Collisional effects in low collisionality plasmas

J.D. Callen[1, a)]

Department of Engineering Physics, University of Wisconsin,
1500 Engineering Drive, Madison, WI 53706-1609 USA

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The effects of collisions are often neglected in theoretical analyses of low collisionality plasmas where the collision rate ν is smaller than the frequency of waves or other physical processes being considered. However, small angle Coulomb collisions scatter the velocity vector $\mathbf{v}$ of charged particles and produce slightly probabilistic rather than fully deterministic charged particle trajectories in a plasma. These diffusive effects produce an effective collision rate $\nu_{\rm eff} \sim \nu/(|\delta\mathbf{v}|/v)^2 \gg \nu$ for relaxation of plasma responses localized to a small region $\delta\mathbf{v}$ in velocity space. In particular, they create narrow dissipative boundary layers in the vicinity of resonant collisionless responses of the plasma to waves which resolve these singular responses and create temporal irreversibility. A new Green-function-based procedure is being developed for exploring these low collisionality effects. This new procedure is first used to explore Coulomb collisional scattering effects on the temporal evolution of the linear Landau damping of Langmuir waves. On collision and longer time scales the relevant plasma kinetic equation becomes an extended Chapman-Enskog type equation. Green function solutions of this kinetic equation can be used to determine self-consistent closures for fluid moment equations. A multiple time scale and systematic small gyroradius and perturbation level analysis has been used to develop descriptions of toroidal magnetically confined plasmas in tokamaks on collision and transport time scales. Some examples of low collisionality closures and their effects on the behavior of tokamak plasmas are noted.

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[a)] callen@engr.wisc.edu; http://homepages.cae.wisc.edu/∼callen

I. INTRODUCTION

Plasmas are often thought of as collisionless media because Coulomb collision rates are much smaller than typical wave frequencies and physical processes in them. However, velocity-space scattering caused by small-angle Coulomb collisions causes temporal irreversibility and dissipative singular layers at resonances in plasma responses. Also, many plasma applications involve investigations of phenomena on collision and longer time scales — for example, electrical resistivity, viscosity-induced damping of flows, and radial transport in magnetized tokamak plasmas such as those in ITER.[1] Finally, the rates at which plasma entropy is increased directly by temporally irreversible collisional effects and in combination with collective plasma phenomena (e.g., microturbulence) are important for determining plasma behavior on time scales of order and longer than the collision time scale.

This paper briefly surveys some recent developments in studies of the effects of collisions in "low collisionality" plasmas, which are defined here as ones in which the Coulomb collision rates are small compared to the characteristic frequencies of waves or other processes. The following sections describe: Coulomb collisional processes in plasmas (II), the comprehensive plasma kinetic equation (PKE, III), a new Green function approach for solving the linearized PKE (IV), exploration of Coulomb collision effects on Landau damping with this procedure (V), fluid moment equations obtained from velocity-space moments of the PKE (VI), procedures for obtaining closure moments (VII), low collisional closures in axisymmetric tokamaks (VIII), and tokamak plasma transport equations (IX). Some recent examples of closures for 3D fields in tokamaks and their effects are discussed in Appendix A. Section X discusses open issues for low collisionality plasma descriptions and XI provides a summary.

II. COLLISIONAL PROCESSES IN PLASMAS

Coulomb collisions and collisional processes are qualitatively very different in a plasma compared to those in a neutral gas. Electrically neutral molecules move on straight-line trajectories between discrete collisional interactions when they come within a molecular force field ($\sim 10^{-10}$ m) of another molecule. Thus, neutral collision processes are often described in terms of average collision frequencies and a "mean free path" between collisions.

The electrostatic Coulomb electric field around a charged particle decreases only as the inverse of the distance squared, i.e., $\mathbf{E}_b = -\boldsymbol{\nabla}\phi_b = (q_b/\{4\pi\epsilon_0\})\,\mathbf{x}/|\mathbf{x}|^3 \propto 1/|\mathbf{x}|^2$. In a plasma a "test" charged electron with charge $q_t = -\,e$ interacts via the electric field forces $q_t\mathbf{E}_b$ due to all the "background" (subscript b) statistically independent charged particles within a collective Debye shielding distance $\lambda_D \equiv \sqrt{\epsilon_0 T_e/n_e e^2}$ in which T_e and n_e are the electron temperature and density. Thus, a charged particle simultaneously experiences Coulomb collisions with the $(4\pi/3)\,n_e\lambda_D^3$ particles within a Debye sphere of it. The parameter $n_e\lambda_D^3$ is very large for many "weakly coupled" plasmas: $\sim 10^4$ in experimental tests of Landau damping, $\gtrsim 10^7$ for fusion-related laboratory magnetic confinement experiments and often $\gg 10^{10}$ for space and astrophysical plasmas. Coulomb collisional effects are dominated not by discrete collision events but by the cumulative effects of the $\sim n_e\lambda_D^3 \gg 1$ simultaneous collisions within a Debye sphere.

As a test charged particle moves past another charged particle it suffers a momentum impulse $m_t\Delta\mathbf{v} \equiv \int_{-\infty}^{\infty} dt\, q_t\mathbf{E}_b\,$. Fokker-Planck coefficients for the average change in the velocity $\langle\Delta\mathbf{v}\rangle$ and its square $\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle$ are obtained[2] by integrating this momentum impulse over all the Coulomb collisions a test particle with velocity $\mathbf{v}$ experiences with all the background charged particles within a Debye sphere. These Coulomb collision processes produce a dynamical frictional force $d\mathbf{v}/dt = \langle\Delta\mathbf{v}\rangle/\Delta t \sim -\,\nu\,\mathbf{v}$ and diffusive scattering $d\,(\mathbf{v}\mathbf{v})/dt = \langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle/\Delta t \sim \nu\,\mathbf{v}\mathbf{v}$ in velocity space at a given spatial position $\mathbf{x}$, to within the usually negligibly small Debye length. Here ν is a characteristic Coulomb collision rate — for collisional drag on the particle's velocity $\mathbf{v}$ and for diffusion of its velocity vector. Thus, in plasmas ν is the rate for diffusive Coulomb-collision-induced scattering of a charged particle's $\mathbf{v}$ through about 90° and consequent loss of its initially directed momentum. Hence, in plasmas it is not appropriate to speak of a mean free path as in neutral gases; rather, one speaks of a collision length $\lambda \sim v/\nu$, for loss of momentum. In low collisionality plasmas these effects cause charged particle trajectories to be straight lines to lowest order but to be continually diffusing in velocity space due to Coulomb-collision-induced scattering.

III. PLASMA KINETIC THEORY

In the Klimontovich approach[3] the microscopic species distribution function is represented by the sum over all its charged particles, which are taken to be delta functions in the 6D $(\mathbf{x},\mathbf{v})$ phase space, along their particle trajectories. In the non-relativistic limit the particle trajectories are governed by Newton's equations of motion. In addition, charged particles suffer cumulative small-angle Coulomb collisions with all other charged particles within a Debye shielding distance λ_D. These collisions produce a Fokker-Planck (F-P) collision operator[4] $\mathcal{C}\{f\} = -(\partial/\partial\mathbf{v})\boldsymbol{\cdot}[(\langle\Delta\mathbf{v}\rangle/\Delta t)f_s] + (1/2)(\partial^2/\partial\mathbf{v}\partial\mathbf{v})\boldsymbol{:}[(\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle/\Delta t)f_s]$, which is the small momentum transfer limit of the Boltzmann collision operator. The F-P collision operator is a diffusion operator in velocity space. The plasma kinetic equation[5] (PKE) for a distribution f_s of a species s of charged particles with charge q_s and mass m_s results from equating the total time derivative of f_s to the microscopic ($|\mathbf{x}| < \lambda_D$) Coulomb collision effects on f_s:

$$\frac{df_s(\mathbf{x},\mathbf{v},t)}{dt} \equiv \frac{\partial f_s}{\partial t} + \mathbf{v}\boldsymbol{\cdot}\frac{\partial f_s}{\partial \mathbf{x}} + \frac{\mathbf{F}_s}{m_s}\boldsymbol{\cdot}\frac{\partial f_s}{\partial \mathbf{v}} = \mathcal{C}\{f_s\}, \quad \text{plasma kinetic equation (PKE).} \qquad (1)$$

Here, $\mathbf{F}_s = q_s(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ is the Lorentz force induced by macroscopic (i.e., $|\mathbf{x}| > \lambda_D$) electromagnetic fields on the s species of charged particles in the plasma. If collective plasma instabilities cause the plasma dielectric to vanish for waves on the Debye scale length and thereby yield long range electric fields around "dressed" test-particles that extend beyond the Debye shielding distance, $\mathcal{C}\{f_s\}$ can be amplified by "instability enhanced collisional effects" and become larger than[6–8] the usual Fokker-Planck collision operator.[4]

When collisions are neglected, the PKE is called the Vlasov[9] equation. The first order partial differential operator in 6D phase space on the left hand side of (1) is known as the "collisionless" Vlasov operator. Its characteristic curves are the deterministic particle trajectories obtained by solving $d\mathbf{x}/dt = \mathbf{v}$ and $d\mathbf{v}/dt = \mathbf{F}_s$ which yield $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{v}_0, t)$ and $\mathbf{v} = \mathbf{v}(\mathbf{x}_0, \mathbf{v}_0, t)$ at times $t > t_0$ for initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$ at time t_0. Such trajectories yield area-preserving mappings as long as there are no nonlinear effects that cause changes in the topology of the 6D phase space (e.g., particle trapping in finite amplitude monochromatic waves) and no dissipative collective plasma instabilities. Thus, solutions of the Vlasov equation usually just propagate the initial distribution forward in time along its particle trajectories (i.e., $f_s \to f_s[\mathbf{x}(\mathbf{x}_0, \mathbf{v}_0, t), \mathbf{v}(\mathbf{x}_0, \mathbf{v}_0, t), t)]$) without any smoothing of f_s; hence, such solutions of the Vlasov equation do not increase the plasma entropy in stable plasmas.[10]

In low collisionality plasmas inclusion of the F-P Coulomb collision operator causes the PKE to become a (diffusive) parabolic second order partial differential equation. Physically, this causes the particle trajectories (characteristic curves) to become slightly probabilistic rather than fully deterministic. In the usual case where the Coulomb collision rate ν is small (i.e., $|\mathcal{C}\{f_s\}| \ll |df_s/dt|$ for most $\mathbf{x}, \mathbf{v}$), the PKE in (1) becomes a singular differential equation. Coulomb collisions relax phenomena that are localized to small regions of velocity space. They thereby create dissipative singular regions where $|df_s/dt| \lesssim |\mathcal{C}\{f_s\}| \sim \nu_{\text{eff}} f_s$ in which $\nu_{\text{eff}} \sim \nu(|\mathbf{v}|^2/|\delta\mathbf{v}|^2) \gg \nu$ is the effective collision rate for the small velocity-space region $\delta\mathbf{v} \ll \mathbf{v}$. Dissipative singular responses provide Coulomb-collision-induced temporal irreversibility that increases the plasma entropy.

IV. GREEN FUNCTION SOLUTION OF LINEARIZED PKE

Many plasma physics studies explore effects of perturbations or inhomogeneity-induced departures of the plasma from a homogeneous equilibrium state. The equilibrium state is determined by setting $\partial f_s/\partial t = 0$ in (1). In an infinite homogeneous plasma with no significant equilibrium electric or magnetic fields, the equilibrium kinetic equation becomes $\mathcal{C}\{f_s\} = 0$. The solution of this collisional equilibrium is an isotropic Maxwellian distribution $f_{s0} = f_{\text{M}s}(v) = (n_{s0}/\pi^{3/2} v_{Ts}^3)\, e^{-v^2/v_{Ts}^2}$ in which $v_{Ts} \equiv \sqrt{2T_s/m_s}$ is the most probable thermal speed of the s species. This lowest order equilibrium is also appropriate for equilibrium electric $\mathbf{E}_0$ and magnetic $\mathbf{B}_0$ fields that are approximately uniform on the collision and gyroradius scale lengths.

