EXTENDED MAGNE TOHYDRODYNAMIC MODELING OF PLASMA RELAXATION DYNAMICS IN THE REVERSED-FIELD PINCH

by

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Ad Majorem Dei Gloriam
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“What does man gain by all the toil at which he toils under the sun?... And I applied my mind to seek and to search out by wisdom all that is done under heaven; it is an unhappy business that God has given to the sons of men to be busy with... For in much wisdom is much vexation, and he who increases knowledge increases sorrow.” (Ecclesiastes 1:3,13,18)

“To get wisdom is better than gold; to get understanding is to be chosen rather than silver.” (Proverbs 16:16)
CONTENTS

I Introduction 1

1 Background 5
   1.1 Magnetic Confinement ................................................. 5
   1.2 The Extended MHD Model ............................................. 7
   1.3 Basic Concepts .................................................... 18
   1.4 The Tearing Mode .................................................. 20
   1.5 The Paramagnetic Pinch .......................................... 26

2 The Reversed-Field Pinch 29
   2.1 The Reversed-Field Pinch ......................................... 29
   2.2 The RFP Dynamo ................................................ 32
   2.3 Flow in the RFP ................................................ 39

3 NIMROD Model 47
   3.1 The Computational Model ........................................ 47
   3.2 NIMROD Spatial Grids ............................................. 50
   3.3 Model Parameters ................................................ 52

II Nonlinear Relaxation Dynamics 57

4 Current Relaxation and Dynamo Drive 58
   4.1 Single-Fluid MHD .................................................. 58
   4.2 Two-Fluid, No Ion Gyroviscosity ................................ 63
   4.3 Two-Fluid with Ion Gyroviscosity ............................... 67
   4.4 Two-Fluid with Ion Gyroviscosity, \( P_m = 0.1 \) .................. 71
   4.5 Discussion ..................................................... 73

5 Momentum Transport 76
   5.1 Parallel and Perpendicular Flows .............................. 76
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2</td>
<td>Momentum Evolution</td>
<td>80</td>
</tr>
<tr>
<td>5.3</td>
<td>Discussion</td>
<td>85</td>
</tr>
<tr>
<td>6</td>
<td>Spectral Power Flow</td>
<td>88</td>
</tr>
<tr>
<td>6.1</td>
<td>Spectral Energy Transfer</td>
<td>88</td>
</tr>
<tr>
<td>6.2</td>
<td>The $m = 1,</td>
<td>n</td>
</tr>
<tr>
<td>6.3</td>
<td>The $m = 0,</td>
<td>n</td>
</tr>
<tr>
<td>6.4</td>
<td>Discussion</td>
<td>98</td>
</tr>
<tr>
<td>7</td>
<td>Helicity Evolution</td>
<td>100</td>
</tr>
<tr>
<td>7.1</td>
<td>Introduction</td>
<td>100</td>
</tr>
<tr>
<td>7.2</td>
<td>Evolution in the NIMROD Model</td>
<td>102</td>
</tr>
<tr>
<td>7.3</td>
<td>Helicity Evolution</td>
<td>106</td>
</tr>
<tr>
<td>7.4</td>
<td>Cross Helicity Evolution</td>
<td>110</td>
</tr>
<tr>
<td>7.5</td>
<td>Profile Relaxation</td>
<td>113</td>
</tr>
<tr>
<td>7.6</td>
<td>Discussion</td>
<td>116</td>
</tr>
<tr>
<td>III</td>
<td>Linear Computations</td>
<td>118</td>
</tr>
<tr>
<td>8</td>
<td>Toroidal Effects on Linear Tearing Modes</td>
<td>120</td>
</tr>
<tr>
<td>8.1</td>
<td>Introduction</td>
<td>120</td>
</tr>
<tr>
<td>8.2</td>
<td>Model System</td>
<td>122</td>
</tr>
<tr>
<td>8.3</td>
<td>Results</td>
<td>126</td>
</tr>
<tr>
<td>8.4</td>
<td>Discussion</td>
<td>136</td>
</tr>
<tr>
<td>9</td>
<td>Resistive Drift Mode</td>
<td>138</td>
</tr>
<tr>
<td>9.1</td>
<td>Model Equilibrium</td>
<td>138</td>
</tr>
<tr>
<td>9.2</td>
<td>Analytic Dispersion Relation</td>
<td>141</td>
</tr>
<tr>
<td>9.3</td>
<td>Computational Results</td>
<td>143</td>
</tr>
<tr>
<td>9.4</td>
<td>Discussion</td>
<td>147</td>
</tr>
<tr>
<td>IV</td>
<td>Conclusions</td>
<td>148</td>
</tr>
<tr>
<td>10</td>
<td>Conclusions</td>
<td>149</td>
</tr>
<tr>
<td>10.1</td>
<td>Conclusions from Nonlinear Studies</td>
<td>149</td>
</tr>
<tr>
<td>10.2</td>
<td>Conclusions from Linear Studies</td>
<td>150</td>
</tr>
<tr>
<td>10.3</td>
<td>Recommendations for Future Work</td>
<td>151</td>
</tr>
</tbody>
</table>
V Appendices

A  Evolution of Momenta in the NIMROD System  154
B  Evolution of Energies in the NIMROD System  159
   B.1 Magnetic Energy ........................................  159
   B.2 Kinetic Energy ..........................................  159
   B.3 Internal Energy ..........................................  160
   B.4 Total Energy of NIMROD Equations .....................  163
C  Evolution of Helicities in the NIMROD System  165
   C.1 Magnetic Helicity .......................................  165
   C.2 Cross Helicity ...........................................  166
   C.3 Kinetic Helicity .........................................  167
D  Spectral Power Flow in the Extended MHD System  168
E  Construction of the Vector Potential  172
F  Gauge-Invariant Definitions of Magnetic Helicity  174
   F.1 Change in Magnetic Helicity with Gauge Transform ....  174
   F.2 Relative Helicity .......................................  175
   F.3 Transformation of the Surface Integral ................  176
G  Constructing the Straight-Field Coordinates  179
H  Cylindrical Phase Relations  181
I  Extended MHD Waves  183
   I.1 Linearization ...........................................  183
   I.2 Ideal System, Uniform Background ......................  187
J  Resistive Drift Dispersion Relation  191
K  Supplemental Plots from Nonlinear Computations  195

References  200
Nonlinear plasma relaxation dynamics in reversed-field pinch (RFP) conditions are investigated using extended magnetohydrodynamic modeling with the NIMROD code [Sovinec et al. JCP 195 (2004)]. The computations display quasi-periodic relaxation events, consistent with the RFP sawtooth cycle. The first event occurs from a saturated single-helicity state and is not representative of the multi-helicity conditions in typical RFP discharges. The MHD and Hall dynamo electric fields cooperate during the initial relaxation, and the change in the plasma flow parallel to the magnetic field is in the direction of the Lorentz force density and opposite to the change in parallel current density. In subsequent events, the MHD dynamo relaxes the parallel current and is opposed by the Hall dynamo, which is smaller in magnitude. The associated Lorentz force density drives changes in flow that are in the same direction as the change in current, in agreement with experimental observations [Kuritsyn et al. POP 055903 (2009)]. Changes in total momentum are due to viscous coupling to the wall; radial transport occurs through Maxwell stresses associated with current relaxation.

The magnetic and hybrid helicity are well conserved relative to magnetic energy over the relaxation events. The helicities change by \( \leq 0.2\% \) of their initial values while the normalized magnetic energy decreases by about 1.5\%. Large changes in cross helicity occur for two-fluid computations, but the hybrid helicity is dominated by magnetic helicity in pinch conditions. The plasma current approaches the relaxed state predicted by variational theories but does not achieve it. The plasma flow develops significant structure in two-fluid computations. The energy in flow perpendicular to the magnetic field is several times larger than in flow along it.

Two linear studies are presented. Linear computations in toroidal geometry show a phase difference of \( 22 - 25^\circ \) between toroidal and poloidal components of magnetic field at the location of probe measurements in the experiment. This phase shift is commensurate with experimental observations for core-resonant tearing modes. A resistive drift wave is identified in linear computations in slab geometry, and the computed growth rates agree with an analytic dispersion relation.
Part I

Introduction
Plasma physics is the study of an ionized gas where some or all of the negatively charged electrons have enough energy to overcome the natural Coulomb forces that bind them to the positively charged ions. The resulting state consists of two (or more) gaseous species of electrically charged particles that are mixed together in space. Charged particles interact with electric and magnetic fields, and these fields are in turn generated by the motion of the charged particles. This interaction gives rise to unique collective behavior, and plasma is fittingly referred to as the fourth state of matter. A detailed treatment of the equations of motion for every single particle in a plasma is, in principle, possible, but in practice it is intractable, both analytically and numerically. Reduced models of the plasma dynamics are therefore essential in any attempt to understand the plasma behavior.

Computational plasma physics utilizes high-performance computer simulation codes to solve the equations of these simplified analytical models numerically in an attempt to understand plasma phenomenon. Numerical modeling is widely used across many fields of science; in plasma physics in particular, it is commonly used for performing equilibrium reconstructions, understanding linear stability properties of plasma profiles, and tracking the nonlinear, turbulent evolution of a plasma. One of the main goals of numerical computation is to develop predictive capabilities. Large-scale experiments are expensive to build and run, which limits the variety of configurations that may be tested and the parameter ranges that may be studied. If a computational model compares favorably with known experiments, it is reasonable to assume that the model will also capture the essential physics for similar devices, and this allows an exploration of novel configurations without the additional experimental overhead. However, knowing the limitations of a model is crucial in understanding where these extrapolations may be expected to be valid and where they are not. In general, numerical modeling requires both verification and validation; the code must be verified to be solving the equations correctly, and the models must be validated by direct comparison to experimental measurements.

Here, we will be concerned primarily with the latter. We will present extended MHD computations of plasma dynamics that are relevant to the reversed-field pinch (RFP) device, a toroidal device capable of confining a hot plasma with magnetic fields. The RFP displays several interesting phenomenon that make it worthy of study in its own right. It is a prototypical driven-damped system with nonlinear phenomenon occurring quasi-periodically over the duration of the discharge. Electromagnetic energy is injected into the plasma via external mechanisms, and the nonlinear dynamics rearrange this energy as the plasma self-organizes into a preferred state. A large portion of the injected energy is used to sustain the magnetic fields that confine the plasma, some of it is dissipated as heat, and some of it is converted into motion of the plasma itself. However, in typical RFP discharges this is not a continuous
process. Instead, the RFP often exhibits sawtooth behavior. Measurables of the plasma, such as electron temperature, will slowly increase in time before a sudden discrete relaxation event occurs, with concomitant changes in the plasma parameters. Understanding the basic physical mechanisms that govern this behavior is crucial to improving our understanding of magnetically confined plasmas and advancing basic scientific knowledge.

