Steady-state Force Balance in the DEBS Code

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The DEBS code advances the induction equation, the energy equation, and the momentum equation with a time centered leap-frog method combined with a predictor-corrector advance for the advective terms. This assures explicit stability as long as the CFL condition $\sqrt{V_A^2 + C_s^2 + V^2}\Delta t/\Delta x < 1$ is satisfied, where V_A is the Alfvén wave speed, C_s is the sound wave speed, and V is the flow speed. Unconditional stability for sound and Alfvén waves with arbitrary time step Δt is achieved by adding a semi-implicit term to the momentum equation. However, the flow CFL condition $V\Delta t/\Delta x <$ must still be satisfied. The resistivity and viscosity are advanced implicitly using fully forward time differencing. These equations can have steady state time-asymptotic solutions with finite flow [e.g., SH (single helicity) and/or QSH (quasi-single helicity) states in the RFP] that in the continuum satisfy the force balance condition

$$\mathbf{V}\cdot\nabla\mathbf{V} = \mathbf{F} + \nu\nabla^2\mathbf{V} \ , \tag{1}$$

where $\mathbf{F} = (\mathbf{J}\times\mathbf{B} - \nabla P)/\rho$ is the ideal MHD force (per unit mass) and the last term is the viscous force. However, time truncation error in the numerical solution may introduce additional forces $\sim \Delta t^N$ that might affect the computed properties of these states if $\Delta t > 0$. Here we quantify the origin and form of these forces.

In the DEBS code the momentum equation has the equivalent continuum form

$$\frac{\partial\mathbf{V}}{\partial t} = -\mathbf{V}\cdot\nabla\mathbf{V} + \mathbf{F} + \nu\nabla^2\mathbf{V} + \sigma\Delta t\nabla^2\frac{\partial\mathbf{V}}{\partial t} \ . \tag{2}$$

The last term is the semi-implicit term that assures stability at arbitrary time step, and $\sigma > 0$ is the semi-implicit coefficient. In the present version of DEBS

this equation is advanced from time step t^n to time step $t^{n+1} = t^n + \Delta t$ using the Lionello algorithm[1], in which the weighted semi-implicit term is included in both predictor and corrector steps. This assures numerical stability in the presence of large flows[1]. The viscous term is advanced separately using operator splitting. The algorithm is

Predictor:

$$\frac{\mathbf{V}^* - \mathbf{V}^n}{\Delta t} = -\mathbf{V}^n \cdot \nabla \mathbf{V}^n + \frac{1}{2}\left(\mathbf{F}^{n+1/2} - \sigma k^2 \Delta t \frac{\mathbf{V}^* - \mathbf{V}^n}{\Delta t}\right) , \tag{3}$$

Corrector:

$$\frac{\mathbf{V}^{**} - \mathbf{V}^n}{\Delta t} = -\mathbf{V}^n \cdot \nabla \mathbf{V}^* + \mathbf{F}^{n+1/2} - \sigma k^2 \Delta t \frac{\mathbf{V}^{**} - \mathbf{V}^n}{\Delta t} , \tag{4}$$

Split Viscosity:

$$\frac{\mathbf{V}^{n+1} - \mathbf{V}^{**}}{\Delta t} = -\nu k^2 \mathbf{V}^{n+1} , \tag{5}$$

where we have written heuristically $\nabla^2 = -k^2$. The viscosity was originally split from the corrector advance to avoid an interaction with the semi-implicit term. Eliminating $\mathbf{V}^*$ and $\mathbf{V}^{**}$ between these equations yields the effective acceleration

$$\begin{aligned} \frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta t} &= \frac{1}{1+\nu k^2 \Delta t}\left[\frac{\mathbf{F}^{n+1/2}}{1+\sigma k^2 \Delta t} - \nu k^2 \mathbf{V}^n - \mathbf{V}^n \cdot \nabla \mathbf{V}^n \right. \\ &\left. \frac{\Delta t}{1+\frac{1}{2}\sigma k^2 \Delta t} \mathbf{V}^n \cdot \nabla \left(\frac{1}{2}\mathbf{F}^{n+1/2} - \mathbf{V}^n \cdot \nabla \mathbf{V}^n\right)\right] . \end{aligned} \tag{6}$$