When small electromagnetic field perturbations $\tilde{\mathbf{E}}(\mathbf{x}, t)$ and $\tilde{\mathbf{B}}(\mathbf{x}, t)$ are introduced in (1), $f_s \to f_{\text{M}s} + \tilde{f}_s$ and the perturbed distribution $\tilde{f}_s(\mathbf{x}, \mathbf{v}, t)$ is governed by the perturbed PKE:

$$\mathcal{L}\{\tilde{f}_s\} = S, \quad \text{in which} \tag{2}$$

$$\mathcal{L}\{\tilde{f}_s\} \equiv \frac{\partial \tilde{f}_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \tilde{f}_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial \tilde{f}_s}{\partial \mathbf{v}} - \mathcal{C}\{\tilde{f}_s\}, \tag{3}$$

$$S = -\frac{q_s}{m_s}(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}})\cdot\frac{\partial f_{\text{M}s}}{\partial \mathbf{v}} - \frac{q_s}{m_s}(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}})\cdot\frac{\partial \tilde{f}_s}{\partial \mathbf{v}}. \tag{4}$$

Here, $\mathcal{L}$ in (3) is a linear, second order partial differential operator in the 7 variables $\mathbf{x}, \mathbf{v}, t$ and $S(\mathbf{x}, \mathbf{v}, t)$ in (4) is the inhomogeneous "source" which includes both linear and quasilinear contributions. The formal solution of (2) is $\tilde{f}_s = \mathcal{L}^{-1}\{S\}$. However, since no general inverse operator $\mathcal{L}^{-1}$ exists, inverse operators must usually be determined for specific applications.

Different approaches are usually used to solve the perturbed PKE in (2)–(4) for collisionless and collisional studies. When collisions are neglected, the solution is obtained using the first order characteristics (particle trajectories) determined from the equations $d\mathbf{x}'/dt' = \mathbf{v}'$

and $d\mathbf{v}'/dt' = (q_s/m_s)\,[\mathbf{E}_0(\mathbf{x}',t') + \mathbf{v}'\times\mathbf{B}_0(\mathbf{x}',t')]$ with initial conditions $\mathbf{x}' = \mathbf{x}$, $\mathbf{v}' = \mathbf{v}$ at time $t' = t$. Equation (2) can then be written as $d\tilde{f}(\mathbf{x}',\mathbf{v}',t')/dt' = S(\mathbf{x}',\mathbf{v}',t')$, which can be integrated along the particle trajectories $\mathbf{x}',\mathbf{v}'$ to yield the "time-history integral" solution

$$\tilde{f}_s(\mathbf{x},\mathbf{v},t) = \tilde{f}_s(\mathbf{x}',\mathbf{v}',t'\!=\!0) \;+\int_0^t dt'\; S(\mathbf{x}',\mathbf{v}',t'). \tag{5}$$

The first term represents the initial condition mapped along unperturbed particle trajectories and the second the response to $\tilde{\mathbf{E}},\tilde{\mathbf{B}}$ perturbations integrated from the initial to present time. For many studies the linear operator in (3) is first averaged over fast particle trajectory time scales (e.g., gyromotion on the cyclotron motion time scale, bounce motion in a periodic electric or magnetic field) before solving for the longer time scale particle drift trajectories, which are then the $\mathbf{x}',\mathbf{v}'$ trajectories along which the mapping and integration in (5) occur.

Collisional studies usually assume collisional effects are dominant or are comparable to bounce or drift time scales in a lowest order perturbed collisional equilibrium. In such cases time-asymptotic solutions of (3) or averaged versions of it with $\partial\tilde{f}_s/\partial t = 0$ are obtained analytically or numerically and then used to determine closures for the heat flux and stress moments — see Sections VI—VIII below. In low collisionality plasmas Coulomb collisional scattering effects often resolve singular regions in the time-asymptotic plasma response.

A general solution of (2) that is analogous to the time history approach but allows for collisional effects (in dissipative singular layers as well as more generally) can be obtained using a Green-function-based approach.[11,12] In this approach the first step is to obtain the solution of the defining equation for the 6D phase space Green function:

$$\mathcal{L}\{G_\nu\} = \delta[\mathbf{x}-\mathbf{x}_0]\;\delta[\mathbf{v}-\mathbf{v}_0]\;\delta[t-t_0]. \tag{6}$$

Here, $G_\nu(\mathbf{x},\mathbf{v},t|\mathbf{x}_0,\mathbf{v}_0,t_0)$ represents the collision-influenced distribution function response at $\mathbf{x},\mathbf{v},t$ to a delta function source at $\mathbf{x}_0,\mathbf{v}_0,t_0$. Also, $\delta[\mathbf{x}-\mathbf{x}_0] = \delta[x-x_0]\,\delta[y-y_0]\,\delta[z-z_0]$ and $\delta[\mathbf{v}-\mathbf{v}_0]$ are three-dimensional (3D) Dirac delta functions. The general Green function solution of (2) is

$$\begin{aligned}
\tilde{f}_e(\mathbf{x},\mathbf{v},t) = {} & \textstyle\int_0^t dt_0 \int d^3x_0 \int d^3v_0\, G_\nu\, S(\mathbf{x}_0,\mathbf{v}_0,t_0) && \text{response to source} \\
& + \textstyle\int d^3x_0 \int d^3v_0\, G_\nu(t_0\!=\!0)\, \tilde{f}_s(\mathbf{x}_0,\mathbf{v}_0,t\!=\!0) && \text{initial condition} \\
& + \int_0^t dt_0 \int d\mathbf{S}_{\mathbf{x}_0}\boldsymbol{\cdot} \left(\int d\mathbf{S}_{\mathbf{v}_0}\boldsymbol{\cdot} \left[G_\nu\,\frac{\partial^2\tilde{f}_s}{\partial\mathbf{v}\,\partial\mathbf{x}} - \tilde{f}_s\,\frac{\partial^2 G_\nu}{\partial\mathbf{v}\,\partial\mathbf{x}}\right]\right) && \text{boundary effects.}
\end{aligned} \tag{7}$$

This is a generalization of the time-history integration in (5) which here includes the diffusive, probabilistic Coulomb collisional scattering effects through the Green function G_ν.

To illustrate the nature of the Green function employed in (7), consider an infinite homogeneous plasma with no equilibrium electric or magnetic fields. Then, the linear operator becomes $\mathcal{L}^\infty\{\tilde{f}_s\} = \partial\tilde{f}_s/\partial t + \mathbf{v}\cdot\partial\tilde{f}_s/\partial\mathbf{x} - \mathcal{C}\{\tilde{f}_s\}$. The lowest order characteristic curves (particle trajectories) of this operator's first derivatives are determined from $d\mathbf{x}/dt = \mathbf{v}$, $d\mathbf{v}/dt = \mathbf{0}$ with initial conditions $\mathbf{x} = \mathbf{x}_0$, $\mathbf{v} = \mathbf{v}_0$ at $t = t_0$, which yield the straight-line real-space trajectories $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\tau$, in which $\tau \equiv t - t_0$. The F-P Coulomb collision operator only involves derivatives in velocity space. Thus, its diffusive effects are separable from these first order straight-line trajectories in real space. For an isotropic (subscript i) collision operator (e.g., for Brownian motion in velocity space) $\mathcal{C}_{\rm i}\{\tilde{f}_s\} = (\nu/2)\nabla_v^2\tilde{f}_s \equiv (\nu/2)\,(\partial/\partial\mathbf{v})\cdot\partial\tilde{f}_s/\partial\mathbf{v}$, the short time ($\nu\tau \ll 1$) probabilistic distribution for an infinite, uniform velocity-space (so there are no boundary conditions) is[11] $e^{-|\mathbf{v}-\mathbf{v}_0|^2/(2\nu\tau v_0^2)}/(2\pi\nu\tau v_0^2)^{3/2}$. For long times ($\nu\tau \gg 1$) the collisional response can be estimated using a Krook (subscript K) collisional damping operator $\mathcal{C}_{\rm K}\{\tilde{f}_s\} \simeq -\,\nu\tilde{f}_s$, for which it is $e^{-\nu\tau}$. Thus, by construction, the Green function that embodies both the lowest order straight-line particle trajectory and collisional scattering or damping effects is

$$G_\nu^\infty = \delta[\mathbf{x}-\mathbf{x}_0-\mathbf{v}_0\tau]\,H(\tau)\times\begin{cases} e^{-|\mathbf{v}-\mathbf{v}_0|^2/(2\nu\tau v_0^2)}/(2\pi\nu\tau v_0^2)^{3/2}, & \nu\tau \ll 1, \\ e^{-\nu\tau}\,\delta[\mathbf{v}-\mathbf{v}_0], & \nu\tau \gg 1, \end{cases} \tag{8}$$

in which $H(\tau) \equiv H(t-t_0)$ is the Heaviside step function. Since $dH(\tau)/dt = \delta[\tau]$ and using the fact that $\lim_{\tau\to 0^+} e^{-|\mathbf{v}-\mathbf{v}_0|^2/(2\nu\tau v_0^2)}/(2\pi\nu\tau v_0^2)^{3/2}$ yields the delta function $\delta[\mathbf{v}-\mathbf{v}_0]$, this Green function satisfies $\mathcal{L}^\infty\{G_\nu^\infty\} = \delta[\mathbf{x}-\mathbf{x}_0]\,\delta[\mathbf{v}-\mathbf{v}_0]\,\delta[t-t_0]$, as desired.

Physically, a Green function is the "propagator" for motion of particles in the 6D phase space. The propagator in (8) includes both the deterministic lowest order straight-line particle trajectories in real space (via the first delta function) and diffusive scattering of the velocity vector $\mathbf{v}$ through $|\delta\mathbf{v}|/v_0 \equiv |\mathbf{v}_0-\mathbf{v}|/v_0 \sim (2\nu\tau)^{1/2} \ll 1$ for short time scales at a rate $\nu_{\rm eff} \sim \nu/|\delta\mathbf{v}/v_0|^2 \gg \nu$, or damping at a rate ν on long time scales.

Unfortunately, the F-P Coulomb collision operator is anisotropic and more complicated than the iostropic operator used to obtain (8). Also, for many problems the real and velocity space characteristic curves are not separable. Finally, in $\nu\tau \sim 1$ regimes, which are often of interest, the Green function is more complicated because boundary conditions in velocity space become important. Nevertheless, the Green function procedure provides a systematic, rigorous and complete solution of the perturbed PKE in (2) that includes Coulomb collisional scattering and longer time scale collisional effects in low collisionality plasmas.

V. COLLISIONAL EFFECTS ON LINEAR LANDAU DAMPING

This new Green function procedure will now be used[12] to explore Coulomb collisional effects on the most fundamental issue in theoretical plasma physics — Landau damping. In a classic theoretical physics paper[13] Landau introduced causality into the determination of the time-asymptotic linear plasma response to a wave by using a Laplace transform. This produced Landau damping of Langmuir waves, which has been confirmed experimentally.[14] It is often thought of as a "collisionless," entropy-producing process because: collisional effects are not directly involved in the derivation; the damping rate is independent of the collision rate; and the wave damping would seem to imply temporal irreversibility.

However, just after a wave is turned on its phase information is transferred to a perturbed distribution of the plasmas' charged particles. That this initial state information still exists has been demonstrated by showing a second wave produces a plasma echo.[15] But when sufficient Coulomb collisions are introduced, the echo is damped.[16,17] Thus, any temporal irreversibility and entropy produced by Landau damping must be due to a collisional process.

Collisional effects on the time-asymptotic Landau damping have been explored previously using various speed-diffusion-based model collision operators[18–21] which neglect the dominant velocity-scattering effect of Coulomb collisions. And a physical interpretation of Landau damping via (quasilinear) wave energy damping has been developed.[22] Further, models of Landau-type damping of a monochromatic large amplitude wave that traps charged particles in its sinusoidal potential have been developed.[23–26] Prior Landau damping studies have been summarized recently.[27] Here, the new Green function procedure[12] for incorporating the diffusive effects of small-angle Coulomb collisional scattering of electrons on the temporal evolution of a small amplitude Langmuir wave in a plasma will be discussed using a simple Lorentz collision F-P collision operator.

Coulomb collisions of a charged particle with other charged particles in a plasma primarily scatter the particle's velocity vector $\mathbf{v}$ at nearly constant speed. This is especially true for superthermal electrons on the tail of the Maxwellian distribution where experimental tests of Landau damping were performed.[14] The characteristic Coulomb collision scattering rate ν is usually much smaller than the Landau damping rate γ_L.

In spherical velocity space coordinates speed $v \equiv |\mathbf{v}|$, "pitch-angle" ϑ and phase angle φ, Coulomb collisions mainly diffuse the pitch-angle through a small angle $\delta\vartheta \equiv \vartheta_0 - \vartheta \sim \sqrt{2\nu t} \ll 1$ in a time $t \ll 1/\nu$. The simplest Coulomb collision operator is that for the Lorentz scattering model in which light electrons scatter off heavier background ions. The Lorentz (subscript L) Fokker-Planck (F-P) Coulomb collision operator for electrons with distribution f_e is (for $\vartheta \ll 1$ and neglecting φ diffusion not relevant for Landau damping)

$$\mathcal{C}_{\rm L}\{\tilde{f}_e\} = \frac{\nu}{2}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\tilde{f}_e}{\partial\vartheta}\right)\right] \simeq \frac{\nu}{2\vartheta}\frac{\partial}{\partial\vartheta}\left[\vartheta\frac{\partial\tilde{f}_e}{\partial\vartheta}\right], \tag{9}$$

in which the Lorentz model Coulomb collision rate is

$$\nu(v) \equiv \frac{4\pi n_e e^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2 m_e^2 v^3} = \omega_p\,\frac{\ln\Lambda}{8\sqrt{2}\,\pi\, n_e\lambda_D^3}\left(\frac{v_{Te}}{v}\right)^3. \tag{10}$$

Here, $\Lambda \equiv \lambda_D/b_{\rm min} \lesssim 12\pi\, n_e\lambda_D^3$ is the ratio of the collective Debye shielding length to the distance of closest approach $b_{\rm min}$ during Coulomb collisions. The $\ln\Lambda$ factor (typically $\gtrsim 10$) represents the cumulative effect of all the charged particle collisions within a Debye sphere. Also, $\omega_p \equiv (n_e e^2/m_e\epsilon_0)^{1/2}$ is the Langmuir electron plasma frequency. Since in typical plasmas $n_e\lambda_D^3 \gg 1$, usually $\nu \sim \omega_p/(n_e\lambda_D^3) \ll \omega_p$.