Five major results are presented in this thesis. First, our nonlinear computations with both a single-fluid and two-fluid model demonstrate multiple discrete relaxation events in experimentally relevant RFP conditions, as discussed in Chapter 4. The quasi-periodic nature of these events is comparable to the sawtooth cycle in experiment. We find that the first relaxation event in these computations is qualitatively distinct from subsequent events as a result of the initial conditions of our computation, and the first event is not expected to be representative of typical sawtooth behavior in the RFP. This behavior is also evident when considering the nonlinear power flow for the dominant mode, as we will see in Chapter 6.

A critical observation is that the dynamo components of the electric field in these subsequent relaxation events differ markedly from the first event in our computations. While the MHD dynamo is found to always relax the parallel current profile, the core dynamics of the Hall dynamo show variability between dynamo and anti-dynamo drive between events. The anti-dynamo nature of the Hall dynamo may help resolve a discrepancy between changes in core plasma flow in past computations and experimental measurements.

Second, significant changes in plasma momentum are associated with the discrete current relaxation events in our computations, and substantial changes occur only in our two-fluid models. The correlated fluctuation-induced Lorentz force density does not change the total momentum, it may only transport it around the plasma. In Chapter 5 we will show that the changes in momentum are due to viscous and gyroviscous coupling to the boundary, and this momentum is then transported radially via the Lorentz force density.

We also examine the evolution of several ideal invariants in our numerical computations in Chapter 7. These quantities are often used in variational theories that attempt to describe the preferred relaxed state of the plasma. Our computations show that the well-known magnetic invariants behave roughly as expected and in agreement with experimental measurements. However, the invariants that are used to capture the effects of plasma flow are found to change considerably as a result of the ion gyroviscosity and other two-fluid effects, and the predicted relaxed flow states are not attained in our computations.

In addition to our nonlinear computations, we also present the results of several linear studies. The first, which is relevant to the Madison Symmetric Torus RFP, provides a possible resolution to a phase shift observed in measurements on the Madison Symmetric Torus. It is found that, within a single-fluid zero-β MHD model, the effects of toroidicity are able to
account for this shift, and this is discussed in Chapter 8.

Lastly, in Chapter 9, we identify a growing resistive drift mode in a simple plasma slab equilibrium, which is expected to be present in the extended MHD system. Drift waves are excited by pressure gradients, and while our present nonlinear computations do not include these gradients, the inclusion of a realistic pressure profile is important for accurate modeling of the RFP dynamics. We confirm here the existence of this mode in our computational model, and we verify the computed growth rates through comparison to an analytic dispersion relation. We find that it is strongly stabilized by magnetic shear, and it is unlikely to be strongly unstable in experimentally relevant conditions.

We will begin with an elementary discussion of some of the basic concepts of magnetically confined plasmas, after which we will derive the extended magnetohydrodynamics model that will be used in this work. This will be followed with a brief introduction to the basic physics of the reversed-field pinch device, and a cursory introduction to the numerical code, NIMROD, that we will be using. This will by no means be complete, but it is hoped that it will provide a sufficient background to make the remainder of the material in the thesis coherent. Much more complete descriptions of magnetic confinement and plasma physics may be found in elementary textbooks, such as Refs. [9, 46, 54]. Those interested in further background on the reversed-field pinch are encouraged to peruse the review article by Bodin and Newton [10] and the excellent textbook on the subject by Ortolani and Schnack [93].
1 BACKGROUND

1.1 Magnetic Confinement

The premise underlying magnetic confinement of plasma is straightforward. In a uniform magnetic field, a charged particle will drift along the field with no change in velocity, while it undergoes simple gyration perpendicular to the field. The parallel dynamics are unconstrained by the magnetic field, while the perpendicular dynamics are heavily constrained. The radius of gyration is called the particle’s Larmor radius, \( \rho_L \equiv v_\perp/|\Omega_c| \) where \( \Omega_c \equiv qB_0/m \) is the particle’s cyclotron frequency. In the limit that the magnetic field becomes very large, \( B_0 \to \infty \) and \( |\Omega_c| \to \infty \), the Larmor radius shrinks to zero. The particle is “stuck” on the field line, and the guiding center of the particle moves freely along the magnetic field.

It is reasonable to assume that the magnitude of the perpendicular velocity is commensurate with the random thermal motion of the particles, \( v_\perp \sim v_{T,s} \), where \( v_{T,s}^2 = T_s/m_s \). With this assumption, the Larmor radius is proportional to the square root of the mass, \( \rho_{L,s} = \sqrt{m_sT_s}/|q_s|B_0 \). Consequently, for species of equal temperature, \( T_i \approx T_e \), the ion Larmor radius is much larger than the electron Larmor radius, \( \rho_{L,i}/\rho_{L,e} = \sqrt{m_i/m_e} \). The simple model of a particle that merely flows along a magnetic field line with no radial excursion will break down for the ions before it breaks down for the electrons. The effects of this finite ion Larmor radius may be included in a fluid model to first order with additional terms in the fluid equations, and we will include some of these effects in our computations.

While the magnetic field constrains the particle motion perpendicular to it, the particle is free to stream along the field line unimpeded by the presence of the magnetic field. Rapid particle losses then occur if the magnetic field lines wander outside of the confinement region. However, if the magnetic field can be made to close back on itself, a particle will, to lowest order, wander along the same field line in a finite region of space. The only topological shape for such a configuration is a torus [46]. A schematic of such a toroidal configuration, the reversed-field pinch, is shown in Fig. 1.1. The RFP itself will be discussed more fully in Chapter 2.

1.1.1 Flux Surfaces and the Safety Factor

For stability purposes, it is necessary to have both a toroidal component of the magnetic field (going the long way around the torus) and a poloidal component (going the short way around). By convention, the poloidal angle is labeled \( \theta \) and the toroidal angle is labeled \( \phi \) or \( \zeta \). Consequently, as the magnetic field travels around the device toroidally, it also wraps around
an axis in the poloidal plane, called the magnetic axis. If the configuration possesses certain symmetry properties, then the magnetic field lines trace out a series of nested surfaces, with the magnetic axis itself being the central “surface” in such a configuration. These surfaces are called flux surfaces. The amount of magnetic flux, either toroidal or poloidal, within such a surface is a constant; the magnetic field does not pierce the surface.

On any given surface, the number of times the field line wraps around in the toroidal direction for any given poloidal circuit in general will depend on the toroidal location. The safety factor is a mathematical quantification of this winding:

\[
q \equiv \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_0^\Theta \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} d\theta. \tag{1.1}
\]

For axisymmetric systems only a single poloidal transit is sufficient to sample the entire surface, and we may take \( \Theta = 2\pi \) [34]. If the toroidal magnetic field is much larger than the poloidal magnetic field, then many toroidal transits occur in a single poloidal transit. One of the leading concepts for toroidal confinement devices, the tokamak, operates in such a magnetic configuration. The tokamak typically has \( q \geq 1 \) everywhere, as a strong toroidal magnetic field helps to stabilize against plasma instabilities. If the toroidal field is comparable to the poloidal field, only a fraction of a toroidal transit will be completed in a given poloidal transit, \( q \leq 1 \). Typically, higher magnetic shear, \( \sim dq/dr \), helps to stabilize these configurations. The specific configuration that we investigate here is the RFP, which derives its name from the fact that the toroidal field vanishes over a particular toroidal surface inside the plasma called the reversal surface; the toroidal field outside of that surface is reversed relative to its on-axis value. Along this surface, \( q = 0 \), and, as we will see, its presence in the plasma plays a crucial role in the nonlinear dynamics that occur in the RFP.

If a field line on a given surface returns to its original location after \( m \) transits poloidally and \( n \) transits toroidally, then \( q = m/n \) and that surface is called a rational surface. Rational
surfaces have special importance in the study of plasma instabilities, with low order rational surfaces often being the most important. On the other hand, when \( q \) is an irrational number, the magnetic field will never return to its original position no matter how long it is followed. Instead, it will ergodically cover the entire flux surface.

The existence of flux surfaces in any magnetic configuration requires a high degree of symmetry, and there is no reason to expect that such surfaces must exist in general. Deviations may be caused by stray external magnetic fields or by symmetry breaking instabilities, and these effects can lead to regions of stochastic magnetic field. In such regions, the field line ergodically covers a volume of space, instead of just a surface, as it winds around the device. Because particle motion along field lines is unconstrained, stochastic regions may readily transport particles and heat across the volume. This is extremely detrimental for confinement, and understanding and mitigating these symmetry breaking perturbations continues to be an area of active research.

### 1.1.2 Cylindrical Approximation

Referring back to Fig. 1.1, we see that while both \( \theta \) and \( \zeta \) are periodic, only \( \zeta \) is symmetric. This is a direct consequence of bending the magnetic field into this geometry; the magnetic field on the inside of the toroidal configuration is larger than the magnetic field on the outside. Toroidal configurations have at least two non-ignorable coordinates.

A very useful approximation is to consider a straight periodic cylinder, which is topologically equivalent to a torus. The toroidal angle becomes the axial direction, \( \zeta = 2\pi z/L \), where \( L = 2\pi R_0 \) is the axial length of the periodic cylinder and \( R_0 \) is the major radius. The (geometric) poloidal angle and minor radial coordinate remain the same. In this configuration, there is symmetry in both of the periodic directions, and axisymmetric configurations become 1-dimensional, depending on the minor radial coordinate alone.

As we will discuss in Chapter 8, the RFP is well-approximated by a cylindrical configuration, and for most of our work here, we will restrict ourselves to such a periodic cylinder geometry. However, in Chapter 8 we will examine the effects of toroidicity on linear tearing modes in RFP-relevant profiles.

### 1.2 The Extended MHD Model

We will now introduce the extended magnetohydrodynamic (MHD) system of equations that we will use to describe the plasma dynamics. The MHD system is essentially a combination of the Navier-Stokes equations for fluid flow with Maxwell’s equations. Starting from the
fundamental plasma kinetic equation, we will walk through a derivation of the two-fluid system. Extended MHD systems are then constructed from specific combinations of the two-fluid equations, and we will describe the system used in the remainder of the thesis, which includes the well-known single-fluid MHD system as a limiting case.