To lowest order in Δt,

$$\frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta t} = \mathbf{F}^{n+1/2} - \nu k^2 \mathbf{V}^n - \mathbf{V}^n \cdot \nabla \mathbf{V}^n + \delta \mathbf{F}_1 , \tag{7}$$

where

$$\delta \mathbf{F}_1 = -k^2 \Delta t \left[\nu \left(\mathbf{F}^{n+1/2} - \nu k^2 \mathbf{V}^n - \mathbf{V}^n \cdot \nabla \mathbf{V}^n\right) + \sigma \left(\mathbf{F}^{n+1/2} - \mathbf{V}^n \cdot \nabla \mathbf{V}^n\right)\right] , \tag{8}$$

is an additional numerical force introduced by time discretization that is $O(\Delta t)$. In steady state $\mathbf{V}^{n+1} = \mathbf{V}^n$, so that the numerical force can affect the physical force balance condition as given by Equation (1), and hence the properties of the steady state solutions. This effect may be significant if $k^2 \Delta t$ is large.

[1]In fact, the original DEBS algorithm that included the semi-implicit term only in the corrector step was shown to be *unconditionally unstable* for finite flows. Stability was attained only if the viscosity was sufficiently large.

It is also possible to treat the viscosity in an unsplit manner. In that case the viscous force is included implicitly on the right hand side of the corrector, Equation (4), $\mathbf{V}^{**} \rightarrow \mathbf{V}^{n+1}$, and Equation (5) is not applied. Then eliminating $\mathbf{V}^*$ as above, to lowest order in Δt the residual numerical force is

$$\delta \mathbf{F}_2 = -\left(\nu + \sigma\right) k^2 \Delta t \left(\mathbf{F}^{n+1/2} - \nu k^2 \mathbf{V}^n - \mathbf{V}^n \cdot \nabla \mathbf{V}^n \right) \ , \tag{9}$$

which is not formally different from Equation (8). However, if all the "physical" forces are in balance at $t^{n+1/2}$, then $\delta \mathbf{F}_2 = 0$ and the unbalanced forces appear only at order Δt^2.

The above analysis applies only to the differencing of the momentum equation. The force $\mathbf{F}^{n+1/2}$, which is determined by the induction and energy equations, also contains truncation errors that are $O(\Delta t)$ and could change the precise form of Equations (8) and (9).

It has been shown rigorously [2] that the marginal stability point ($\omega = 0$) of the linearized, incompressible visco-resistive MHD equations in doubly periodic cylindrical geometry depends only on the Pinch Parameter $\Theta_0 = B_\theta(a)/B_0$ and the Hartmann number $H = 1/\sqrt{\nu\eta}$, where ν and η are the suitably normalized kinematic viscosity and resistivity. It has also been shown [3] that, when inertia (the total time derivative $d\mathbf{V}/dt$) can be neglected, the properties of non-linear steady states of the force-free ($\nabla P = 0$) visco-resistive MHD equations, such as SH (single helicity) and QSH (quasi-single helicity) states, also depend only on these same two dimensionless parameters. It was further speculated that the *transition* from steady SH states to dynamical MH (multi-helicity dynamo) states also depends only on Θ_0 and H; this was supported by computational results [3].

Some recent results with DEBS using the split viscosity algorithm given by Equations (3-5), for the same parameters used in Reference [3], indicated that, while the marginal stability point was determined primarily by H, the transition from SH from MH depended on *both* H and the Lundquist number S. However, further calculations are showing that the SH $\rightarrow$ MH transition seems to lose dependence on S as Δt is reduced. This implies that the force balance condition is being affected by residual numerical forces such as those given by Equation (8). The marginal stability point remains relatively independent of S. These results will be reported in a future note.

In Figure 1 we plot N_s, the number of $n = 1$ modes participating in the dynamo, as a function of the Hartmann number $H = S/\sqrt{P_M}$ for three different values of the Lundquist number: $S = 3.3 \times 10^3$, $S = 3 \times 10^4$, and $S = 10^5$, for $\Theta_0 = 1.9$ and $R/a = 4$ The calculations used a maximum time step of $\Delta t_{max} = 0.5\tau_A$. Both marginal stability, at $H_{marg} \sim 100 - 200$, and transition from SH ($N_s = 1$) to MH ($N_s > 2.5$), at $H_{crit} \sim 1000 - 3000$, occur at a specific value of H independent of S, in substantial agreement with References [2] and [3]. Previous calculations

using $\Delta t_{max} = 5\tau_A$ showed a strong S dependence for H_{crit}. This was due to the residual force resulting from time truncation error.