Landau damping is explored most simply in an infinite, uniform unmagnetized plasma that has only a perturbed electrostatic electric field force: $\tilde{\mathbf{F}}_e = -\,q_e\boldsymbol{\nabla}\tilde{\phi}$. Then, the perturbed distribution $\tilde{f}_e(\mathbf{x},\mathbf{v},t)$ is governed by the Landau (superscript L) linearized PKE:

$$\mathcal{L}^{\rm L}\{\tilde{f}_e\} \equiv \frac{\partial\tilde{f}_e}{\partial t} + \mathbf{v}\boldsymbol{\cdot}\frac{\partial\tilde{f}_e}{\partial\mathbf{x}} - \mathcal{C}_{\rm L}\{\tilde{f}_e\} = -\frac{q_e}{T_e}(\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\phi})f_{\rm Me}. \tag{11}$$

Neglecting collision effects and assuming Fourier perturbations $\tilde{\phi}(\mathbf{x},t) = \hat{\phi}(\mathbf{k},\omega)e^{i(\mathbf{k}\boldsymbol{\cdot}\mathbf{x}-\omega t)}$ for $\tilde{\phi}$ and a similar form for $\tilde{f}_e$, this equation yields

$$\hat{f}_e = \frac{(q_e/T_e)\,(\mathbf{k}\boldsymbol{\cdot}\mathbf{v})\,\hat{\phi}\,f_{\rm Me}}{\omega - \mathbf{k}\boldsymbol{\cdot}\mathbf{v}} = \frac{e\hat{\phi}}{T_e}\,f_{\rm Me}\,\frac{u}{u - V_\varphi}, \tag{12}$$

which is singular for particles with $u \equiv \mathbf{k}\boldsymbol{\cdot}\mathbf{v}/k \simeq V_\varphi = \omega/k$, the wave phase speed.

Since the solution in (12) is real, Vlasov[9] concluded there is no damping of wave-like perturbations in low collisionality plasmas. Landau[13] also neglected collision effects but used a Laplace transform for the time domain to introduce causality in the definition of the singularity in (12) by deforming the inverse Laplace transform contour to always be below the singularity in (12). Using $u = (\omega + i\delta)/k$, Landau's formalism yields the Plemelj formula

$$\lim_{\delta\to 0} \frac{u}{u-(V_\varphi+i\delta)} = \mathcal{P}\left\{\frac{u}{u-V_\varphi}\right\} + i\pi u\,\delta[u-V_\varphi]. \tag{13}$$

Here, $\mathcal{P}$ is the (real) principal value operator and $\delta[x]$ is the Dirac delta function.

The perturbed electron density induced by $\tilde{\phi}$ is obtained by integrating $\hat{f}_e$ in (12) over velocity space using (13). For $V_\varphi \gg v_{Te}$ this yields[5,28] $\tilde{n}_e \equiv \int d^3v\,\tilde{f}_e = -\,n_{e0}\,(e\tilde{\phi}/T_e)\,[v_{Te}^2/2V_\varphi^2 + (3/4)v_{Te}^4/V_\varphi^4 + \cdots + i\sqrt{\pi}\,(V_\varphi/v_{Te})e^{-V_\varphi^2/v_{Te}^2}]$. The dispersion relation for $k\lambda_D \ll 1$ Langmuir waves is obtained by substituting this $\tilde{n}_e$ into the perturbed Poisson's equation $-\nabla^2\tilde{\phi} = \tilde{\rho}_q/\epsilon_0 \simeq -\,\tilde{n}_e e/\epsilon_0$ in which the $\mathcal{O}\{m_e/m_i\}$ smaller ion contribution has been neglected. Solving this dispersion relation for $\omega \equiv \omega_R + i\,\gamma$ yields $\omega_R \simeq \omega_p\,[1 + (3/2)k^2\lambda_D^2 + \cdots]$ and the Landau damping rate $\gamma_L \simeq -\,[\omega_p/(k\lambda_D)^3]\,(\pi/8)^{1/2}\,e^{-1/(2k^2\lambda_D^2)-3/2}$. The imaginary delta function in (13) produces the Landau damping rate γ_L, which is usually smaller than ω_p (e.g., $\gamma_L/\omega_p \simeq 0.02$ for $k\lambda_D \simeq 0.3$ and $V_\varphi/v_{Te} \simeq 2.7$ in Ref. 14) but much larger than ν.

In spherical velocity-space coordinates $u = v\cos\vartheta$. Thus, if the collision operator in (9) operates on the $\hat{f}_e$ solution in (12), it yields $\mathcal{C}_{\rm L}\{\hat{f}_e\} \sim \nu\,\sin^2\vartheta/(u-V_\varphi)^3$, which is even more singular than $\hat{f}_e$ itself. The Green function procedure will now be used to obtain a solution of (2) that includes collisional effects and resolves the singular region where $u \simeq V_\varphi$.

The solution of (2) can be obtained in terms of a Green function $G_\nu^{\rm L}(\mathbf{x},\vartheta|\mathbf{x}_0,\vartheta_0;t-t_0)$ that solves the equation $\mathcal{L}^{\rm L}\{G_\nu^{\rm L}\} = \delta[\mathbf{x}-\mathbf{x}_0]\,\delta[(\vartheta-\vartheta_0)^2/2]\,\delta[t-t_0]$. This Green function represents the response at $\mathbf{x},\vartheta,t$ to a delta function source at $\mathbf{x}_0,\vartheta_0,t_0$. For situations where the initial condition for $\tilde{f}_e$ vanishes, the infinite medium Green function solution of (11) is

$$\tilde{f}_e = \int_0^t\! dt_0 \int\! d^3x_0 \int_0^\pi\! \sin\vartheta_0 d\vartheta_0\, G_\nu^{\rm L}(\mathbf{x},\vartheta|\mathbf{x}_0,\vartheta_0;t-t_0)\,\frac{e}{T_e}\Big[(\mathbf{v}\cdot\boldsymbol{\nabla}\tilde{\phi})f_{\rm Me}\Big]_{\mathbf{x}_0,\vartheta_0,t_0}. \tag{14}$$

As above, the lowest order deterministic particle trajectories are given by $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\tau$ in which $\tau \equiv t - t_0 \geq 0$. Pitch-angle scattering effects of the second derivative diffusion operator $\mathcal{C}_{\rm L}\{\tilde{f}_e\}$ in (11) are again separable. The probability distribution for infinite-domain, short time scale ($\nu\tau \ll 1$) Coulomb-collision-induced diffusion effects is $e^{-(\vartheta-\vartheta_0)^2/(2\nu\tau)}/(\nu\tau)$.

By construction, the Green function for the differential operator $\mathcal{L}^{\rm L}$ in (11) is thus

$$G_\nu^{\rm L}(\mathbf{x},\vartheta|\mathbf{x}_0,\vartheta_0;\tau) = \delta[\mathbf{x}-\mathbf{x}_0-\mathbf{v}_0\tau]\,\frac{e^{-(\vartheta-\vartheta_0)^2/(2\nu\tau)}}{\nu\tau}H(\tau),. \tag{15}$$

Using $\lim_{\tau\to 0^+} e^{-(\vartheta-\vartheta_0)^2/(2\nu\tau)}/(\nu\tau) = \delta[(\vartheta-\vartheta_0)^2/2]$, operating on $G_\nu^{\rm L}$ with the linear operator $\mathcal{L}^{\rm L}$ shows $G_\nu^{\rm L}$ satisfies the defining equation for this Green function for $\vartheta_0 \ll 1$, which will be sufficient here. Equation (15) is the propagator for charged particle trajectories which includes both the deterministic motion $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\tau$ and probabilistic diffusion of the velocity-space pitch-angle difference $\delta\vartheta \equiv \vartheta_0 - \vartheta \sim \sqrt{2\nu\tau}$ induced by Coulomb collisions.

After substituting the Green function solution in (15) into (14), the d^3x_0 integration can be performed using the particle trajectory delta function to replace $\mathbf{x}_0$ with $\mathbf{x} - \mathbf{v}_0\tau$. Then, using $t_0 = t - \tau$ and omitting the common factor $e^{i(\mathbf{k\cdot x}-\omega)\tau}$, the solution in (14) yields

$$\hat{f}_e = \frac{e\hat{\phi}}{T_e} f_{\text{M}e} \int_0^t d\tau \int_0^\pi \frac{\sin\vartheta_0 d\vartheta_0}{\nu\tau}\, i\,(\mathbf{k\cdot v}_0)\, e^{i\Phi}, \tag{16}$$

in which the effective wave-particle phase factor is $i\Phi = -\,i\,(\mathbf{k\cdot v}_0 - \omega)\tau - (\vartheta_0 - \vartheta)^2/2\nu\tau$. The last term here is caused by collisional pitch-angle scattering. When this dissipative phase is integrated over ϑ_0 and ϑ, in combination with the requirement $\vartheta_0 \ll 1$ they limit these angles to $\max\{\vartheta_0, \vartheta\} \sim \sqrt{\nu\tau} \ll 1$. Thus, $\mathbf{k\cdot v}_0 = kv\cos\vartheta_0 \simeq kv\,(1 - \vartheta_0^2/2)$, $ku \equiv \mathbf{k\cdot v} \simeq kv\,(1 - \vartheta^2/2) \simeq ku(1 + \mathcal{O}\{\nu\tau\})$ and $\sin\vartheta_0 \simeq \vartheta_0$. Then, the effective wave-particle phase is

$$i\Phi \simeq -\,i\,(ku - \omega)\tau - \frac{(\vartheta_0 - \vartheta)^2}{2\nu\tau} + i\,ku\tau\,\frac{\vartheta_0^2 - \vartheta^2}{2}. \tag{17}$$

Here, $\omega - ku$ represents the Doppler-shifted wave frequency, $ku\tau\,(\vartheta_0^2 - \vartheta^2)/2$ is due to the differences of $\mathbf{k\cdot v}_0$ and kv from ku caused by $\vartheta_0, \vartheta \neq 0$ and $i\,(\vartheta - \vartheta_0)^2/(2\nu\tau)$ results from Coulomb collisional pitch-angle scattering effects. The combined effects of the latter two terms cause Φ to become complex and yield a damped response for times longer than

$$\tau_{\text{damp}} \equiv \frac{1}{\sqrt{\nu ku}} \simeq \frac{1}{\sqrt{\nu\omega}}, \quad \text{Coulomb collision damping time.} \tag{18}$$

This leads to temporal irreversibility[12] of all particles from the wave for $t > \tau_{\text{damp}}$ and an effective collision rate of $\nu_{\text{eff}} \sim \nu/\delta\vartheta^2 \sim \nu/(\nu\tau_{\text{damp}}) \sim \sqrt{\nu\omega} \gg \nu$. This Coulomb-collision-induced damping causes[12] a dissipative singular region where resonant particles have $|\omega - ku|\,\tau_{\text{damp}} \lesssim 1$. In (18) and below ku has been estimated by ω because $\hat{f}_e$ is highly peaked at $ku \simeq \omega\,(1 + \mathcal{O}\{1/\omega\tau_{\text{damp}}\}) \simeq \omega\,(1 + \mathcal{O}\{\sqrt{\nu/\omega}\})$.