1.2.1 The Plasma Kinetic Equation

The equations of motion for a single particle, labeled with subscript $k$, are given by

$$\frac{d\mathbf{x}_k}{dt} = \mathbf{v}_k$$

$$m_k \frac{d\mathbf{v}_k}{dt} = \mathbf{F}_k$$

where $\mathbf{F}_k$ is the force acting on a charged particle. In the non-relativistic limit, the electromagnetic forces take the form $\mathbf{F}_k = q_k (\mathbf{E} + \mathbf{v}_k \times \mathbf{B})$.

Charged particles are acted upon by both electric and magnetic fields, and the charges themselves represent a source of electric field. Similarly, the motion of the charged particle constitutes an electric current, which is itself a source for magnetic fields. This interaction is described by Maxwell’s Equations [62],

$$\nabla \cdot \mathbf{E} = \rho_q / \epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t},$$

where $\rho_q = \sum_n n_s q_s$, $\mathbf{J} = \sum_n n_s q_s \mathbf{v}_s$, and $\epsilon_0$ and $\mu_0$ are the permittivity and permeability of free space, respectively. In differential form, Maxwell’s equations represent a statistical averaging over discrete particles, and the charge density and current density here are to be interpreted as continuous, rather than discrete, entities. This is akin to taking the inter-particle spacing to be much smaller than the length scales of interest; a more complete description of this is given in Ref. [62].

Owing to the very long range Coulomb interaction among particles, any local accumulation of particles of a given charge very quickly attracts particles with the opposite charge. This cancels the original charge density, and, as a result, plasmas are very effective at shielding out local accumulations of charge. Because of this extremely effective shielding, plasmas are very often quasi-neutral; the charge density of the ion species is very nearly equal to the
charge density of the electron species at any given point, \( n_e \approx Z n_i \), where \( Z \) is the effective ion charge. In what follows, we will assume a hydrogenic plasma with unit charge, so \( n_i \approx n_e \), and we may use a common density, \( n = n_e = n_i \). This has important ramifications for the current density. Taking the divergence of Ampère’s law, Eq. (1.7), we find

\[
\nabla \cdot J = \frac{\partial}{\partial t} \left[ \epsilon_0 \nabla \cdot E \right]
\]

so that in the quasi-neutral limit \( \nabla \cdot J \approx 0 \). Consequently, we will neglect the displacement current term and replace Eq. (1.7) with the pre-Maxwell Ampère’s law:

\[
\mu_0 J = \nabla \times B.
\]

If an oscillatory electric field is applied in a plasma, then the electrons, being by far the lighter component, will respond much more rapidly than the ions; they will quickly move so as to counteract the potential from the applied field. However, if the field is oscillated quickly enough, even the mobile electrons will be unable to respond to the effects of the disturbance, which may propagate freely through the plasma. This critical frequency, inversely proportional to the mass, is the plasma frequency:

\[
\omega_{\text{pe}}^2 \equiv \frac{n_0 e^2}{\epsilon_0 m_e} \quad \quad \quad \quad \quad \omega_{\text{pi}}^2 \equiv \frac{n_0 e^2}{\epsilon_0 m_i}.
\]

Note that \( \omega_{\text{pi}}^2 \ll \omega_{\text{pe}}^2 \), so that the main frequency of interest here is the electron frequency. For frequencies greater than \( \omega_{\text{pi}} \), the electrons may still effectively cancel out the electric field, while the ions are essentially fixed in place. Irrotational electric field perturbations oscillating at \( \omega < \omega_{\text{pe}} \) are attenuated over a length scale called the skin depth: \( d_e \equiv \frac{\omega_{\text{pe}}}{\omega} \). The ion skin depth, \( d_i \equiv \frac{\omega_{\text{pi}}}{\omega} \), is also of interest for lower frequency dynamics. Owing to the mass disparity, \( d_i = \frac{m_e}{m_i} d_e \). For length scales on the order of \( d_i \), the electrons may respond to changes in the electric field, while the ions are unable to do so and basically remain fixed. As a result, in an unmagnetized plasma, the electron and ion motion is uncoupled on scales smaller than the ion skin depth, \( d_i \); to resolve scales below the ion skin depth accurately, we must treat the electrons and ions as separate species.

In principle, a variant of Eqs. (1.2)-(1.3) for each particle in the plasma coupled with Eqs. (1.4)-(1.7) constitutes a closed system of equations; the evolution can be determined from just the initial conditions of each particle and the initial electric and magnetic field configurations. In practice, however, the number of particles in even a very small system renders this intractable. Instead, we will perform an ensemble average of microscopic variations
of a given plasma state. Such a procedure was implicitly used in obtaining Maxwell’s equations in the differential form presented above.

After carrying out this averaging procedure, we find the plasma kinetic equation,
\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{\mathbf{F}_s}{m_s n_s} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = C(f_s)
\] (1.11)

where the \(s\) subscript now differentiates between different species in the plasma. The plasma kinetic equation describes the evolution of the distribution function, \(f_s(x, \mathbf{v}, t)\), which represents the probability of finding a particle of species \(s\) in the 6-dimensional space \([x, x + dx] \times [\mathbf{v}, \mathbf{v} + d\mathbf{v}]\) at time \(t\). The operator on the right hand side, \(C(f_s)\), is the collision operator. It describes the effects of collisions between particles of the same species and collisions with all other species. One of the main tasks of kinetic theory is to develop useful models for the collision operator.

While we have reduced the system size considerably from our original \(6N\) single-particle equations of motion, the distribution function itself is 6-dimensional and the system remains quite large. Fluid theory further reduces the dimensionality of the system by taking moments over all of velocity space. While information is lost in this procedure, fluid theory remains surprisingly accurate for many plasma phenomena of interest, and it forms the backbone of the MHD system.

1.2.2 The Fluid Equations

Fluid variables are defined by taking moments of the distribution function. The density of particles at a given point in physical space may be found by integrating the distribution function over all of velocity space,
\[
n_s(x, t) = \int d\mathbf{v}' f_s(x, \mathbf{v}', t).
\] (1.12)

The total number of particles in the system is found by integrating this over all space, 
\(N_s(t) = \int d\mathbf{x}' n_s(x', t)\). In a similar fashion, the fluid velocity may be defined as
\[
n_s \mathbf{v}_s(x, t) = \int d\mathbf{v}' \mathbf{v}' f_s(x, \mathbf{v}', t).
\] (1.13)
Higher order fluid variables are defined similarly. The pressure, stress tensor, and heat flux are given by

\begin{align}
p_s (x, t) &= (\Gamma - 1) \frac{m_s}{2} \int \! d\mathbf{v}' \, |\mathbf{v}' - \mathbf{v}_s (x, t)|^2 f_s (x, \mathbf{v}', t) \\
P_s (x, t) &= m_s \int \! d\mathbf{v}' \, \mathbf{v}' \mathbf{v}' f_s (x, \mathbf{v}', t) \\
q_s (x, t) &= \frac{m_s}{2} \int \! d\mathbf{v}' \, |\mathbf{v}' - \mathbf{v}_s (x, t)|^2 (\mathbf{v}' - \mathbf{v}_s (x, t)) f_s (x, \mathbf{v}', t)
\end{align}

respectively, where \( \Gamma \) is the ratio of specific heats.

The evolution of these fluid quantities is found by taking appropriate moments of the plasma kinetic equation. However, each moment equation contains terms that are governed by a higher order moment of Eq. (1.11), so that formally all moments of the kinetic equation are required to represent the kinetic equation exactly. In practice, only a few low order moments are kept. The higher order terms may be dropped completely in what is called truncation, or they may be described in terms of the lower order moments via closure relations. Closure relations are always approximate, and the search for more accurate closures is an ongoing area of investigation in fluid theory.

For our purposes, we will only consider the first three moments of Eq. (1.11), which yield the density evolution equation, the momentum evolution equation, and the temperature (energy) evolution equation:

\begin{align}
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) &= 0 \\
m_s n_s \left[ \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right] &= n_s q_s (E + \mathbf{v}_s \times \mathbf{B}) - \nabla p_s - \nabla \cdot \mathbf{\Pi}_s + \mathbf{R}_s \\
\frac{n_s}{\Gamma_s - 1} \left[ \frac{\partial T_s}{\partial t} + \mathbf{v}_s \cdot \nabla T_s \right] &= -n_s T_s \nabla \cdot \mathbf{v}_s - \mathbf{\Pi}_s : \nabla \mathbf{v}_s - \nabla \cdot \mathbf{q}_s + Q_s
\end{align}

where \( \mathbf{\Pi}_s \equiv \mathbf{P}_s - p_s \mathbf{I} \), and \( p_s = n_s T_s \). With the quasi-neutrality condition, \( n_i = n_e \), and Eq. (1.17) provides only one independent equation. Note that we have absorbed the Boltzmann constant, \( k_B \), into the temperature so that \( T_s \) has units of energy. The terms \( \mathbf{R}_s \) and \( Q_s \) are moments of the collision operator:

\begin{align}
\mathbf{R}_s (x, t) &\equiv \int \! d\mathbf{v}' \, m_s \mathbf{v}' C (f_s) \\
Q_s (x, t) &\equiv \int \! d\mathbf{v}' \, \frac{m_s}{2} |\mathbf{v}'|^2 C (f_s) - \mathbf{v}_s \cdot \mathbf{R}_s.
\end{align}

Note that \( \mathbf{R}_s \) represents the effects of collisional friction on the momentum, and \( Q_s \) represents collisional heating, and the exact specification of these requires detailed knowledge of the
collision operator. The terms $\Pi_s$ and $q_s$ are related to higher order moments of the distribution function (Eq. (1.15) and Eq. (1.16)), and their evolution is formally governed by additional moments of the plasma kinetic equation. Closure relations are needed in order to describe $R_s$, $Q_s$, $q_s$, and $\Pi_s$ in terms of the lower order moments $n_s$, $v_s$, and $T_s$.