We finally remark on the possible significance of the residual time truncation force in the numerical simulation of sawtooth oscillations in the RFP. These are dynamical states that show repeated bursts of mode growth and nonlinear in-

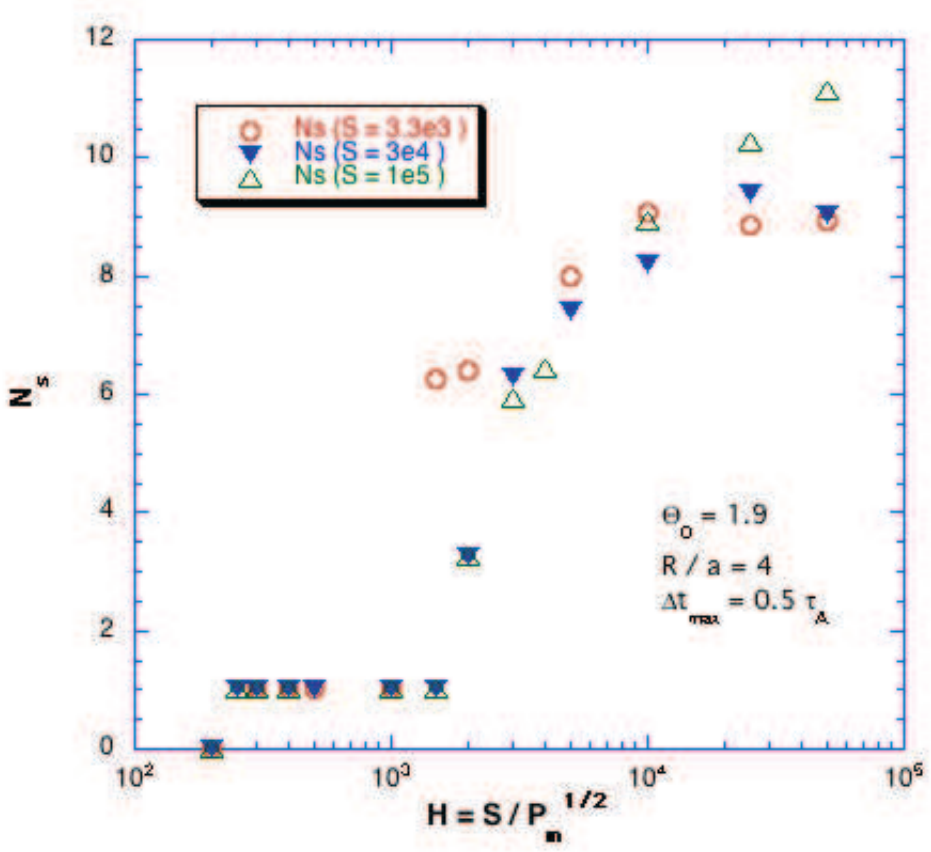

Figure 1: Summary of results with $\Delta t_{max} = 0.5\tau_A$: the time average of N_s vs. H for $\Theta_0 = 1.9$ and $R/a = 4$ for $S = 3.3 \times 10^3$, $S = 3 \times 10^4$, and $S = 10^5$. All values of S have a marginal stability point at $H_{marg} = 1.8 \times 10^2$; below this value all states are stable. The transitions from SH ($N_s = 1$ to QSH ($1 < N_s < 2.5$), and MH ($N_s > 2.5$) all occur at $H_{crit} \sim 1000 - 3000$.

teraction (the sawtooth crashes) separated by longer periods of relatively steady state behavior. The present algorithm in DEBS requires that the CFL condition $kV\Delta t < 1$ be satisfied, so the time step becomes small as the flow becomes large, and vice versa. The velocity V peaks during the dynamical crash phase of the sawtooth, so the time step becomes correspondingly small and the residual numerical force becomes smaller. However, during the relatively steady periods between crashes the flow becomes small and Δt increases, possibly up to its maximum allowed value. The residual numerical force therefore may become larger between sawtooth crashes, and may be affecting the properties of these states. This needs to be checked by further computations with smaller maximum time step.

The problem with the residual force can apparently be solved by "un-splitting" the viscosity and using it directly in the corrector. These changes are presently being made in the DEBS code, and results will be reported soon. Hopefully this will allow the use of large time steps during the relatively quiescent periods between the sawtooth flux generation events in RFP dynamics, and a more efficient and accurate determination of the transition from SH to MH states.

References

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