Neglecting the $\mathcal{O}\{\vartheta^2, \vartheta_0^2\}$ corrections to the $\mathbf{k\cdot v}_0$ in (16) and setting it to ku, (16) becomes

$$\hat{f}_e \equiv \frac{e\hat{\phi}}{T_e} f_{\text{M}e}\, I_\nu^{\text{L}}, \tag{19}$$

$$I_\nu^{\text{L}}(\Delta_u, \epsilon, kut, \vartheta) \equiv \int_0^{kut} dz\, i\, e^{-i\Delta_u z} \int_0^\infty \frac{\vartheta_0 d\vartheta_0}{\epsilon z}\, e^{-(\vartheta_0-\vartheta)^2/(2\epsilon z) + iz(\vartheta_0^2 - \vartheta^2)/2}. \tag{20}$$

Three dimensionless parameters have been defined in this $I_\nu^{\rm L}$ time history integral which includes the effects of Coulomb-collision-induced pitch-angle scattering:

$$z \equiv ku\tau, \quad 2\pi \times \text{number of wavelengths traversed,} \tag{21}$$

$$\Delta_u \equiv \frac{u - \omega/k}{u} = 1 - \frac{V_\varphi}{u}, \quad \text{relative speed, and} \tag{22}$$

$$\epsilon \equiv \frac{\nu}{ku}, \quad \text{small relative collisionality.} \tag{23}$$

The $I_\nu^{\rm L}$ integral[12] is simplest for $\vartheta = 0$, for which it can be integrated over ϑ to yield

$$I_{\nu 0}^{\rm L} \equiv I_\nu^{\rm L}(\Delta_u, \epsilon, kut, \vartheta\!=\!0) = i\int_0^{kut} dz \, \frac{e^{-i\,\Delta_u z}}{1 - i\epsilon z^2}. \tag{24}$$

This distribution function response is oscillatory for $|\Delta_u|/\epsilon^{1/2} \gg 1$ and in phase with the Doppler-shifted wave at all Δ_u for $t \ll \tau_{\rm damp}$. However, it is damped as $\tau_{\rm damp}^2/t^2$ for all Δ_u for times longer than the Coulomb collisional pitch-angle scattering damping time $\tau_{\rm damp}$.

Coulomb collisional effects can be neglected by taking the $\epsilon \to 0$ limit in (20) for which the ϑ_0 integration yields unity since $\lim_{\epsilon\to 0^+} e^{-(\vartheta-\vartheta_0)^2/(2\epsilon z)}/(\epsilon z) = \delta[(\vartheta - \vartheta_0)^2/2] = \delta[\vartheta - \vartheta_0]/\vartheta_0$. Then, the $I_\nu^{\rm L}$ integral yields

$$I_\nu^{\rm L}(\epsilon\!\to\!0) = \frac{[1 - \cos(kut\,\Delta_u)] + i\sin(kut\,\Delta_u)]}{\Delta_u}. \tag{25}$$

When $kut\,\Delta_u \gg 1$, the average of the real part over its sinusoidal period $2\pi/(kut\,\Delta_u)$, which represents "phase mixing," yields $1/\Delta_u$. However, the un-averaged real part of (25) vanishes at $\Delta_u = 0$. The imaginary part of (25) is singular for speeds near the wave phase speed (i.e., for $\Delta_u \ll 1/kut$). Since $\sin(kut\,\Delta_u)/\Delta_u$ is a correlation function, its $kut \gg 1$ limit is a delta function: $\lim_{kut\to\infty} \sin(kut\,\Delta_u)/\Delta_u \doteq \pi\,\delta[\Delta_u]$. Thus, the phase mixed collisionless $I_\nu^{\rm L}$ yields $\lim_{kut\to\infty} \overline{I}_\nu^{\rm L}(\epsilon\!=\!0) = \mathcal{P}\{1/\Delta_u\} + i\pi\,\delta[\Delta_u]$, which is the Landau result in (13).

When Coulomb collision effects are included, key properties of $I_\nu^{\rm L}$ can be obtained for times long compared to the damping time $\tau_{\rm damp}$ for which the upper limits of the z integrals in (20) and (24) become infinite. In this limit successive integration by parts for large $|\Delta_u|/\epsilon^{1/2}$ yields $I_{\nu 0}^{\rm L} = (1/\Delta_u)\,(1 + \mathcal{O}\{\epsilon/\Delta_u^2\})$ for $t \gg \tau_{\rm damp}$. More generally, it can be shown[12] that $\lim_{t\gg\tau_{\rm damp}} I_{\nu 0}^{\rm L}(\sqrt{\epsilon} \ll 1) = \mathcal{P}\,\{1/\Delta_u\} + i\pi\,\delta[\Delta_u]$. This result also applies to $I_\nu^{\rm L}$ for small $\vartheta \neq 0$. Thus, the Coulomb-collision-based $I_\nu^{\rm L}$ also yields the Landau result in (13).

The preceding Fourier-based analysis has shown that when a Green function approach is used and Lorentz model Coulomb collisional scattering effects are included, the singular factor $u/(u - V_\varphi)$ in (12) is replaced by the time history integral $I_\nu^{\rm L}$ defined in (20). Both the phase mixed collisionless and long time collisional limits of $I_\nu^{\rm L}$ yield the Landau-based prescription in (13) for resolving the $u \simeq V_\varphi$ singularity. Thus, the Landau damping rate γ_L obtained using (13) is also obtained using a Green function-based approach that includes Coulomb collisional pitch-angle scattering effects. Note that the Laplace transform procedure is not needed here to introduce causality.

Does Landau damping increase entropy? Addressing this question requires a quasilinear-type analysis of the last term in (4) that is beyond the linear analysis discussed here. However, the instantaneous $\tilde{f}_e$ response in (19) and (20) is in phase with the applied potential for all $\Delta_u \equiv (u - V_\varphi)/u$ for times up until the Coulomb-collision-induced damping time $\tau_{\rm damp}$, after which it is collisionally damped. This suggests entropy changes little for $t < \tau_{\rm damp}$ but then increases primarily due to collisional scattering in the resonant velocity space region $|\Delta_u| \lesssim \sqrt{\epsilon}$ for $t > \tau_{\rm damp}$ due to Coulomb-collision-induced temporal irreversibility.

When a wave is monochromatic and has a finite amplitude, nearly resonant electrons are trapped in its sinusoidal potential. The "oscillation" frequency for such trapped electrons is[23–26] $\omega_{\rm osc} \equiv k\,(e\hat{\phi}/m_e)^{1/2} = (kv_{Te}/\sqrt{2})\,(e\hat{\phi}/T_e)^{1/2}$. This nonlinear trapping becomes dominant when resonant electrons can complete a bounce period. Thus, the linear Coulomb collision effects discussed here are applicable for times up to the minimum of $\tau_{\rm damp}$ or $2\pi/\omega_{\rm osc}$.

In the preceding analysis a Lorentz pitch-angle scattering model was used. In general, both speed diffusion and pitch-angle scattering effects induced by Coulomb collisions need to be considered. A model that includes both these effects is presented in Ref. 12. The results presented in this section are also obtained for this more complete model, with the exception that the collision-induced damping rate is changed. For pitch-angle scattering at rate $\nu_\perp$ and speed diffusion at rate $\nu_\parallel$ the effective collision damping rate is changed to[12]

$$1/\tau_{\rm damp} \to \nu_{\rm eff} \equiv (\nu_\perp \omega/2)^{1/2} + (\nu_\parallel \omega^2/2)^{1/3} \;\gg\; \nu. \tag{26}$$

The key physical effect of Coulomb collisions in Landau damping is that they change the lowest order deterministic charged particle trajectories into slightly diffusive, probabilistic ones. This dissipative process is caused by Coulomb collisional scattering of the charged particle velocity $\mathbf{v}$ and provides irreversibility in Landau damping for $t \gg \tau_{\rm damp}$. The novel Green function procedure presented here for exploring short time scale collisional scattering effects on the temporal evolution of responses to waves and wave-particle resonances in low collisionality plasmas should also be useful for other plasma applications.

VI. FLUID MOMENT EQUATIONS

For time scales on the order of or longer than collision times it is convenient to use fluid moment descriptions of plasmas. The velocity-space moments $\int d^3v\,(1, \mathbf{v}, v^2)$ of (1) yield fluid moment equations for the species density, momentum and energy for each plasma species s:

$$\text{density:}\quad (\partial/\partial t + \mathbf{V}_s \cdot \boldsymbol{\nabla})\, n_s = -\, n_s \boldsymbol{\nabla} \cdot \mathbf{V}_s, \tag{27}$$

$$\text{momentum:}\quad m_s n_s (\partial/\partial t + \mathbf{V}_s \cdot \boldsymbol{\nabla})\, \mathbf{V}_s = n_s q_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) - \boldsymbol{\nabla} p_s - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_s + \mathbf{R}_s, \tag{28}$$

$$\text{energy:}\quad (3/2)\,(\partial/\partial t + \mathbf{V}_s \cdot \boldsymbol{\nabla})\, p_s = -\,(5/2)\, p_s \boldsymbol{\nabla} \cdot \mathbf{V}_s - \boldsymbol{\nabla} \cdot \mathbf{q}_s - \boldsymbol{\pi} : \boldsymbol{\nabla} \mathbf{V}_s + Q_s. \tag{29}$$

Here, $(n_s, \mathbf{V}_s, p_s) \equiv \int d^3v\,(1, \mathbf{v}/n_s, m_s v_r^2/3)\, f_s$ are the species (density, flow velocity, isotropic pressure), $T_s \equiv p_s/n_s$ is the species temperature and $(\mathbf{R}_s, Q_s) \equiv \int d^3v\,(m_s \mathbf{v}, m_s v_r^2/3)\, \mathcal{C}\{f_s\}$ are the Coulomb-collision-induced (momentum, energy) exchange with other plasma species. The $\mathbf{R}_s$ term is the dynamical friction force density. In these definitions $\mathbf{v}_r \equiv \mathbf{v} - \mathbf{V}_s$ and $v_r \equiv |\mathbf{v}_r|$ are the "random" velocity and speed of particles relative to the species flow velocity $\mathbf{V}_s$. The "closure" moments needed to complete these equations for each species are

$$\text{conductive heat flux:}\quad \mathbf{q}_s \equiv \int d^3v\, \mathbf{v}_r\, (m_s v_r^2/2)\, f_s, \tag{30}$$

$$\text{stress tensor:}\quad \boldsymbol{\pi}_s \equiv \int d^3v\, m_s (\mathbf{v}_r \mathbf{v}_r - v_r^2 \mathbf{I}/3)\, f_s. \tag{31}$$

The macroscopic plasma quantities $n_s, \mathbf{V}_s, T_s, p_s$ are only physically meaningful for times longer than the collision time scale $1/\nu$. If appropriate closure moments for the heat flux $\mathbf{q}$ and stress tensor $\boldsymbol{\pi}$ can be determined for specific applications, these equations are exact and together with Maxwell's equations provide a complete description of the plasma for such applications. In many applications (e.g., for the collisional effects on Landau damping discussed in the preceding section) Coulomb-collision-induced pitch-angle scattering causes a larger effective collision rate: $\nu_{\rm eff} \sim \nu/\delta\vartheta^2 \gg \nu$. For such applications useful closures and plasma descriptions can sometimes be obtained for time scales shorter than or of order $1/\nu$.

The fluid moment equations in (27)–(29) are the basis for magnetohydrodynamic (MHD) type descriptions of plasmas. Summing these density and momentum equations and the masses times them over species taking account of plasma quasineutrality ($\rho_q \equiv \sum_s n_s q_s = 0$) and the fact that the Coulomb collision operator is momentum conserving so $\sum_s \mathbf{R}_s = \mathbf{0}$

yields, for the MHD model in the center of mass (really momentum) frame, the equations:

$$\text{mass density:} \quad (\partial/\partial t + \mathbf{V\cdot\nabla})\,\rho_m = -\rho_m \mathbf{\nabla\cdot V} \tag{32}$$

$$\text{charge density:} \quad \mathbf{\nabla\cdot J} = 0 \tag{33}$$

$$\text{plasma momentum:} \quad \rho_m\,(\partial/\partial t + \mathbf{V\cdot\nabla})\,\mathbf{V} = \mathbf{J\times B} - \mathbf{\nabla}P - \mathbf{\nabla\cdot\Pi} \tag{34}$$

$$\text{Ohm's law:} \quad \mathbf{E} + \mathbf{V\times B} = \eta\,\mathbf{J} + \frac{\mathbf{J\times B} - \mathbf{\nabla}p_e - \mathbf{\nabla\cdot}\boldsymbol{\pi}_e}{n_e e} - \frac{m_e}{e}\left(\frac{\partial}{\partial t} + \mathbf{V}_e\mathbf{\cdot\nabla}\right)\mathbf{V}_e. \tag{35}$$

Here, $\rho_m \equiv \sum_s m_s n_s \simeq m_i n_i$ is the mass density, $\mathbf{V} = \sum_s m_s n_s \mathbf{V}_s/\rho_m \simeq \mathbf{V}_i$ the mass-weighted plasma flow velocity, $P \equiv \sum_s (p_s - m_s n_s |\check{\mathbf{V}}_s|^2/3)$ the total isotropic plasma pressure, and $\mathbf{\Pi} \equiv \sum_s [\boldsymbol{\pi}_s + n_s m_s(\check{\mathbf{V}}_s\check{\mathbf{V}}_s - \mathbf{I}|\check{\mathbf{V}}_s|^2/3)]$ the plasma stress tensor in which $\check{\mathbf{V}}_s \equiv \mathbf{V}_s - \mathbf{V}$. In general, in addition the energy (or pressure) equations (29) for each species s are included in the equations to be solved in a complete "extended MHD" description of plasmas. Note again that these fluid moment equations provide an exact description of plasma behavior — providing appropriate closure relations can be determined for the application of interest.