For our purposes, we will use a closure first developed for magnetized plasmas by Braginskii [13], following the Chapman-Enskog [23] procedure. The distribution function is expanded as

$$f_s(x, v, t) = \sum_{i=0}^{\infty} \epsilon^i f_{s,i} \tag{1.22}$$

for a small parameter $\epsilon \ll 1$ and inserted into the plasma kinetic equation. This equation is then solved iteratively, order by order, to whatever arbitrary accuracy is required. To lowest order, and in the absence of external forcing, the distribution function must be Maxwellian, $f_{s,0} = f_{s,M}$, where

$$f_{s,M}(x, v, t) = \frac{n_s}{\pi^{3/2}} \left( \frac{m_s}{2T_s} \right)^{3/2} \exp \left[ -\frac{m_s}{2T_s} |v - v_s|^2 \right] \tag{1.23}$$

and $C(f_{s,M}) = 0$. Non-Maxwellian distribution functions relax to a Maxwellian distribution within a characteristic collision time, $\tau_s = 1/\nu_s$, where the collision frequency is given by

$$\nu_e = \frac{\sqrt{2} q_i^2 q_e^2 n_i \ln \Lambda}{4\epsilon_0^2 \sqrt{\pi^3 m_e T_e^3}} \quad \nu_i = \frac{\sqrt{2} q_i^4 n_i \ln \Lambda}{4\epsilon_0^2 \sqrt{\pi^3 m_i T_i^3}} \tag{1.24}$$

The $O(\epsilon)$ terms in the plasma kinetic equation relate $f_{s,1}$ to $f_{s,0}$ via an integro-differential equation. Expansion techniques can turn this into a set of algebraic equations that can be used to compute $f_{s,1}$ to any desired accuracy. Lastly, $f_{s,1}$ is used to calculate $R_s$, $Q_s$, $q_s$, and $\Pi_s$.

In the usual Chapman-Enskog treatment of a classical gas, the small parameter used in the expansion is the ratio of the particle mean-free-path to the length scale of variation (or equivalently, the ratio of collision frequency to the frequency of the dynamics of interest). Particles experience many collisions in a classical gas before traveling any appreciable distance. By necessity, this keeps the distribution function very nearly Maxwellian at all points in space, and the Chapman-Enskog closure approach is satisfied.

In a plasma, however, the mean time between collisions increases with increasing temperature, $\tau_{\text{coll}} \sim T^{3/2}$. For higher temperature plasmas, we do not expect the collisions to be very frequent, and, as a result, the distribution functions may be very far from Maxwellian. Note that in a strongly magnetized plasma the gyromotion of a particle around a magnetic
field line is very fast, with frequency \( \Omega_s = q_s B / m_s \). If we restrict ourselves to dynamics that are much slower than this gyrofrequency, \(|\omega| \ll |\Omega_s|\), then the gyromotion keeps the particles very nearly Maxwellian in the direction perpendicular to the magnetic field, validating the Chapman-Enskog approach for the perpendicular dynamics. The gyromotion does not restrict the behavior along the magnetic field, but fortunately the most violent MHD behavior occurs in the perpendicular direction. Although the Chapman-Enskog approach for the parallel dynamics is not strictly valid, fluid-based plasma models at least provide a starting point for understanding the perpendicular dynamics.

Carrying out the full Chapman-Enskog approach is beyond the scope of this work. We will instead present, without further derivation, the results of such a closure scheme for a single-ion-species plasma with \( Z = 1 \). Interested readers are encouraged to peruse Refs. [13, 22, 65] for further details.

The collisional friction terms are approximated as a simple resistivity

\[
\mathbf{R}_e \approx e n_e \eta \mathbf{J} = -\mathbf{R}_i
\]

where \( \eta = (m_e/n_e e^2)\nu_e \) is an isotropic resistivity and \( \nu_e \) is the electron collision frequency. Note that at high temperatures a plasma is an excellent conductor of electrical current \( (\nu_e \sim T_e^{-3/2}) \). In principle, the resistivity that results from the Chapman-Enskog closure is anisotropic, but the difference between parallel and perpendicular resistivities only amounts to roughly a factor of 2 \((\eta_\parallel \sim 0.51 \eta_\perp)\) and will be neglected here. We have also neglected the small thermal force from parallel electron temperature gradients.

The collisional heating terms, \( Q_s \), reflect exchange of thermal energy between species and also collisional heating of electrons via the electrical resistivity

\[
Q_i = 3\nu_e m_e/m_i (T_e - T_i) \quad Q_e = -Q_i - \mathbf{R} \cdot (\mathbf{v}_e - \mathbf{v}_i). \tag{1.26}
\]

Our computations will assume that the species’ temperatures are equal, and we will further neglect the effects of the resistive heating of the electrons, so that \( Q_i = 0 = Q_e \). As we will see, this results in a loss of energy conservation in our final equations, but it does not significantly factor into the dynamics.

The ion viscous stress tensor has significant contributions from gyromotion, and perpendicular and parallel flows,

\[
\mathbf{\Pi}_i \approx \mathbf{\Pi}_{\text{gyr}} + \mathbf{\Pi}_\perp + \mathbf{\Pi}_\parallel. \tag{1.27}
\]

King et al. [69] showed that ion gyroviscosity has a considerable impact on the evolution of
tearing modes in pinch profiles. We will present simulations both with and without the ion gyroviscosity. We will take

$$\Pi_{\text{gyr}} = \frac{m_ip_i}{4e|B|} \left[ \hat{b} \times W \cdot (I + 3\hat{b}\hat{b}) - (I + 3\hat{b}\hat{b}) \cdot W \times \hat{b} \right]$$

(1.28)

(see Ref. [22]) where $W = \nabla v + (\nabla v)^T - 2/3I (\nabla \cdot v)$ and $\hat{b} \equiv B/|B|$. It is important to note that the gyroviscosity is non-dissipative [65]. Note that we have neglected contributions to the ion gyroviscosity from the heat flux, $q_i$, but these are expected to be small for the dynamic conditions considered here. The parallel and perpendicular viscosities will be modeled with a simple isotropic viscosity, $\Pi_{\parallel} + \Pi_{\perp} \approx \Pi_{\text{iso}}$, where

$$\Pi_{\text{iso}} = -m_i n \nu_{\text{iso}} W$$

(1.29)

and $\nu_{\text{iso}}$ is a spatial constant. This is undoubtedly a gross oversimplification, as the two contributions have completely different scalings with plasma temperature, but the tearing dynamics we are most interested in are largely perpendicular to $B$.

Lastly, the heat flux, $q_s$, is given by

$$q_s = -n_s \chi_s \nabla T_s.$$ 

(1.30)

Formally, the thermal conduction in a magnetized system is anisotropic with respect to the magnetic field,

$$\chi_s = \chi_{s,\parallel} \hat{b}\hat{b} + \chi_{s,\perp} (I - \hat{b}\hat{b}),$$

(1.31)

but for simplicity we will take

$$q_s = -n_s \chi \nabla T_s$$

(1.32)

where $\chi$ is a spatial constant. Again, this is a gross oversimplification; rapid parallel thermal conduction is very important in understanding thermal transport, especially in regions of stochastic magnetic field which are prevalent in the RFP, but the highly anisotropic nature necessarily complicates the system even further. As such, we neglect it in our computations, choosing to focus on the current relaxation dynamics rather than detailed thermal transport effects.

We have now specified the higher order moments in terms of the lower. Formally, Eqs. (1.17)-(1.19), combined with the closure relations outlined above, constitute a closed
system, and comprise what we will refer to as the two-fluid system of equations. Systems of extended magnetohydrodynamic equations can be constructed from this system with various approximations made along the way.

### 1.2.3 Extended Magnetohydrodynamics

The electron and ion momentum equations may be added together to find an equation governing the evolution of the center of mass flow,

\[
\sum_s m_s n_s \left[ \frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s \right] = \left( \sum_s n_s q_s v_s \right) \times B - \nabla \left( \sum_s p_s \right) - \nabla \cdot \left( \sum_s \Pi_s \right) \tag{1.33}
\]

where we have used quasi-neutrality \((\sum_s n_s q_s = 0)\) and the fact that \(\sum_s R_s = 0\). The quasi-neutrality condition, for a \(Z = 1\) plasma, gives \(n_i = n_e \equiv n\).

The center of mass velocity and the current density are given by

\[
v \equiv \frac{\sum_s m_s v_s}{\sum_s m_s} = \frac{1}{1 + \frac{m_e}{m_i}} \left( v_i + \frac{m_e}{m_i} v_e \right) \quad J \equiv \sum_s n_s q_s v_s = ne \left( v_i - v_e \right). \tag{1.34}
\]

The inverse relationship, expressing the species’ velocities in terms of \(v\) and \(J\) is readily found:

\[
v_i = v + \left[ \frac{m_e}{1 + \frac{m_e}{m_i}} \right] \frac{1}{ne} J \\
v_e = v - \left[ \frac{1}{1 + \frac{m_e}{m_i}} \right] \frac{1}{ne} J. \tag{1.35}
\]

Note that \(v_i = v + \mathcal{O} \left( \frac{m_e}{m_i} \right)\) and \(v_e = v - \frac{1}{ne} J + \mathcal{O} \left( \frac{m_e}{m_i} \right)\). Using these in the sum of the species’ momentum equations, we find

\[
(m_i + m_e) n \frac{\partial v}{\partial t} + n [m_i v_i \cdot \nabla v_i + m_e v_e \cdot \nabla v_e] = J \times B - \nabla p - \nabla \cdot \left( \sum_s \Pi_s \right). \tag{1.36}
\]

With a little algebra, it follows that the advective term is given exactly by

\[
m_i v_i \cdot \nabla v_i + m_e v_e \cdot \nabla v_e = (m_i + m_e) v \cdot \nabla v + \frac{m_e}{ne} J \cdot \nabla \left( \frac{1}{ne} J \right) + m_e \left( v - v_i \right) \cdot \nabla \left( \frac{1}{ne} J \right). \tag{1.37}
\]

Defining \(m \equiv \sum_s m_s\), the sum of momentum equations becomes simply the plasma momentum
equation:

\[
\begin{align*}
mn \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] &= \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \left( \sum_s \Pi_s \right) \\
&- \frac{m_e}{e} \mathbf{J} \cdot \nabla \left( \frac{1}{ne} \mathbf{J} \right) + m_e n \left( \frac{m_e}{1 + \frac{m_e}{m_i}} \right) \frac{\mathbf{J}}{ne} \cdot \nabla \left( \frac{1}{ne} \mathbf{J} \right).
\end{align*}
\] (1.38)

This equation is exact so far.

