-ambipolar neoclassical particle fluxes and the viscous forces they induce.

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