In the "ideal MHD" plasma model, all the dissipative processes (i.e., those that cause $\mathbf{R}_s, Q_s, \boldsymbol{\pi}_s, \mathbf{q}_s \neq \mathbf{0}$) are neglected in (32)–(35) and (29). Then, the energy equations yield an isentropic plasma equation of state: $(\partial/\partial t + \mathbf{V\cdot\nabla})(P/\rho_m^\Gamma) = 0$ in which $\Gamma = 5/3$ for three-dimensional plasma applications. The ideal MHD model is valid and used for studying plasma behavior on time scales shorter than the Coulomb collision time scale $1/\nu_{\text{eff}}$.

VII. DETERMINATION OF CLOSURES

The most general and widely used closures are those employed in the "Braginskii" collisional fluid moment equations[29] for the evolution of n_s, $\mathbf{V}_s$ and p_s in magnetized plasmas. They are derived using the Chapman-Enskog procedure[30,31] for short collision lengths and gyroradii compared to inhomogeneity scale lengths of the plasma and magnetic equilibrium.

The Chapman-Enskog Ansatz posits that the species distribution function can be decomposed into two parts — a time-dependent Maxwellian $f_{\text{M}s}$, whose spatial and temporal dependences come from its dependence on the fluid moment quantities $n_s(\mathbf{x},\mathbf{v},t)$, $\mathbf{V}_s(\mathbf{x},\mathbf{v},t)$ and $T_s(\mathbf{x},\mathbf{v},t)$, plus a kinetic distortion F_s (effects beyond those embodied in $n_s, \mathbf{V}_s, T_s$):

$$f(\mathbf{x},\mathbf{v},t) = f_{\text{M}s}(\mathbf{x},\mathbf{v},t) + F_s(\mathbf{x},\mathbf{v},t), \quad \text{in which} \tag{36}$$

$$f_{\text{M}s}(\mathbf{x},\mathbf{v},t) = f_{\text{M}s}[n_s(\mathbf{x},t),\mathbf{V}_s(\mathbf{x},t),T_s(\mathbf{x},t)] = \frac{n_s(\mathbf{x},t)\,e^{-m_s[\mathbf{v}-\mathbf{V}_s(\mathbf{x},t)]^2/2T_s(\mathbf{x},t)}}{[2\pi T_s(\mathbf{x},t)/m_s]^{3/2}}. \tag{37}$$

Substituting this Ansatz into the full PKE in (1) and using the fluid moment equations in (27)–(29) yields an extended Chapman-Enskog equation for the kinetic distortion:[32,33]

$$\begin{aligned}\frac{dF_s}{dt} - \mathcal{C}\{F_s\} \equiv \frac{\partial F_s}{\partial t} + \mathbf{v}\cdot\frac{\partial F_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial F_s}{\partial \mathbf{v}} - \mathcal{C}\{F_s\} \\ = \frac{m_s}{T_s}\left(\mathbf{v}_r\mathbf{v}_r - \frac{v_r^2}{3}\mathbf{I}\right)\boldsymbol{:}\boldsymbol{\nabla}\mathbf{V}_s\, f_{\mathrm{M}s} - \left(\frac{m_s\mathbf{v}_r^2}{2T_s} - \frac{5}{2}\right)\mathbf{v}_r\cdot\boldsymbol{\nabla}T_s\,\frac{f_{\mathrm{M}s}}{T_s} \\ + \mathbf{v}_r\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_s - \mathbf{R}_s)\,\frac{f_{\mathrm{M}s}}{p_s} - \left(\frac{m_s v_r^2}{3T_s} - 1\right)(\boldsymbol{\nabla}\cdot\mathbf{q}_s - \boldsymbol{\pi}_s\boldsymbol{:}\boldsymbol{\nabla}\mathbf{V}_s + Q_s)\,\frac{f_{\mathrm{M}s}}{p_s}. \quad (38)\end{aligned}$$

Note that this is still an exact plasma kinetic equation since no approximations or truncations have been introduced. Therefore, this equation for the kinetic distortion F_s is just a recast plasma kinetic equation that has used the Chapman-Enskog Ansatz. Note also that, by construction, the kinetic distortion F_s does not yield any density, flow velocity or pressure contributions: $\int d^3v\,(1, \mathbf{v}, m_s v_r^2)\, F_s = 0$. Thus, closures for the heat flux $\mathbf{q}_s$ and stress tensor $\boldsymbol{\pi}_s$ obtained by the appropriate velocity-space moments of accurate F_s solutions of (38) are guaranteed to be consistent with the fluid moment equations in (27)–(29).

In the original Chapman-Enskog procedure,[30,31] which is used in obtaining the closures for the Braginskii equations,[29] it is assumed collisions are dominant to lowest order so the terms on the last line of (38) can be neglected. However, these terms often must be kept in determining closures for low collisionality plasma applications where the Coulomb collision rate is smaller than the frequency of waves or other physical processes being considered.

While closures used in the Braginskii equations are very useful and widely used for collision-dominated plasmas, they are not always applicable. The main issue for many low collisionality plasmas is that the collision lengths $\lambda \sim v_T/\nu$ are longer than equilibrium inhomogeneity scale lengths or distances to plasma boundaries. This is especially true in magnetically confined laboratory plasmas where collision lengths are usually longer than the length or toroidal circumference of the device. Also, in general, physical effects that occur wherever plasma particles can reach in the time scale of interest should be included in the plasma description. Since no general closures are available for low collisionality plasmas, closures often must be developed for the specific process or application being addressed.

In fusion-related magnetically-confined plasmas the gyro- (or cyclotron) frequency $\omega_{cs} \equiv q_s B/m_s$ in the magnetic field is large compared to frequencies and damping rates of interest. Thus, the d/dt operator in (1) and (38) is often averaged over the gyromotion angle φ, whose effect is represented by $(q_s/m_s)(\mathbf{v}\times\mathbf{B})\cdot(\partial f_s/\partial \mathbf{v}) = -\,\omega_{cs}\,\partial f_s/\partial\varphi$ in (1) and (38). When the particle's gyroradius $\varrho \equiv v/\omega_c$ is also much smaller than all gradient scale lengths perpendicular to the magnetic field [i.e., $\varrho_* \equiv (v_{Ts}/\omega_{cs})(d\ln p/d\mathbf{x}_\perp, d\ln B/d\mathbf{x}_\perp) \sim \varrho_s/L_\perp \ll 1$], the natural velocity-space variables are the magnetic moment $\mu \equiv m_s v_\perp^2/2B$ and guiding center energy $\varepsilon_g = m_s v_\parallel^2/2 + \mu B + q_s\phi$. Then, using for the spatial coordinate the particle's gyromotion-averaged guiding center position $\mathbf{x}_g$ and velocity-space coordinates μ, ε_g, the gyromotion-averaged (overbar) $d\bar{f}_s/dt$ and resultant "drift-kinetic" equation[34] become

$$\frac{d\bar{f}_s(\mathbf{x}_g,\mu,\varepsilon_g,t)}{dt} \equiv \frac{\partial f_s}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_d)\boldsymbol{\cdot}\frac{\partial \bar{f}_s}{\partial \mathbf{x}_g} + \frac{d\mu}{dt}\frac{\partial \bar{f}_s}{\partial \mu} + \frac{d\varepsilon_g}{dt}\frac{\partial \bar{f}_s}{\partial \varepsilon_g} = \bar{\mathcal{C}}\{\bar{f}_s\}. \tag{39}$$

Here, $\mathbf{v}_\parallel \equiv (\mathbf{B}\boldsymbol{\cdot}\mathbf{v})\,\mathbf{B}/B$ is the parallel velocity, $\mathbf{v}_d \equiv \sum \mathbf{F}_g \times \mathbf{B}/q_s B^2$ is the order ϱ_* smaller drift velocity caused by the forces $\mathbf{F}_g$ on the particle's guiding center due to $q_s\mathbf{E}$, $-\mu\boldsymbol{\nabla}B$ and $-m_s v_\parallel^2\boldsymbol{\kappa}$ forces in which $\boldsymbol{\kappa} \equiv (\mathbf{B}/B)\boldsymbol{\cdot}\boldsymbol{\nabla}(\mathbf{B}/B)$ is the magnetic-field-curvature vector. The magnetic moment μ is usually a good adiabatic invariant so $d\mu/dt$ is typically negligible. Finally, $d\varepsilon_g/dt \simeq q_s\,\partial\phi/\partial t + \mu\,\partial B/\partial t + q_s(\mathbf{v}_\parallel + \mathbf{v}_d)\boldsymbol{\cdot}\mathbf{E}$ in which $d/dt = \partial/\partial t + (\mathbf{v}_\parallel + \mathbf{v}_d)\boldsymbol{\cdot}\partial/\partial\mathbf{x}_g$. As indicated by the overbar, the Fokker-Planck Coulomb collision operator $\bar{\mathcal{C}}\{\bar{f}_s\}$ is also averaged over the gyromotion. When the gyromotion-averaged drift-kinetic and Coulomb collision operators are used in the extended Chapman-Enskog equation for the gyro-averaged kinetic distortion $\bar{F}_s$ in (38), the right side of this equation (last two lines) is also averaged over the gyromotion.[32,33]

For magnetized plasmas the most comprehensive formalism for the d/dt operator in the plasma kinetic equation (1) is the gyrokinetic formalism.[35] It expands the drift-kinetic formalism by allowing for perturbations with perpendicular wavelengths on the order of the gyroradii ϱ_s of the charged particles. Gyro-averaged forms of the Fokker-Planck Coulomb collision operator have been developed.[36] To date no kinetic version of the gyro-averaged Chapman-Enskog version of the perturbed kinetic equation has been developed for the gyrokinetic formalism.

VIII. LOW COLLISIONALITY CLOSURES, EFFECTS IN TOKAMAKS

Tokamaks with their topologically toroidal magnetic field geometry provide an important paradigm for developing useful closures for magnetically confined plasmas because: 1) they are 2D (toroidally axisymmetric) to lowest order; 2) there are many tokamak plasma experiments worldwide; and 3) the international magnetic fusion experiment ITER[1] currently being built employs this geometry. Tokamaks plasmas are low collisionality plasmas since collision lengths of charged particles traveling along their helical toroidal magnetic field lines are usually longer than the tokamak's toroidal circumference. However, radial excursions of particles from the lowest order toroidally axisymmetric nested magnetic flux surfaces due to gyromotion and drifts off the flux surfaces are much smaller than the tokamak's minor radius (i.e., $\varrho_* \ll 1$). Also, electromagnetic field and distribution function perturbations are similarly small. Thus, a systematic small gyroradius and perturbation expansion can be used to develop[37,38] tokamak plasma descriptions on successively longer time scales that ultimately extend to plasma confinement on time scales much longer than collision times.

The fastest time scales in tokamaks are those for plasma oscillations (Langmuir waves) and gyromotion. However, since the lowest order distribution functions are Maxwellians, these phenomena usually do not produce any instabilities or other problemmatic effects. Plasma oscillations enforce charge neutrality ($\rho_q = 0$ so $\boldsymbol{\nabla\cdot}\mathbf{J} = 0$) and the gyromotion can be averaged over. The next shortest tokamak time scales are those for Alfvén waves, which are governed by ideal MHD. Since ideal MHD macroinstabilities[39] grow on the very short Alfvénic time scales ($t_{\rm A} \sim L/c_A \lesssim 10^{-7}$ s), their stability boundaries produce important constraints (limits on toroidal current in the plasma relative to the magnetic field strength and plasma pressure relative to the magnetic field energy density) on operational regimes in tokamaks. In ideal MHD stable tokamak plasmas, compressional Alfvén waves perpendicular to the magnetic field enforce the plasma radial force balance $\mathbf{J}_0\boldsymbol{\times}\mathbf{B}_0 = \boldsymbol{\nabla}P_0 = \boldsymbol{\nabla}P_0(\psi)$. Together with the Maxwell equations $\boldsymbol{\nabla\times}\mathbf{B}_0 = \mu_0\mathbf{J}_0$ and $\boldsymbol{\nabla\cdot}\mathbf{B}_0 = 0$ this yields the 2D Grad-Shafranov equilibrium[39] for the axisymmetric nested toroidal equilibrium magnetic flux surfaces $\psi(\mathbf{x})$ which satisfy $\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\psi = 0$. They also enforce radial ion force balance for each ion species ($0 = \boldsymbol{\nabla}\psi\boldsymbol{\cdot}[n_i q_i(\mathbf{E}_0 + \mathbf{V}_i\boldsymbol{\times}\mathbf{B}_0) - \boldsymbol{\nabla}p_i]$) and incompressible flows within a flux surface.