The electron momentum equation can be rearranged to solve for the electric field

\[
\mathbf{E} = -\mathbf{v}_e \times \mathbf{B} - \frac{1}{ne} (-\mathbf{R}_e + \nabla p_e + \nabla \cdot \Pi_e) - \frac{m_e}{e} \left[ \frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla \right] \mathbf{v}_e.
\] (1.39)

Using our closure relation for \( \mathbf{R}_e \), and our expression for \( \mathbf{v}_e \) in terms of \( \mathbf{v} \) and \( \mathbf{J} \), we find an expression called the generalized Ohm’s law:

\[
\begin{align*}
\mathbf{E} &= -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} + \frac{1}{ne} (\mathbf{J} \times \mathbf{B} - \nabla p_e) - \frac{1}{ne} \nabla \cdot \Pi_e \\
&- \frac{m_e}{e} \left[ \frac{\partial}{\partial t} \left( \mathbf{v} - \frac{\mathbf{J}}{ne} \right) + \left( \mathbf{v} - \frac{\mathbf{J}}{ne} \right) \cdot \nabla \left( \mathbf{v} - \frac{\mathbf{J}}{ne} \right) \right] - \frac{m_e}{e} \left( \frac{m_e}{1 + \frac{m_e}{m_i}} \right)^2 \frac{\mathbf{J}}{ne} \cdot \nabla \left( \frac{\mathbf{J}}{ne} \right) \\
&- \left( \frac{m_e}{1 + \frac{m_e}{m_i}} \right) \left( \frac{m_e}{e} \left[ \frac{\partial}{\partial t} \left( \frac{\mathbf{J}}{ne} \right) + \left( \mathbf{v} - \frac{\mathbf{J}}{ne} \right) \cdot \nabla \left( \frac{\mathbf{J}}{ne} \right) + \frac{\mathbf{J}}{ne} \cdot \nabla \left( \mathbf{v} - \frac{\mathbf{J}}{ne} \right) \right] + \frac{\mathbf{J} \times \mathbf{B}}{ne} \right).
\end{align*}
\] (1.40)

Like the summed momentum equation above, this equation remains exact. Eq. (1.38) and Eq. (1.40) exactly reproduce the dynamics of the original two-fluid system.

These combinations of the species’ momentum equations would seem to be an unnecessary complication to the analysis if they remained exactly in the form above. What is often done, then, is that terms that are \( \mathcal{O} \left( \frac{m_e}{m_i} \right) \) are thrown out, greatly simplifying the resulting equations. However, it should be pointed out that the original two-fluid system conserves energy exactly. Throwing out some of these terms while keeping others may have the unintended side effect of breaking conservation of energy in an extended MHD system [68].

The extended MHD system of equations that we will use throws away terms that are \( \mathcal{O} \left( \frac{m_e}{m_i} \right) \), ignores the anisotropic electron stress tensor, and approximates the electron inertia term with only the time derivative of the current density [81]. Then Eq. (1.38) and Eq. (1.40)
reduce to

\[
mn \left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] = J \times B - \nabla p - \nabla \cdot \Pi_i \quad (1.41)
\]

\[
E = -v \times B + \eta J + \frac{1}{ne} (J \times B - \nabla p_e) + \frac{m_e}{ne^2} \frac{\partial J}{\partial t}. \quad (1.42)
\]

This extended MHD system also requires the density evolution equation, Eq. (1.17), the
temperature evolution equations, Eq. (1.19), and the closure relations previously mentioned.
In the limit that the time between particle collisions approaches \( \infty \), the viscous drag, \( \Pi_{iso} \),
and the electrical resistivity, \( \eta J \), both approach zero. This is called the ideal limit, and as we
will see in the next section, this restricts the allowable motions of the plasma.

1.2.3.1 The Generalized Ohm’s Law

The terms in the generalized Ohm’s law, Eq. (1.42), merit some additional consideration.
The first and second terms on the right hand side are present in single-fluid models, while
the third, fourth, and fifth terms stem from two-fluid effects. We will refer to these last
three terms as two-fluid terms. The sum of the first and third terms on the right hand side
represents the electron fluid velocity, \( v_e \times B = v \times B - \frac{1}{ne} J \times B + O\left(\frac{me}{mi}\right) \). If the remaining
terms are negligible compared to this, then the magnetic field is said to be frozen into the
electron fluid.

The \( v \times B \) term is called the MHD term, for reasons that will become apparent momentarily.
If this term is much larger than the \( \frac{1}{ne} J \times B \) term, called the Hall term, then the electron
velocity is approximately equal to the center of mass velocity, \( v_e \approx v = v_i + O\left(\frac{me}{mi}\right) \). In this
case, both electrons and ions move with the common center of mass velocity. The \( \eta J \) term
represents plasma resistivity, the \( \nabla p_e \) term on the right hand side of Eq. (1.42) is referred to
appropriately as the electron pressure term, and the \( \frac{\partial J}{\partial t} \) term is the electron inertia.

1.2.3.2 Force Balance

Eq. (1.41) is often referred to as the plasma momentum equation, and the right-hand-side
describes forces that act on the center of mass flow. For magnetically confined systems,
the magnetic energy is much larger than the kinetic energy, and the system remains in
approximate force balance, \( J \times B \approx \nabla p \) during the evolution. Significant imbalances in the
force density would drive motion on a fast Alfvénic time scale. This does not occur in the
RFP and other magnetic confinement devices, and the MHD dynamics may be considered as
an evolution through a sequence of nearby equilibrium states.
1.2.3.3 Single-Fluid MHD

All extended MHD systems are extensions of the well-known single-fluid MHD system, which is valid when both species move at approximately the center of mass velocity. Weighting the Hall term to the MHD term in the generalized Ohm’s law, and recalling that the system is nearly in force balance, \( \mathbf{J}_\perp \sim \mathbf{B} \times \nabla p/|\mathbf{B}|^2 \), we find

\[
\frac{1}{m_e} \mathbf{J} \times \mathbf{B} \approx \left( \frac{V_{T,i}}{V_0} \right) k \rho_{L,i},
\]

where \( V_0 \) is a characteristic velocity. If we assume \( V_{T,i}/V_0 \sim O(1) \), then the additional Hall terms are negligible when \( k \rho_{L,i} \ll 1 \). The electron pressure and electron inertia terms also represent additional two-fluid effects, and may be ignored in the same limit, so that the single-fluid generalized Ohm’s law is given by

\[
\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J}.
\]

Formally, in the single-fluid MHD system, the ion gyroviscosity must also be discarded. It enters the extended MHD system at the same order as the two-fluid effects in the generalized Ohm’s law, and it is inconsistent to keep it if we have discarded those terms. Reversing this logic, it is also inconsistent to keep the two-fluid terms in the generalized Ohm’s law but neglect the ion gyroviscosity, unless the ions are cold, \( p_i = 0 \). Extended MHD systems that result from taking cold ions but keeping the two-fluid terms in the generalized Ohm’s law are often called Hall MHD systems.

1.3 Basic Concepts

Before delving into the specifics of the reversed-field pinch configuration, it will be useful to introduce several additional concepts for magnetically confined plasmas. For simplicity and specificity, in what follows we will restrict ourselves to the consideration of a straight cylindrical system that is periodic along the axial direction. In addition, everything considered in this section is assumed to be axisymmetric. Toroidal effects will be ignored here.

1.3.1 Pinch Effect

It is well known that for two infinitely long thin parallel wires, the Lorentz force between them is attractive if the currents are in the same direction, and it is repulsive if they are in opposite directions. The magnetic field generated by the first wire exerts a force on the charges moving
in the second wire, via the Lorentz force density $\mathbf{J}_2 \times \mathbf{B}_1$. Similarly, the magnetic field from the second wire, $\mathbf{B}_2$, exerts a force on the charges in the first wire, $\mathbf{J}_1 \times \mathbf{B}_2$. If we generalize this to many infinitely long thin wires, all carrying current in the same direction, we find that the wires will be drawn together. This is known as the pinch effect [5]. It can be thought of as an excess of magnetic pressure outside the current carrying region, where the magnetic fields from each individual wire add synergistically, compared to inside the current carrying region where the field from one wire in general is opposed by an equal and opposite field from another wire.

There is nothing unique to the wires, and the same phenomenon occurs in plasmas, which, as we have seen, are generally very good conductors of electrical current. A linear plasma column may be viewed as the continuous limit of a discrete set of conducting wires. The plasma column will pinch down upon itself as current is directed along it, just as the wires are themselves brought together via their mutual Lorentz attraction.

### 1.3.2 Frozen Flux

If a uniform axial magnetic field that is aligned with the direction of the plasma current, $\mathbf{B}_0 = B_0 \hat{z}$, is added to the plasma column we have just described, then there is no additional Lorentz force as a result. As the plasma pinches down on itself, the magnetic flux through a surface

$$\Phi_B \equiv \int \mathbf{B} \cdot \mathbf{n} \, dA. \quad (1.45)$$

evolves as

$$\frac{d\Phi_B}{dt} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, dA + \int \mathbf{B} \cdot (\mathbf{v} \times d\ell). \quad (1.46)$$

The first term accounts for the change in the local magnetic field through the flux surface, and the second accounts for the motion of the surface itself. The magnetic field evolves according to Faraday’s law, Eq. (1.6), and here we will take the ideal single-fluid MHD Ohm’s law, $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$. Then it follows that

$$\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, dA = -\int \mathbf{B} \cdot (\mathbf{v} \times d\ell). \quad (1.47)$$

This exactly cancels with the second term in Eq. (1.46), and we find that $\frac{d}{dt} \Phi_B = 0$. The fluid motion carries the magnetic flux with it, and the flux may be considered to be frozen into the perfectly conducting fluid.
1.3.3 Magnetic Reconnection

If the magnetic flux is frozen perfectly into the fluid, then fluid motions may twist up the magnetic field into much more complex shapes. However, bending magnetic field lines requires energy, which, in this case, comes from the motion of the fluid itself. Kinetic energy in the flow is converted into magnetic energy of the wound-up field. If the fluid is perfectly conducting, this process may be continued indefinitely as long as kinetic energy is added to the plasma, and the energy in the magnetic field could grow increasingly larger with time.

However, physical processes are non-ideal so it is necessary to examine what happens if we instead allow a small amount of resistivity in the plasma. Using the single-fluid MHD Ohm’s law for a resistive plasma, $E = -v \times B + \eta J$, the change in magnetic flux through a surface becomes

$$\frac{d\Phi_B}{dt} = - \int \eta J \cdot d\ell.$$ (1.48)

The magnetic flux diffuses through the surface. In many cases, the resistivity is small, so that, for small currents, the rate at which the flux diffuses is also small.