Many interesting phenomena arise in tokamak plasmas on the longer collision time scales — collisional equilibration toward Maxwellian distributions, distinctions between the behavior of particles that are trapped and untrapped [in the $|\mathbf{B}_0| \sim B_0(1 - \epsilon\cos\theta)$ magnetic field wells along the helical magnetic field lines], shear-Alfvén, resistive and diamagnetic-flow-driven ("drift") waves and instabilities etc. On this time scale equilibrium flows are a factor of ρ_* smaller than their thermal speeds and incompressible ($\boldsymbol{\nabla\cdot}\mathbf{V}_s = 0$), and flow only within magnetic flux surfaces. Thus, from a combination of radial force balance equations and flow incompressibilities they can be written in terms of their components parallel to ($\|$) and cross ($\wedge$, perpendicular to but within a flux surface) the equilibrium magnetic field:

$$\mathbf{V}_s = V_\| \mathbf{B}_0/B_0 + \mathbf{V}_\wedge, \quad V_\| \equiv \mathbf{B}_0 \boldsymbol{\cdot} \mathbf{V}_s/B_0, \quad \mathbf{V}_\wedge \equiv \mathbf{B}_0 \times \boldsymbol{\nabla} \Phi_0/B_0^2 + \mathbf{B}_0 \times \boldsymbol{\nabla} p_s/(n_s q_s B_0^2). \quad (40)$$

Since $\mathbf{E}_0 = -\boldsymbol{\nabla}\Phi_0 = -\,\mathbf{e}_\rho d\Phi_0/d\rho$ the first contribution to $\mathbf{V}_\wedge$ is the $\mathbf{E}_0 \times \mathbf{B}_0$ flow. The last contribution is the "diamagnetic" flow caused by the radial pressure gradient of species s. The effects of collision-induced closures on parallel electron and ion flows are discussed next.

Flux-surface-averages (FSA, $\langle ... \rangle$) of key plasma equations are relevant in low collisionality tokamak plasmas because the electron collision length ($\lambda_e \equiv v_{Te}/\nu_e \sim 10^3$ m) is usually much longer than the toroidal circumference ($\gtrsim 10$ m). The FSA of the parallel (to $\mathbf{B}_0$) equilibrium electron force balance in (28) yields the tokamak parallel Ohm's law:[37,38]

$$0 = \langle \mathbf{B}_0 \boldsymbol{\cdot} [-\, n_e e \mathbf{E}^A - \boldsymbol{\nabla\cdot}\, \boldsymbol{\pi}_{e\|} + \mathbf{R}_e] \rangle \implies \langle \mathbf{B}_0 \boldsymbol{\cdot} \mathbf{J}_0 \rangle \;=\; \sigma_\|^{\rm SP} \left[\langle \mathbf{B}_0 \boldsymbol{\cdot} \mathbf{E}^A \rangle + \frac{\langle \mathbf{B}_0 \boldsymbol{\cdot} \boldsymbol{\nabla\cdot}\, \boldsymbol{\pi}_{e\|} \rangle}{n_e e} \right]. \quad (41)$$

Here, $\mathbf{E}^A = -\,\partial \mathbf{A}/\partial t \simeq -\,(\partial \Psi/\partial t)\boldsymbol{\nabla}\zeta$ is the inductive electric field induced by temporal increases in the magnetic flux Ψ in the Ohmic transformer, $\sigma_\|^{\rm Sp}$ is the Spitzer parallel electrical conductivity[29] induced by the collisional friction force $R_{e\|} = -m_e n_e \nu_e (V_{e\|} - V_{i\|}) \sim n_e e J_\|/\sigma_\|^{\rm Sp}$ and $\langle \mathbf{B}_0 \boldsymbol{\cdot} \boldsymbol{\nabla\cdot} \boldsymbol{\pi}_{e\|} \rangle$ is the "neoclassical" parallel viscous force closure[40] induced by parallel stress in the electron fluid. It is caused by collisions of the untrapped electrons which carry FSA flows along field lines with trapped particles.

The electron closure produces viscous drags at a rate $\mu_e \lesssim \nu_e$ on the $\overline{\mathbf{E}}^A$- and diamagnetic-flow-induced electron flows. Thus, the $\langle \mathbf{B}_0 \boldsymbol{\cdot} \boldsymbol{\nabla\cdot} \boldsymbol{\pi}_{e\|} \rangle$ equilibrium ($t > 1/\nu_e \sim 10^{-5}$ s) closure[40] reduces the electrical conductivity and yields a neoclassical parallel resistivity $\eta_\|^{\rm nc} \simeq (1 + \mu_e/\nu_e)/\sigma_\|^{\rm Sp}$. It also causes a plasma diamagnetic-flow-induced "bootstrap" current $\langle \mathbf{B}_0 \boldsymbol{\cdot} \mathbf{J}_{\rm bs} \rangle \propto -\,(\mu_e/\nu_e)\, dP_0/d\psi$. Time-dependent effects in the tokamak Ohm's law for $t \lesssim 1/\nu_e$ have been explored using a dynamic closure for $-\,\langle \mathbf{B}_0 \boldsymbol{\cdot} \boldsymbol{\nabla\cdot} \boldsymbol{\pi}_{e\|} \rangle$ that was obtained[41] by solving the Chapman-Enskog drift-kinetic equation (38) for the untrapped electron distribution function distortion F_e^{ut} using a procedure analogous to the Green-function approach (with velocity-space boundary conditions) discussed above.

Because of the greater mass of the ions ($\sqrt{m_i/m_e} \geq \sqrt{m_D/m_e} \sim 60$), the ion collision time scale ($t_i \sim 1/\nu_i \sim 10^{-3}$ s) is a factor of 100 longer than the electron collision time scale, but still much shorter than the transport time scale ($\tau_E \sim$ s). The FSA of the lowest order parallel component of the total plasma momentum equation (34) yields[37,38]

$$\rho_m \frac{\partial \langle B_0 V_\| \rangle}{\partial t} \simeq -\sum_s \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{s\|} \rangle \simeq -\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\|} \rangle. \tag{42}$$

As indicated at the end, the electron parallel viscous force is much smaller [$\sim (m_e/m_i)^{1/2} \lesssim 1/60$] than the analogous ion force and can be neglected. The lowest order parallel ion viscous force closure has no effect on the ion flow in the toroidal direction (ζ, long way around the torus) — because of the lowest order toroidal axisymmetry of tokamaks. However, since parallel flow along field lines is composed of both toroidal and poloidal (θ, short way around the torus) ion flows, the parallel ion viscous force closure $\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\|} \rangle \sim \rho_m \mu_i (\mathbf{V}_i \cdot \boldsymbol{\nabla}\theta / \mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta)$ damps the poloidal ion flow at a rate $\mu_i \lesssim \nu_i$ to an ion-temperature-gradient-driven value.[40] Time-dependent effects on the parallel (poloidal) ion flows have been explored using a dynamic closure for $\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\|} \rangle$ analogous to that developed for the electrons.[41] Combined with the MHD-enforced radial ion force balance, on time scales longer than the ion collision time (i.e., for $t > 1/\nu_i$) this yields the equilibrium FSA toroidal ion flow rotation frequency

$$\Omega_\mathrm{t} \equiv \langle \mathbf{V}_i \cdot \boldsymbol{\nabla}\zeta \rangle = -\left(\frac{d\Phi_0}{d\psi} + \frac{1}{n_i q_i} \frac{dp_i}{d\psi} \right) + \frac{c_p}{q_i} \frac{dT_i}{d\psi} = \frac{1}{R B_\mathrm{p}} \left(E_\rho - \frac{1}{n_i q_i} \frac{dp_i}{d\rho} + \frac{c_\mathrm{p}}{q_i} \frac{dT_i}{d\rho} \right). \tag{43}$$

The first two terms on the right hand side here are caused by the equilibrium $\mathbf{E}_0 \times \mathbf{B}_0$ and ion diamagnetic flows while the last term is due to the residual poloidal ion flow ($c_p \lesssim 1.17$ is a collisionality-determined numerical coefficient[40]) enforced by the closure $-\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\|} \rangle$. The result in (43) relates the plasma toroidal rotation frequency Ω_t to the radial electric field, but does not determine either of them. Equation (46) below provides a transport equation that can be solved for Ω_t, which then determines the radial electric field via (43).

Unstable resistive MHD modes[42] and drift waves[43] also develop on the electron and ion collision time scales. These instabilities typically have small amplitudes and short radial correlation lengths compared to the plasma radius. The main effects of these fluctuations (quantities denoted by tildes over them, e.g., $\tilde{\phi}$) are to produce microturbulence that creates longer time scale "anomalous" transport whose effects will be taken into account below.

MHD-like $e^{i(m\theta - n\zeta - \omega t)}$ tearing modes cause $\tilde{\mathbf{J}} \cdot \mathbf{B}_0 \neq 0$ and hence $\tilde{\mathbf{B}} \cdot \boldsymbol{\nabla}\psi \neq 0$ which produces resistivity-driven reconnection of field lines at rational flux surfaces where $q(\psi) \equiv \mathbf{B}_0 \cdot \boldsymbol{\nabla}\zeta / \mathbf{B}_0 \cdot \boldsymbol{\nabla}\theta = m/n$ (ratio of integers). Unstable tearing modes with low m/n (e.g., 2/1) can bifurcate the magnetic field topology by introducing 3D magnetic island structures at the rational surfaces. Plasma transport in the vicinity of such islands flattens the T_e profile there, short-circuits transport in this region and reduces plasma confinement in proportion to the width of the island.[44] There are two types of tearing modes. Classical[42] tearing modes are derived using resistive MHD, which adds resistivity effects to the ideal MHD model. They are driven unstable by a large enough radial gradient of the toroidal current density. Neoclassical tearing modes (NTMs) are derived from the neoclassical MHD equations[45] which add the parallel viscous force closures $-\langle \mathbf{B}_0 \boldsymbol{\cdot} \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\pi}_{s\|} \rangle$ to resistive MHD. They are driven by a sufficiently large bootstrap current produced by the radial plasma pressure gradient and, if not controlled, can produce significant additional limits on the plasma pressure that can be achieved in tokamak plasmas.[46,47]

IX. TOKAMAK PLASMA TRANSPORT EQUATIONS

Plasma transport in tokamaks takes place "'radially" across the nested magnetic flux surfaces ψ in MHD stable plasmas. However, in the fluid moment equations (27)–(29) all quantities are evaluated at laboratory positions $\mathbf{x}$ and the partial time derivatives are at constant $\mathbf{x}$. Thus, the fluid moment equations need to be transformed from laboratory to magnetic flux coordinates. The axisymmetric magnetic field in a tokamak can be written as $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ with $\mathbf{A} = \psi_{\rm t} \boldsymbol{\nabla} \theta - \psi_{\rm p} \boldsymbol{\nabla} \zeta$ in which $2\pi\, \psi_{\rm t}(\mathbf{x}, t)$ and $2\pi\, \psi_{\rm p}(\mathbf{x}, t)$ are the toroidal and poloidal magnetic fluxes. The tokamak equilibrium magnetic field is written as $\mathbf{B}_0 \equiv \mathbf{B}_{\rm t} + \mathbf{B}_{\rm p} = I \boldsymbol{\nabla} \zeta + \boldsymbol{\nabla} \zeta \times \boldsymbol{\nabla} \psi_{\rm p} = \boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} [q(\psi)\theta - \zeta]$. In tokamaks the toroidal magnetic flux is relatively constant in time since it is determined mainly by steady currents flowing in toroidal magnetic field coils; thus, $\partial \psi_{\rm t} / \partial t |_{\mathbf{x}} \simeq 0$. In contrast, the poloidal magnetic flux can change on the transport time scale relative to the nearly stationary toroidal flux:[37,40]

$$\dot{\psi}_{\rm p} \equiv \left. \frac{\partial \psi_{\rm p}}{\partial t} \right|_{\psi_{\rm t}} = D_\eta \, \Delta^+ \psi_{\rm p} - S_{\psi_{\rm p}}, \quad S_{\psi_{\rm p}} = \frac{\partial \Psi_{\rm p}}{\partial t} + \frac{\eta_\|^{\rm nc}}{I \langle R^{-2} \rangle} \langle \mathbf{B}_0 \boldsymbol{\cdot} \mathbf{J}_{\rm bs} \rangle. \tag{44}$$

Here, $D_\eta \equiv \eta_\|^{\rm nc} / \mu_0$ is the magnetic field diffusivity caused by the parallel neoclassical resistivity, Δ^+ is a second order cylindrical-like differential operator in ρ and $S_{\psi_{\rm p}}$ represents the sources of poloidal magnetic flux produced by the Ohmic transformer and bootstrap current.

Collision-induced neoclassical (drift-orbit-induced) plasma transport[40,48] in tokamaks is calculated relative to poloidal magnetic flux surfaces — because to lowest order the canonical toroidal angular momentum $p_\zeta \equiv \mathbf{e}_\zeta \boldsymbol{\cdot} (m_s \mathbf{v} + q_s \mathbf{A}) = m_s R^2 \mathbf{v} \boldsymbol{\cdot} \boldsymbol{\nabla} \zeta - q_s \psi_{\rm p}$ ties the particles to a particular poloidal flux function value $\psi_{\rm p}$ to lowest order in $\rho_* \ll 1$. Here, $\mathbf{e}_\zeta \equiv$

$R^2\boldsymbol{\nabla}\zeta = R\,\hat{\mathbf{e}}_\zeta$ is the (covariant) toroidal angular base vector. Microinstabilities in the plasma and the transport they induce are also calculated relative to $\psi_{\rm p}$. Thus, in order to describe radial plasma transport in tokamak plasmas, the moment equations in (27)–(29) are first transformed[37,38,40] from laboratory coordinates to poloidal magnetic flux coordinates. However, because the toroidal flux is nearly constant in time, the radial coordinate that is used is the toroidal-flux-based average minor radius $\rho \equiv \sqrt{\psi_{\rm t}/\pi B_{\rm t0}}$ (m) in which $B_{\rm t0}$ is the toroidal magnetic field strength at the center of the plasma (magnetic axis). After adding sources of density S_n, momentum $\mathbf{S}_m$ and energy S_E and taking the FSA, the tokamak transport equations for the plasma density n and total toroidal angular momentum density $L_{\rm t} \equiv \rho_m \langle R^2\rangle\,\Omega_{\rm t}$, and species pressures p_s are[38]

$$\text{density:}\quad \frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}}\langle n\rangle V' + \dot{\rho}_{\psi_{\rm p}}\frac{\partial\langle n\rangle}{\partial\rho} + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\Gamma) = \langle \overline{S}_n\rangle, \tag{45}$$

$$\text{tor. mom.:}\quad \frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} L_t V' + \dot{\rho}_{\psi_{\rm p}}\frac{\partial L_t}{\partial\rho} + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\,\overline{\Pi}_{\rho\zeta}) = \langle \mathbf{e}_\zeta\boldsymbol{\cdot}\big(\overline{\mathbf{J}\times\mathbf{B}} - \boldsymbol{\nabla\cdot}\overline{\boldsymbol{\Pi}} + \overline{\mathbf{S}}_m\big)\rangle, \tag{46}$$

$$\text{energy:}\quad \frac{3}{2}\langle p_s\rangle\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}}\ln\langle p_s\rangle V'^{5/3} + \frac{3}{2}\dot{\rho}_{\psi_{\rm p}}\frac{\partial\langle p_s\rangle}{\partial\rho} + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\Upsilon_s) + \langle\boldsymbol{\nabla\cdot}\mathbf{q}^{\rm pc}_{s*}\rangle = \overline{Q}_{s\rm net}. \tag{47}$$

Here, $V' \equiv dV/d\rho$ (m^2) is the radial derivative of the volume $V(\rho)$ (m^3) of the ρ surface. Further, $V'\langle n\rangle d\rho$ and $V' L_{\rm t} d\rho$ are the number of plasma particles and plasma toroidal angular momentum between the ρ and $\rho + d\rho$ flux surfaces. Both are adiabatic (isentropic) plasma properties that are conserved in the absence of closures and transport time scale sources. Similarly, $\ln\langle p_s\rangle V'^{5/3} d\rho$ is the collisional entropy density between the ρ and $\rho + d\rho$ flux surfaces, which is also a conserved quantity in the absence of closures and sources. Further, $\dot{\rho}_{\psi_{\rm p}} \equiv -\,\dot{\psi}_{\rm p}/\psi'_{\rm p}$ in which $\psi'_{\rm p} \equiv \partial\psi_{\rm p}/\partial\rho = RB_{\rm p}$ takes account of $\psi_{\rm p}$ surface motion relative to the $\psi_{\rm t}$-based radial coordinate ρ.

The closure and source contributions to the tokamak plasma transport equations in (45)–(47) will now be be discussed sequentially — first those in the density, then toroidal rotation and finally energy equations. The preceding discussion has considered the radial and parallel components of the species momentum equations. The linearly-independent third component considered will be the toroidal component. The species s force balance equation in (28) can be written in the form $\mathbf{0} = n_s q_s \mathbf{V}_s\times\mathbf{B}_0 + \sum_j \mathbf{F}_{sj}$. The toroidal ($\mathbf{e}_\zeta$) component of this equation shows force densities $\mathbf{F}_{sj}$ that have toroidal components induce radial density fluxes: $q_s\mathbf{e}_\zeta\boldsymbol{\cdot}\mathbf{V}_s\times\mathbf{B}_0 = q_s n_s\mathbf{V}_s\boldsymbol{\cdot\nabla}\psi_{\rm p} = -\,\mathbf{e}_\zeta\boldsymbol{\cdot}\sum_j\mathbf{F}_{sj}$. Thus, a FSA toroidal torque density $T_{s\zeta} \equiv \langle\mathbf{e}_\zeta\boldsymbol{\cdot}\mathbf{F}_s\rangle = \langle R\,\hat{\mathbf{e}}_\zeta\boldsymbol{\cdot}\mathbf{F}_s\rangle$ induces a FSA radial density flux $\Gamma_s \equiv \langle n_s\mathbf{V}_s\boldsymbol{\cdot\nabla}\rho\rangle = -\,T_{s\zeta}/q_s\psi'_{\rm p}$.

There are a large number of force densities $\mathbf{F}_{sj}$ in tokamak plasmas that have toroidal components and hence induce radial density fluxes. Electron and ion torques per unit charge that are equal in magnitude but opposite in sign produce equal density fluxes and hence no radial current in the plasma; these are intrinsically ambipolar fluxes Γ_s^a. Eight intrinsically ambipolar density fluxes have been identified:[37] 7 collision-based ones (classical,[29] neoclassical,[37,40,48] paleoclassical[37,49] etc.) plus that caused by microturbulence-induced fluctuating $\tilde{\mathbf{E}}\times\mathbf{B}_0$ flows ($\Gamma^a_{\tilde{n}\tilde{V}_\rho} = \overline{\tilde{n}_s\tilde{\mathbf{E}}\times\mathbf{B}_0/B_0^2\cdot\boldsymbol{\nabla}\rho}$) (see Erratum in Ref. 37) in which the overline indicates an average over the toroidal symmetry angle ζ. Unequal electron and ion torques per unit charge produce non-ambipolar density fluxes Γ_s^{na}. Eight nonambipolar density fluxes caused by toroidal torques induced by microturbulence-induced Reynolds and Maxwell stresses, polarization flows from toroidal rotation transients, momentum sources due to neutral beam injection etc. have been identified.[37]

The FSA plasma toroidal rotation equation in (46) results from[37] setting the net radial current induced by all the nonambipolar fluxes to zero. Solving this equation for the plasma toroidal angular momentum density L_t determines the plasma toroidal rotation frequency $\Omega_\mathrm{t}^{\mathrm{amb}}$ and thus from (43) the radial electric field $E_\rho^{\mathrm{amb}} \equiv -\,d\Phi_0^{\mathrm{amb}}/d\rho$ required for net ambipolar transport in the plasma. The electron and ion density fluxes become equal for this ambipolar radial electric field. Thus, the total plasma ambipolar density flux Γ in (45) is given by[37]

$$\Gamma = \Gamma_e^a + \Gamma_e^{na}(E_\rho^{\mathrm{amb}}) = \Gamma_i^a + \Gamma_i^{na}(E_\rho^{\mathrm{amb}}). \tag{48}$$

Note that the FSA plasma density $\langle n\rangle$ in (45) is either the electron density $\langle n_e\rangle$ or the total ion density $\langle \sum_i n_i\rangle$, which are equal because of the ambipolar transport constraint. The FSA source $\langle \overline{S}_n\rangle$ on the right of (45) represents the net ambipolar particle source in the plasma produced by ionization, neutral beam injection etc.

For this fluid-based model the radial flux of plasma toroidal angular momentum in (46) induced by the Reynolds stress closure caused by microturbulence is[37]

$$\overline{\boldsymbol{\Pi}}_{\rho\zeta} \equiv \sum_s \overline{\boldsymbol{\pi}}_{s\rho\zeta}, \quad \overline{\boldsymbol{\pi}}_{s\rho\zeta} = m_s\langle n_s\rangle\,\langle\,\boldsymbol{\nabla}\rho\cdot\overline{\tilde{\mathbf{V}}_s\tilde{\mathbf{V}}_s}\cdot\mathbf{e}_\zeta\rangle + \langle\boldsymbol{\nabla}\rho\cdot\overline{\boldsymbol{\pi}}_{\wedge s}\cdot\mathbf{e}_\zeta\rangle. \tag{49}$$

As before, the overline indicates an average over the toroidal angle. The $\overline{\boldsymbol{\pi}}_{\wedge s}$ term represents the (collisionless) gyroviscous stress.[37] A microturbulence-induced contribution $\langle\mathbf{e}_\zeta\cdot\overline{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}\,\rangle$ to the right hand side of (46) produces the corresponding Maxwell stress effect; an externally-induced magnetic flutter contribution to $\langle\mathbf{e}_\zeta\cdot\overline{\mathbf{J}\times\mathbf{B}}\rangle$ is discussed in Appendix A. Finally, the

$\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \overline{\boldsymbol{\Pi}} \rangle$ and $\langle \mathbf{e}_\zeta \cdot \overline{\mathbf{S}}_m \rangle$ terms on the right of (46) represent effects of toroidal viscous forces (examples are discussed in Appendix A) and externally supplied toroidal momentum sources. More details about the toroidal momentum equation (torque balance) are given in Appendix A which discusses the primary torques caused by 3D magnetic fields in tokamaks.[50]

Determining the closure contributions to radial energy transport is straightforward because there is no ambipolar constraint involved. The heat flux for each species is

$$\Upsilon_s \equiv \left\langle \left(\overline{\mathbf{q}}_s + (5/2)\langle n_s \rangle \, \overline{\tilde{T}_s \tilde{\mathbf{V}}_s} \right) \cdot \boldsymbol{\nabla} \rho \right\rangle + (5/2)\langle T_s \rangle \, \Gamma. \tag{50}$$

Here, $\langle \overline{\mathbf{q}}_s \cdot \boldsymbol{\nabla} \rho \rangle$ represents mainly the FSA classical,[29] neoclassical,[38,40,48] and axisymmetric paleoclassical[49] radial heat fluxes. The term with $\overline{\tilde{T}_s \tilde{\mathbf{V}}_s}$ in it is caused by microturbulence. The other two terms in (50) represent the convective heat fluxes caused by the ambipolar density flux Γ. The final transport contribution to (47) is caused by resonant helical (subscript $*$) paleoclassical transport processes and has a non-standard transport operator:[49]

$$\langle \boldsymbol{\nabla} \cdot \mathbf{q}_{s*}^{\rm pc} \rangle = -\,\frac{M_s}{V'} \frac{\partial^2}{\partial \rho^2} \left(V' \overline{D}_\eta \, \frac{3}{2} \, \langle p_s \rangle \right) + \frac{3}{2} \, \dot{\rho}_{\psi_*} \frac{\partial \langle p_s \rangle}{\partial \rho} \tag{51}$$

Here, $M_e \lesssim 10$ is a factor representing the distance, relative to the poloidal periodicity length, over which the electron temperature is equilibrated as it diffuses radially with the helical magnetic flux ψ_*. In most tokamak plasmas M_i is small;[49] thus, this paleoclassical ion contribution usually can be neglected. Also, $\dot{\rho}_{\psi_*} \equiv -\, q \, (\partial \dot{\psi}_{\rm p}/\partial \rho)/q' \psi'$ reflects ρ motion to stay on the same ψ_* surface during poloidal flux transients.[38,49]

The $\overline{Q}_{s\rm net}$ term on the right hand side of (47) represents the sum of all the FSA energy sources and sinks for the s species — electron-ion collisional energy transfer, ohmic heating, and auxiliary, externally supplied heat sources (S_E) introduced via energetic neutral beams and radio frequency waves, radiative energy losses etc.

X. OPEN ISSUES

There are four classes of open issues regarding development of fluid-moment-based descriptions of low collisionality plasmas. This first class concerns the need for temporally evolving Green functions that include collisional effects in combination with the lowest order deterministic particle trajectories — beyond the "simple" Landau damping Green function presented in Section V. For example, Green functions would be useful for: more complete

Fokker-Planck Coulomb collision operators, longer time scales, plasmas in uniform magnetic fields, the drift-kinetic equation, the gyrokinetic equation and relativistic plasmas.