However, if the field changes over a very short spatial scale, proportionately large currents exist as a result: $\mu_0 J = \nabla \times B \rightarrow \mu_0 J \sim \frac{\partial}{\partial x} B$. In a 2D system, such a structure constitutes a current sheet. Magnetic flux may diffuse very quickly through this narrow current sheet, and the topology of the magnetic field changes as a result. Such a process is called magnetic reconnection. After reconnection, the magnetic field lines are under a great deal of tension, and releasing this tension converts the stored magnetic energy into kinetic and thermal energy of the plasma fluid.

1.4 The Tearing Mode

The dominant instability in the RFP, the tearing mode, operates through magnetic reconnection. The free energy source for this reconnection comes from gradients in the parallel current density, and tearing modes in the RFP nonlinearly saturate by coupling to other stable modes and relaxing the parallel current profile globally. Here, we will briefly discuss the tearing mode within the single-fluid MHD model to highlight some of the key features of these instabilities. A derivation from the two-fluid framework is beyond the scope of this work, so we will only mention the two-fluid extensions to tearing behavior and comment on its applicability to our nonlinear simulations.
1.4.1 Single-Fluid Tearing

The tearing mode is a resistive instability [47] with global structure that reconnects magnetic field lines locally at a surface where the stabilizing line-bending response of the magnetic field may be minimized. These locations are called resonant surfaces, and they are quantized by a minimum in the derivative along the magnetic field, $\mathbf{B} \cdot \nabla \to 0$. If perturbations are decomposed in a Fourier representation, $\tilde{f} = \sum_{m,n} \tilde{f}_{m,n}(r) \exp [im\theta + in\zeta]$, then the parallel derivative vanishes when

$$B \cdot \nabla \tilde{f} = \sum_{m,n} inB \cdot \nabla \theta \left( \frac{m}{n} + \frac{B \cdot \nabla \zeta}{B \cdot \nabla \theta} \right) \tilde{f}_{m,n}e^{im\theta+in\zeta} = 0,$$

where $B \cdot \nabla \zeta/B \cdot \nabla \theta$ is related to the safety factor $q$ through Eq. (1.1). In a cylindrical system, Eq. (1.49) is separable for every $m$ and $n$, and the parallel derivative vanishes at surfaces where $q = -m/n$, i.e. at surfaces where $q$ is a rational number.

For plasmas with very low resistivity, the ideal MHD system remains an excellent approximation for the plasma behavior. However, the ideal equations become singular at locations where the parallel derivative vanishes, and additional effects such as resistivity are needed to resolve the singularity. These effects are only needed in a narrow region around the resonance, and a boundary layer problem is the most natural way of addressing this dichotomy. Outside of the resonant surface, the ideal MHD equations are solved yielding a solution on either side of it. The non-ideal system is solved in a very narrow region around the resonance, called the inner layer, and the solutions are matched to obtain a dispersion relation.

In the external region, a simplifying assumption is to take $\tilde{B}_x$ to be a constant across the layer, known as the constant-$\psi$ approximation. The solution may then be quantified by

$$\Delta' \equiv \left[ \frac{1}{\tilde{B}_x} \frac{d\tilde{B}_x}{dx} \right]^{x=0+}_{x=0-},$$

the discontinuity in the first derivative of the perturbed magnetic field. As we will see, the growth rate of the tearing mode depends explicitly on $\Delta'$, so that stability is determined completely from the solution in the external region. The tearing stability factor depends intrinsically on the magnetic field profiles, and it is often computed numerically with the use of MHD stability codes. However, for some simple configurations, an analytic solution is possible, which gives some insight into the basic behavior of tearing modes.

An analytic expression for the stability parameter, $\Delta'$, may be obtained for the sheet
pinch equilibrium in slab geometry \[9\]

\[ B_{0y}(x) = B_0 \tanh \left( \frac{x}{L_s} \right) \quad B_{0y}' = \frac{B_0}{L_s} \text{sech}^2 \left( \frac{x}{L_s} \right). \] (1.51)

With boundary conditions \( \tilde{B}_x = 0 \) at \( x \to \pm \infty \), the solution of the outer layer equation (neglecting plasma inertia) yields

\[ \Delta' = \frac{2}{L_s} \left( \frac{1}{k_y L_s} - k_y L_s \right). \] (1.52)

The stability parameter, \( \Delta' \), is largest for the longest wavelength (smallest \( k_y \)) modes, and it is stable for \( k_y L_s \geq 1 \). However, the stability parameter alone is insufficient to determine the growth rate of the mode. To obtain the growth rate, \( \gamma \), one needs to solve the equations in the inner layer.

A detailed solution of the inner layer equations is beyond the scope of this work; we will instead present results outlined in Hazeltine and Meiss [54] merely to point out some of the key features. Hazeltine and Meiss formulate the inner layer equations as

\[ E + \left( \frac{x}{x_A} \right)^2 E - (x x_R)^2 \left( \frac{E'}{x^2} \right)' = \frac{1}{\Delta'} \int_{-\infty}^{\infty} dx \frac{E'}{x} \] (1.53)

where \( x_A \equiv \frac{y}{k_f' \|V_A} \) is the shear-Alfvén width, and \( x^2_R \equiv \frac{\eta \gamma}{\mu_0 \gamma} \) is the resistive skin depth. The first term in this equation represents the plasma inertia, the second represents the magnetic line-bending, the third represents resistive diffusion, and the fourth represents competition between the tearing and kink drive. Taking \( E' \sim E/\delta \), where \( \delta \) is the inner layer width, we have two unknown quantities, \( \gamma \) and \( \delta \), that can be determined by balancing any three of the four terms above. First, ignoring resistivity leads to a singular Alfvén mode, which is stable. Second, neglecting the line-bending term leads to simple resistive diffusion on a slow time scale. Fortunately, the latter two options for the balance describe more interesting dynamics.

The kink term may be justifiably neglected if \( \Delta' \) is large. This results in a layer width and growth rate that scale as

\[ \delta \sim x_A \sim x_R \quad \gamma \sim \left( k_f' V_A \right)^2 \left( \frac{\eta}{\mu_0} \right)^{1/3}. \] (1.54)

This is the resistive kink mode, which has a growth rate that scales as \( \gamma \sim S^{-1/3} \).

The tearing mode is found from the fourth and final choice for balance, which neglects the inertial term. This results in balance between line-bending, diffusion, and the kink drive,
and the layer width and growth rate here scale as

$$
\delta \sim (x_A x_R)^{1/2} \sim \left[ \frac{\gamma}{(k' V_A)^2 \mu_0} \right]^{1/4}, \quad \gamma \sim (\Delta')^{4/5} \left( \frac{\eta}{\mu_0} \right)^{3/5} \left( k' V_A \right)^{2/5}.
$$

(1.55)

Note that if we crudely assume $\Delta' \sim 1/k$, as for the sheet pinch, then the growth rate scaling with wavenumber is $\gamma \sim k^{-2/5}$, and it is reasonable to conclude that the fastest growing modes are those with the longest wavelength. However, this simple analysis is incomplete. A more complete analysis that does not make the constant-$\psi$ approximation shows that the growth rate has a maximum near $\gamma_{\text{max}} = \gamma(k_0)$ where $k_0 \sim \eta^{1/4}$ [9]. The growth rate then decreases with decreasing $k$.

1.4.1.1 General Dispersion Relation for Kink and Tearing Modes

In the preceding analysis, we made use of the constant-$\psi$ approximation to simplify the analysis considerably. This is only valid for $\Delta' \delta \ll 1$, and it breaks down otherwise [9]. More generally, a detailed analysis (see Coppi et al. [28], Ara et al. [4], Migliuolo et al. [77]) that employs Fourier transform techniques can be used to transform the inner layer equations. The small $k$ (large $x$) limit of the exact solution can be matched to the large $k$ (small $x$) limit of the similarly transformed external solution. Term by term matching then yields a dispersion relation which is valid in general for any value of $\Delta'$; it describes the ideal kink, resistive kink, and tearing modes. In cylindrical geometry, this gives

$$
\hat{\lambda} = \hat{\lambda}_H \frac{\hat{\lambda}^{9/4}}{8} \Gamma \left[ \frac{(\hat{\lambda}^{3/2} - 1)}{4} \right] \Gamma \left[ \frac{(\hat{\lambda}^{3/2} + 5)}{4} \right]^{-1},
$$

(1.56)

where $\hat{\lambda} = \lambda/\epsilon^{1/3}$ is a normalized growth rate and $\hat{\lambda}_H \sim -1/\Delta'$ is a normalized ideal stability parameter (details in Ara et al. [4]). The numerical solution of this dispersion relation is shown in Fig. 1.2. The small $\Delta'$ tearing modes exist for $\hat{\lambda}_H \ll 0$, and this corresponds to configurations which are very far from ideal instability. The ideal kink exists in the opposite limit, $\hat{\lambda}_H \gg 0$, and the resistive kink exists between these two extremes.

1.4.2 Two-Fluid Tearing

As we have seen, the single-fluid MHD system breaks down at scales below the ion Larmor radius, $\rho_{L,i}$. Within this model, the tearing layer width scales as $\delta \sim \eta^{2/5}$, so that as the plasma conditions become more ideal, $\eta \to 0$, the tearing layer widths are predicted to become
quite narrow. Consequently, when the tearing layer width $\delta$ approaches $\rho_{L,i}$, we expect that additional two-fluid effects will become important in the dynamics.

The standard tearing treatment discussed previously is applicable when when $\rho_{L,i} \ll \delta$, where $\delta$ is the layer width, such that the ion FLR effects may be neglected. In this regime, if $\gamma \ll \nu_e$, where $\nu_e = (\eta/\mu_0) / d_e^2$ is the collision frequency, then the tearing mode is classified as collisional. When $\rho_{L,i} \gtrsim L$, the finite Larmor radius effects of the ions become important. In this limit, Drake and Lee [37] identify a semi-collisional regime when $\gamma \ll \nu_e$ and a collisionless regime when $\gamma \gg \nu_e$. Electron inertia allows for collisionless reconnection when the tearing layer width becomes smaller than the electron skin depth, $d_e = c/\omega_{pe}$ [8, 37, 38].

Low-current plasmas, which are more amenable to insertable probes, are typically also lower temperature. As a result, the plasmas in these discharges are much more likely to be in the collisional or semi-collisional tearing regimes, rather than the collisionless regime. In addition, computational practicalities presently limit our nonlinear RFP simulations to modest values of plasma resistivity, so we will primarily be concerned with collisional and semi-collisional tearing.

Another salient feature of two-fluid models is the addition of drift waves into the system. Drift waves are stable waves that propagate as a result of thermodynamic gradients and the difference in electron and ion velocities in the two-fluid system, and these waves may affect the growth of the tearing mode. The earliest derivation is given by Coppi [27], who shows that for sufficiently large drift frequencies, the tearing mode can be stabilized. Ara et al. [4] derive a general dispersion relation for cylindrical systems,

$$\left[ \hat{\lambda} (\hat{\lambda} - \hat{\lambda}_i) \right]^{1/2} = \frac{\hat{\lambda}_H}{8} \Lambda^{9/4} \frac{\Gamma \left[ \left( \Lambda^{3/2} - 1 \right) / 4 \right]}{\Gamma \left[ \left( \Lambda^{3/2} + 5 \right) / 4 \right]},$$

which is analogous to the previous single-fluid MHD dispersion relation (Eq. (1.56)). Complete
stabilization of the drift-tearing mode is then found for sufficiently large values of the diamagnetic drift frequencies [77].

The diamagnetic drifts rely on pressure gradients for drive, but as the tearing mode evolves nonlinearly into a magnetic island, the pressure gradients in the vicinity of the resonant surface are expected to drop off dramatically. The presence of the island enables rapid thermal conduction across the region, and pressure gradients are unable to be supported [107]. Consequently, the stabilization from these effects is not expected to be significant during the nonlinear evolution. King et al. [69] identify a similar drift stabilization mechanism for pinch profiles that results from the ion gyroviscous stress, where the leading order contributions are expressed as drifts originating from $\nabla |B|$ and poloidal curvature. In contrast to the diamagnetic drifts, these drifts do not vanish in the nonlinear island evolution phase and substantially reduce the saturated magnetic island width.

The drifts identified by King et al. [69] are significant only in an intermediate regime where the tearing dynamics are not completely separated from the response of the ion fluid. When the ion response is decoupled, the tearing growth rates are increased as a result of transitioning to the faster electron dynamics, and complete stabilization through these drift effects is not realized. This faster tearing growth was missed in the treatment of Ara et al. [4] and Migliuolo et al. [77] as a result of the neglect of electron compressibility, which formally eliminates Hall effects from the system [130]. Electron compressibility is included in later treatments, most notably in Mirnov et al. [81] and Ahedo and Ramos [1]. This allows coupling to the kinetic Alfvén and whistler waves and increases the growth rate as the tearing dynamics become dominated by the faster electron fluid. However, in these treatments the diamagnetic drift effects are excluded as a result of uniform equilibrium pressure. King and Kruger [72] extend the analysis of Ahedo and Ramos [1] to include the finite pressure gradients, and analytic dispersion relations are derived asymptotically for several regimes classified by $\beta$ and a normalized skin depth, $d_i/\delta$.

Our nonlinear computations here have uniform equilibrium pressure in order to eliminate the effects of interchange drive and retain only the current-driven dynamics, and there are no equilibrium diamagnetic flows. However, our two-fluid computations indicate that non-MHD effects are significant. We see enhanced growth rates in our two-fluid computations without ion gyroviscosity, and the relaxation events are much more regular than in the single-fluid limit. When the ion gyroviscosity is also included, the magnetic activity following the initial event is greatly reduced. However, as we mentioned previously, the gyroviscous response scales as $p_i$, and it is expected to be small near the plasma edge with a realistic pressure profile. Because the dominant nonlinear coupling in the RFP occurs through both core-resonant and edge-resonant modes, a modification of the gyroviscous response for the edge-resonant modes
may have important effects on the relaxation dynamics.

1.5 The Paramagnetic Pinch

A knowledge of the basic plasma pinch will be useful in understanding the behavior of the RFP, and we will close this chapter with a brief description of the paramagnetic pinch. Consider an initially uniform axial magnetic field embedded in a uniform plasma surrounded by a perfectly conducting wall. The plasma carries no current initially, but we will drive it through an inductively applied electric field. As we drive axial current in the plasma, the plasma will pinch together as a result of the pinch effect. If the plasma itself is a nearly perfect conductor, then the initial axial magnetic field must be carried inward by the plasma fluid as it constricts. Because the boundary of the plasma column is a perfect conductor, the total axial magnetic flux must remain constant. Consequently, the axial magnetic field will peak on axis.

This evolution may be understood as follows. Initially, there is no current and the magnetic field is uniform, \( B_0 = B_{z,0} \hat{z} \). As axial current is driven along the initial magnetic field lines, this current, \( J_0 = J_{z,0} \hat{z} \), gives rise to an additional poloidal magnetic field, \( B_1 = B_{\theta,1} \hat{\theta} \). As we saw in Sec. 1.1, the motion of the charge carriers may be approximated as simple flow along magnetic field lines. We will therefore posit that the current here remains always parallel to the magnetic field, \( \mu_0 J = \lambda B \), where \( \lambda \) is a measure of the parallel current,

\[
\lambda \equiv \frac{\mu_0 J \cdot B}{|B|^2}.
\]

The parallel current flowing along the azimuthal field lines constitutes a new poloidal current, \( J_1 = J_{\theta,1} \hat{\theta} \), which also has an associated axial magnetic field, \( B_2 = B_{z,2} \hat{z} \). This field adds to the initial axial magnetic field in the center of the plasma column and subtracts from it outside, explaining the peaking of magnetic flux on axis.

The inductively applied electric field will only drive as much current as can be sustained against resistive losses in the plasma. However, if the current drive is maintained for a time much longer than the equilibration time of the plasma and there are no plasma instabilities and \( |\nabla p| \ll |B|^2/2\mu_0 a \), then we will eventually come to an Ohmic steady-state with a fixed magnetic field profile. This configuration is known as a paramagnetic pinch [99]. In the periodic cylinder under consideration here, this steady-state is described by

\[
\frac{\partial B_\theta}{\partial t} = -\frac{\partial E_z}{\partial r} = 0 \quad \frac{\partial B_z}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) = 0 \tag{1.59}
\]
with the solution $E_\theta = 0$ and $E_z = V_{\text{loop}}/2\pi R_0$ where $V_{\text{loop}}$ is the inductive “toroidal” loop voltage which is spatially constant in our periodic cylinder approximation. Similarly, the decomposition of the single-fluid MHD Ohm’s law, $\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J}$, becomes

\begin{align*}
E_\theta &= -v_r B_z + \frac{\eta}{\mu_0} \lambda B_\theta \\
E_z &= -v_r B_\theta + \frac{\eta}{\mu_0} \lambda B_z.
\end{align*}

(1.60)

One may solve this system for the steady-state parallel current profile,

$$
\lambda_{ss} (r) = \lambda (0) B_z (0) \frac{B_z (r)}{| \mathbf{B} (r) |^2},
$$

(1.61)

and the radial velocity,

$$
v_{r,ss} (r) = -\frac{\eta}{\mu_0} \lambda (0) B_z (0) \frac{B_\theta (r)}{| \mathbf{B} (r) |^2},
$$

(1.62)

which represents the inward pinch flow of the plasma. There is compression associated with this pinch flow, so that the density evolution is not truly steady-state. However, the pinch flow scales directly with resistivity and we expect that this contribution only provides a small contribution to the overall radial transport in magnetically confined plasmas. We will see that the pinch effects are generally negligible in our computations even at values of plasma resistivity that are significantly higher than in experiment. The parallel current on axis may be related to the loop voltage and the resistivity as

$$
\lambda (0) = \frac{\mu_0}{\eta} \frac{1}{B_z (0)} \frac{V_{\text{loop}}}{2\pi R_0} \quad \rightarrow \quad V_{\text{loop}} = 2\pi R_0 \lambda (0) B_z (0) \frac{\eta}{\mu_0}.
$$

(1.63)

This provides a useful reference for determining the value of loop voltage that is needed to drive a given amount of parallel current on axis for a given resistivity and magnetic field in the Ohmic steady-state.

The paramagnetic pinch is the natural final configuration of a driven plasma in the absence of any instabilities [95], but it is not a minimum magnetic energy state. The axial loop voltage injects energy continuously into the plasma via the Poynting flux on the boundary, $\mathbf{E} \times \mathbf{B}/\mu_0$, and this energy is transported radially inward where it is resistively dissipated by the plasma [113]. The parallel current profile in this Ohmic steady-state, $\lambda_{ss}$, is quite peaked for RFP conditions, however. Unless the driven parallel current is small enough to produce a final state with insufficient drive for instability, it is unlikely that the plasma will achieve a paramagnetic pinch state before instabilities affect the evolution.

Lastly, note that an axisymmetric paramagnetic pinch can not support a reversed axial
magnetic field. At most, the axial magnetic field at the boundary may drop to zero, but it cannot cross it. To see this, consider the projection of the Ohm’s law onto the magnetic field,

\[ \mathbf{E} \cdot \mathbf{B} = \eta \mathbf{J} \cdot \mathbf{B} \quad \rightarrow \quad B_z = \frac{\eta}{\mu_0} \frac{2\pi R_0}{V_{\text{loop}}} \lambda |\mathbf{B}|^2. \quad (1.64) \]

If \( B_z \) is zero at some location in the plasma, it follows that \( \lambda \) must also vanish there. If \( B_z = 0 \), the parallel current density is simply \( \lambda = \mu_0 J_\theta B_\theta / |\mathbf{B}|^2 \). The field \( B_\theta \) is inductively driven, and it does not reverse in the plasma, \( B_\theta > 0 \), so then \( J_\theta \) must vanish. However, from Ampère’s law, \( J_\theta = -\frac{d}{dr} B_z \). It follows that \( B_z \) vanishes when its derivative vanishes, i.e. at an extremum of \( B_z \). Therefore, \( B_z = 0 \) is the minimum, and \( B_z \) may not reverse in the paramagnetic pinch.
2 THE REVERSED-FIELD PINCH

2.1 The Reversed-Field Pinch

The reversed-field pinch [10] (RFP) is a class of toroidal magnetic confinement device with a magnetic field that is largely generated by currents within the plasma itself. These currents are driven inductively by a transformer that links the center of the device, and the discharge duration is typically limited by the available volt-seconds in the transformer. In the RFP, the toroidal and poloidal magnetic fields are comparable, and the toroidal magnetic field at the boundary is reversed relative to its direction on the axis of the device. Early research into toroidal pinches [15, 16, 91] showed that improved stability properties and plasma confinement are associated with having this reversed axial magnetic field at the edge of the device. Toroidal field reversal may be achieved through appropriate application of external fields [115], but in many cases the plasma dynamics alone is sufficient to cause field reversal; the latter situation is referred to as self-reversal.