Second, closures need to be obtained by taking velocity-space moments of the general low collisionality Green function solution given in (7). The distribution function to be used in this evaluation should be the kinetic distortion F_s obtained from solving the perturbed extended Chapman-Enskog plasma kinetic equation in (38). Closures obtained from such comprehensive Green function solutions would be useful for linear Landau damping, and Chapman-Enskog versions of the drift-kinetic and gyrokinetic equations. These closures may also be useful for developing comprehensive "extended MHD" descriptions of magnetically confined toroidal plasmas that can be solved numerically.

Third, it would be useful to develop appropriately averaged fluid moment equations through transport time scales for other applications. For example, they could be developed as modest extensions of those that have been developed for tokamaks (Section VIII) for plasmas in stellarators and reversed field pinches, which are also closed toroidal magnetic field systems. In addition, it would be useful to develop plasma transport equations for plasmas confined in open-ended magnetic systems such as magnetic mirrors, plasmas in the earth's magnetosphere, and most generally space and astrophysical plasmas.

Fourth, it would be helpful if rigorous procedures for including effects that are often thought to be outside the realm of fluid moment equations could be developed. Or, if that is not feasible, develop criteria for regions of plasmas where a fluid moment description is not feasible and procedures for interfacing the fluid moment description to those regions. An example of such effects is the need for descriptions within a collision length of sheath-type boundaries where magnetic fields intersect limiting surfaces or between regions with vastly different plasma parameters. Other examples include bifurcated island structures in the magnetic field (or in a phase space of higher dimensionality), mixtures of magnetic islands,[44] flutter[51–53] and stochasticity,[54] and perhaps regions characterized by fractal dimensions[55] or localized perhaps avalanche-type behavior.[56]

XI. SUMMARY

A new general framework for developing comprehensive fluid moment descriptions of low collisionality plasmas has been proposed in this paper. The first step in this procedure is to obtain a Green function solution of (6) for the application-specific linear operator $\mathcal{L}$ that produces the appropriate lowest order deterministic plus Coulomb-collision-induced probabilistic charged particle trajectories. The kinetic source function S for (2) should be determined from the extended Chapman-Enskog form of the plasma kinetic equation in (38) for the kinetic distribution distortion F_s — to ensure closure relations obtained from these kinetic solutions are consistent with the fluid moment equations. Then, the initial value solution of the perturbed plasma kinetic equation in (2) or (38) is determined by the Green function solution in (7). Finally, this perturbed kinetic distribution solution is used to calculate the closure relations for the heat flux $\mathbf{q}_s$ and stress tensor $\boldsymbol{\pi}_s$ defined in (30) and (31). [Since the closure relations $\mathbf{q}_s$ and $\boldsymbol{\pi}_s$ also appear as sources in the extended Chapman-Enskog kinetic equation (38), the closure relations obtained from (30) and (31) may depend on $\mathbf{q}_s$ and $\boldsymbol{\pi}_s$. Thus, these implicit closure equations for $\mathbf{q}_s$ and $\boldsymbol{\pi}_s$ may need to solved to obtain the net closures, as was done, for example, in Ref. 41.] The resultant closure relations are used in the then exact fluid moment equations for n_s, $\mathbf{V}_s$ and p_s in (27)–(29), which can be used to develop comprehensive extended-MHD-type plasma descriptions on ideal MHD, collisional and transport time scales for specific applications.

This first steps in such a procedure have been illustrated in Section IV by exploring the effects of Coulomb-collision-induced velocity-space scattering on linear Landau damping in an infinite, homogeneous, unmagnetized plasma. The relevant Green function that includes lowest order straight-line particle trajectories and diffusive pitch-angle scattering is given in (15) and the resultant perturbed distribution function in (19)–(23). This solution shows that the perturbed response is in phase with the applied potential perturbation up to the collision-induced damping time in (18) $\tau_{\text{damp}} \sim 1/\sqrt{\nu\omega}$, after which the response dissipatively decorrelates from the initial perturbation and becomes irreversible. This collision-based solution reproduces the usual Landau prescription in (13) for resolving the $u \simeq V_\varphi$ singular response for both short time scales via a phase mixing approximation and long time scales via collisional scattering effects, without the equations needing to be Laplace transformed.

A multiple time scale analysis of a systematic small gyroradius and perturbation level expansion has been used to obtain equations governing tokamak plasmas on ideal MHD, collisional and transport time scales. In ideal MHD stable plasmas this yields: determination of the axisymmetric equilibrium magnetic flux surfaces and an ion radial force balance equation (43) on the Alfvén time scale, electron-collision-induced parallel Ohm's law (41) and ion collisional damping (42) of the parallel/poloidal flow on collision times scales, and finally radial transport equations (45)–(47) for the FSA plasma density and toroidal angular momentum (torque) density and species pressures on time scales long compared to collision times. For complete self-consistent transport time scale solutions these $\langle n \rangle$, $L_{\rm t} \equiv \rho_m \langle R^2 \rangle \, \Omega_{\rm t}$ and $\langle p_s \rangle$ equations all need to be solved simultaneously — not just for $\langle n \rangle$, $\langle p_s \rangle$ relying on experimental measurements of $\Omega_{\rm t}$. The "mean field" effects of microturbulence on the parallel Ohms law, poloidal ion flow, density fluxes, and toroidal momentum and energy transport are all included self-consistently in the comprehensive version of these tokamak transport equations.[38] Closures for including the effects of various 3D magnetic field on tokamak plasmas are discussed in Appendix A.

The widely used Braginskii closures[29] are applicable for all collision-dominated plasmas and widely used for them. And they are often approximately "adapted" for low collisionality applications. The novel Green function solution procedure developed in this paper provides a comprehensive, rigorous framework for including Coulomb collisional effects in low collisionality plasmas, and determination of useful closures and the resultant long time scale transport equations. Hopefully the development of this new framework for including low collisionality effects will spur development of new Green functions, closures and transport equations for additional applications beyond the Landau damping and 3D field tokamak applications discussed here.

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Appendix A: Closures for 3D fields And Their Effects

The magnetic field structure in tokamaks is never purely 2D axisymmetic. Small three-dimensional (3D) field components are introduced by a variety of effects — finite toroidal field coils, field errors due to misalignment of coils during assembly, tearing instabilities (e.g., NTMs[45–47]), externally imposed fields etc. A recent topical review paper[50] discussed the effects of various types of 3D magnetic perturbations on tokamak plasmas. An annotated version of the general torque balance equation in (46) that emphasizes the effects of the various possible gyroradius-small 3D fields ($\delta B^{\rm 3D}/B_0 \lesssim 10^{-2}$) on the plasma toroidal angular momentum density $L_t \equiv \sum_{\rm ions} m_i n_i \langle R^2 \mathbf{V}_i \cdot \boldsymbol{\nabla}\zeta\rangle$ which yields $\Omega_t(\rho,t) \equiv L_t/\rho_m\langle R^2\rangle$ is

$$\underbrace{\frac{\partial L_t}{\partial t}}_{\text{inertia}} \simeq -\underbrace{\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}^{\rm 3D}_{i\|}\rangle}_{\text{NTV from } \delta B} + \underbrace{\langle \mathbf{e}_\zeta \cdot \overline{\delta \mathbf{J} \times \delta \mathbf{B}}\rangle}_{\text{FEs, flutter}} - \underbrace{\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}_{i\perp}\rangle}_{\text{cl, neo, paleo}} - \underbrace{\frac{1}{V'}\frac{\partial}{\partial \rho}\left(V' \pi_{i\rho\zeta}\right)}_{\text{Reynolds stress}} + \underbrace{\langle \mathbf{e}_\zeta \cdot {\textstyle\sum_s} \bar{\mathbf{S}}_{sm}\rangle}_{\text{mom. sources}}. \tag{A1}$$

In the absence of 3D field effects, in the hot core of tokamak plasmas the toroidal momentum induced by unidirectional tangential energetic neutral beams that are often used to heat tokamak plasmas is usually balanced by the diffusive radial transport of toroidal momentum caused by the microturbulence-induced ion Reynolds stress $\pi_{i\rho\zeta}$ which is defined in (49). The perpendicular ion viscosities $\bar{\boldsymbol{\pi}}_{i\perp}$ caused by collision-induced classical, neoclassical[40,48] and paleoclassical[37,38,49] processes are usually small in the core but can be important in the lower ion temperature region near the plasma edge.

Most non-resonant 3D fields cause toroidal drags on the plasma toroidal rotation. These effects are represented by closures of the form $\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}}^{\rm 3D}_{i\|}\rangle$. One low collisionality, long-time-scale 3D closure whose effects have been tested experimentally is neoclassical toroidal viscosity (NTV, see Ref. 50 for references), which is induced by Coulomb collisional scattering of radial drifts of the "banana" trajectories of trapped ions caused by 3D-fields:[50,57,58]

$$-\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}^{\rm 3D}_{i\|}\rangle = -\,\mu_{\|\rm t}(\Omega_{\rm t})\,(\delta B^{\rm 3D}/B_0)^2\,\rho_m\langle R^2\rangle\,(\Omega_{\rm t} - \Omega_*). \tag{A2}$$

Here, $\mu_{\|\rm t}(\Omega_{\rm t})\,(\delta B^{\rm 3D}/B_0)^2$ is a viscous damping frequency caused by the toroidal (3D) variation of $|\mathbf{B}|$ along field lines and $\Omega_* \sim [1/(q_i\psi'_{\rm p})](dT_i/d\rho)$ is a (typically negative) "offset" rotation frequency. This NTV torque contributes to the $-\,\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \overline{\boldsymbol{\Pi}}\rangle$ term on the right hand side of (46) and is the first term on the right hand side of (A1). Experimental tests of it have validated[50] the $\delta B^{\rm 3D}$-induced damping of the toroidal torque $L_{\rm t}$ and rotation frequency $\Omega_{\rm t}$ to the offset frequency Ω_*, and the peak in $\mu_{\|{\rm t}}(\Omega_{\rm t})$ at the $\Omega_{\rm t}$ where the corresponding radial electric field [see (43)] nearly vanishes and thus no longer limits the radial excursions of the 3D-field-induced drifts of the bananas. The magnetic field ripple caused by the discrete number of toroidal field coils has similar effects and has been similarly validated experimentally.[50]

Magnetic field errors introduce low n resonant components which induce dissipative torques $\langle \mathbf{e}_\zeta \boldsymbol{\cdot} \overline{\delta \mathbf{J} \times \delta \mathbf{B}} \rangle$ in thin layers around $q = m/n$ rational surfaces.[59] When the field errors are small or the very low resistivity plasma rotates toroidally very rapidly and thereby prevents "penetration" of the field error to the rational surface, these effects are negligible. However, for large enough field errors and low enough toroidal plasma rotation, the penetration threshold is exceeded and the tokamak plasma bifurcates (in a few ms) to a state with no rotation at the rational surface and a growing "locked mode" is induced which often leads to a "major disruption" (collapse of the plasma toroidal current on a very fast time scale $\sim$ ms). The latest theory for and experimental tests of the penetration criteria are discussed in Ref. 50.

Recently, externally applied 3D fields have been used[60] to stabilize ideal MHD "peeling ballooning" (P-B) instabilities which cause deleterious, repetitive edge localized modes (ELMs) in the pedestal at the edge of tokamak plasmas. A new "magnetic flutter" model has recently been developed[52,53,61] to explore plasma transport effects induced by the non-resonant component of the 3D fields which can change profiles of plasma parameters in the pedestal and thereby stabilize P-B instabilities. In this model the radial component $\delta B_\rho \equiv \delta \mathbf{B} \boldsymbol{\cdot} \boldsymbol{\nabla} \rho \neq 0$ of the 3D fields in the pedestal plasma cause non-resonant radial flutter of magnetic field lines between rational surfaces. This causes 3D modifications of the parallel flow and heat flow of barely passing (untrapped) electrons. On the transport time scale, these 3D-flutter-induced effects cause a FSA nonambipolar radial electron density flux $\Gamma_e^{\delta B_\rho}$, a FSA toroidal Maxwell-stress $\langle \mathbf{e}_\zeta \boldsymbol{\cdot} \overline{\delta \mathbf{J}_\| \times \delta \mathbf{B}_\rho} \rangle$ and a FSA radial electron heat flux $\Upsilon_e^{\delta B_\rho}$. Since the flutter-induced toroidal torque can be quite significant in tokamak pedestals, it is important to determine the increase in the radial electric field it induces and the consequent changes in the total density flux in (48) and electron heat transport.[61] Validation tests of the magnetic flutter model are just beginning; they are promising so far.[52,53,61,62]

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