Our computations are performed for parameters similar to the Madison Symmetric Torus [33] (MST), a reversed-field pinch device at the University of Wisconsin-Madison. MST is the second-largest RFP in the world, and it hosts an extensive suite of diagnostics allowing for detailed measurements of the plasma dynamics. Typical magnetic field profiles for an RFP plasma in MST are shown in Fig. 2.1(a). The toroidal field is peaked in the center and reverses near the edge of the plasma, where the poloidal field is large. Because the toroidal and poloidal magnetic fields are comparable in the RFP, the safety factor is less than unity, as can be seen in Fig. 2.1(b). There are many low order rational surfaces, where \( q = -\frac{m}{n} \) for integer \( m \) and \( n \), in the core of the plasma. At each surface, the \((m, n)\) tearing mode

![Figure 2.1: Typical (a) magnetic field profiles and (b) safety factor in the MST RFP. Images from Den Hartog et al. [30], Fig. 1 and Fig. 3.](image)
is resonant. The reversal surface, where $q = 0$, exists near the edge of the plasma, and all $m = 0$ modes are resonant here.

The level of reversal in an RFP may be quantified by the reversal parameter,

$$F \equiv \frac{\langle B_z \rangle |_{r=a}}{\langle B_z \rangle_{\text{vol}}},$$

(2.1)

the ratio of the averaged toroidal magnetic field at the boundary of the device and the averaged toroidal field over the whole volume of the device. When $F > 0$, the plasma is not reversed, and when $F < 0$ it is. A similarly useful parameter for RFP discharges is

$$\Theta \equiv \frac{\langle B_\theta \rangle |_{r=a}}{\langle B_z \rangle_{\text{vol}}},$$

(2.2)

a dimensionless measure of the toroidal current in the plasma.

The first attempts at understanding the self-reversal process in the RFP [120] began by considering variational theories of plasma relaxation [127]. Variational theories predict the relaxed state of the plasma by minimizing free energy while maintaining the initial value of robust constraints. The robust constraints are often selected from among the ideal invariants of the system. The quantities that suffer the least dissipation in the resistive system are selected as the constraints, while one or more of the strongly dissipated quantities are minimized.

Within the single-fluid MHD system, the ideal invariants are the total energy of the system, $W = W_B + W_K + W_P$, and the magnetic helicity, a topological measure of the linkedness of the magnetic field lines in a volume [6, 85, 127]. Mathematically, it is

$$\mathcal{K} \equiv \int \mathbf{A} \cdot \mathbf{B} \, d^3x,$$

(2.3)

where $\mathbf{A}$ is the magnetic vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. This quantity is well-defined only if the magnetic field does not leave the volume, that is, $\mathbf{B} \cdot \hat{n} dA = 0$ on the surface of the volume. If this is not the case, then a gauge transformation, $\mathbf{A} \to \mathbf{A} + \nabla \chi$, may change the value of the magnetic helicity, rendering it a meaningless concept. Gauge-invariant definitions of the magnetic helicity [6, 41, 93, 97, 129] rely on a relative magnetic helicity

$$\mathcal{K}_{\text{rel}} \equiv \int (\mathbf{A} \mp \mathbf{A}') \cdot (\mathbf{B} \pm \mathbf{B'}) \, d^3x$$

(2.4)

where $\mathbf{A}'$ and $\mathbf{B}' = \nabla \times \mathbf{A}'$ are reference fields. The different definitions of relative magnetic helicity may be shown to be equivalent, and the quantity itself is gauge-invariant (see Appendix F).
In the single-fluid MHD system, the magnetic helicity in any flux surface is a robust invariant of the motion and is perfectly conserved in the ideal system. There exists an infinity of constraints, one for every possible flux surface. However, even a small amount of resistivity breaks the invariance of the magnetic helicity. Taylor [120] posited that in the presence of small resistivity, only the global magnetic helicity remains well conserved relative to the magnetic energy. This was later justified theoretically [121], and experimental measurements on MST demonstrate that the magnetic helicity is more robustly conserved than the magnetic energy during relaxation [63]. At a relaxation event, the magnetic energy decreases between 4.0% and 10.5%, while the magnetic helicity decreases between 1.3% and 5.1%.

The solution of the variational problem that minimizes the magnetic energy and conserves global magnetic helicity predicts that

$$\mu_0 J = \lambda_0 B$$

where $\lambda_0$ is a global constant. That is, the current is everywhere parallel to the magnetic field, and the factor of proportionality is everywhere constant; there are no gradients in the $J_\parallel / B$ profile. In general, Eq. (2.5) admits both axisymmetric and non-axisymmetric solutions. The relaxed state is found from among these as the state with the lowest energy.

In cylindrical geometry, for sufficiently small values of the driven current, $\lambda_0 a \leq 3.11$, the relaxed state is the axisymmetric one [96]. This is called the Bessel Function Model (BFM),

$$B_z = B_0 J_0 (\lambda_0 r) \quad B_\theta = B_0 J_1 (\lambda_0 r),$$

where $J_0$ and $J_1$ are Bessel functions. With sufficiently large current drive, $\lambda_0$, the axial magnetic field will pass through zero at some point in the plasma, and the field at the wall will be reversed relative to its on-axis value. The critical value for onset of field reversal is readily found to be $\lambda_0 a \approx 2.4$. The parallel current parameter, $\lambda_0$, may be related to the dimensionless current, $\Theta$, in the BFM as $\lambda_0 = 2\Theta/a$, where $a$ is the minor radius. Reversal is predicted by the BFM for $\Theta \gtrsim 1.2$, which agrees qualitatively with experimental measurements [10] which show an increasing tendency for reversal to occur as $\Theta$ increases.

Despite the agreement in predicting the onset of field reversal, the BFM does not agree with experiments in other aspects. Most notably, the BFM predicts that the parallel current in the fully relaxed state is completely flat, but this is not borne out in experimental observations where the parallel current typically falls to zero towards the edge of the plasma [14, 122]. In addition, the Taylor theory does not address the plasma pressure or the plasma flow. We will discuss various extensions to the Taylor theory that attempt to incorporate these effects with additional constraints in Chapter 7.
2.2 The RFP Dynamo

The variational approach says nothing about how the plasma gets to the relaxed state, it only predicts where the dynamics should lead it. Understanding the reversal process itself requires additional considerations. Before delving into those considerations, we note that the reversal of the axial magnetic field can not be sustained by the axisymmetric fields themselves, and in particular, the Taylor state can not be maintained against the effects of resistive diffusion indefinitely. To see this, it suffices to consider the flux inside the reversal surface, \( q = 0 \). Recalling Eq. (1.48), the toroidal flux inside this surface evolves as

\[
\frac{d\Phi_B}{dt} = -\int E|_{q=0} \cdot d\ell = -\int \eta|J|| d\theta. \tag{2.7}
\]

The conversion to the parallel component merits some further discussion. For reference, when comparing with results from MST, where discharges can have either positive or negative \( \lambda \), detailing the effects of positive and negative \( E_\parallel \) is worthwhile. First, consider the case when \( \lambda > 0 \). Here, \( B_\theta > 0 \) and at the reversal surface \( E \cdot d\ell = E_\parallel d\theta \sim \eta J_\parallel d\theta \), where \( J_\parallel > 0 \) since \( \lambda > 0 \). In the opposite case, when \( \lambda < 0 \), we have \( B_\theta < 0 \) and \( E \cdot d\ell = -E_\parallel d\theta \sim -\eta J_\parallel d\theta \). But \( J_\parallel < 0 \) now, because of our choice of \( \lambda \), so that \( E \cdot d\ell \sim \eta|J|| d\theta \).

In both cases, the toroidal flux will diffuse through the reversal surface on a resistive time scale, and field reversal will eventually be lost. However, this contradicts measurements in RFP experiments where reversal is maintained longer than the flux diffusion time, that is, \( E_\parallel = \mathcal{E}_\parallel + \eta J_\parallel \). The anomalous electric field, \( \mathcal{E} \), is referred to as a dynamo electric field, and it results from the nonlinear dynamics of the plasma that act to sustain the reversed-field state. The sign of \( \mathcal{E}_\parallel \) and the sign of \( \lambda \) (i.e. \( J_\parallel \)) determine whether the dynamo acts to increase the parallel current or decrease it. If \( \lambda > 0 \), then \( \mathcal{E}_\parallel > 0 \) results in a smaller \( J_\parallel \) than would exist in the absence of the dynamo electric field; that is, the dynamo reduces current and is more properly termed anti-dynamo. In the same situation, if \( \mathcal{E}_\parallel < 0 \), then a larger \( J_\parallel \) can be supported against resistive decay, and the dynamo electric field drives current. The opposite holds when \( \lambda < 0 \). This is summarized in Table 2.1.

<table>
<thead>
<tr>
<th>( \lambda &gt; 0 )</th>
<th>( \mathcal{E}_\parallel &gt; 0 )</th>
<th>dynamo</th>
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<tbody>
<tr>
<td>( \lambda &lt; 0 )</td>
<td>( \mathcal{E}_\parallel &lt; 0 )</td>
<td>anti-dynamo</td>
</tr>
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The origin of the dynamo electric field may be understood through mean-field analysis.
Consider first the decomposition of the $-\mathbf{v} \times \mathbf{B}$ term in the generalized Ohm’s law, Eq. (2.9). There are two contributions to the axisymmetric component of the mean-field decomposition of this term. The first, $-\langle \mathbf{v} \rangle \times \langle \mathbf{B} \rangle$, results from the interaction of the axisymmetric velocity and axisymmetric magnetic field, but this term does not contribute to the electric field parallel to $\langle \mathbf{B} \rangle$. The second, which results from correlated fluctuations of velocity and magnetic field, also gives rise to an axisymmetric electric field, the MHD dynamo \[ E_{\text{MHD}} \equiv -\langle \mathbf{\tilde{v}} \times \mathbf{\tilde{B}} \rangle. \] (2.10)

The MHD dynamo is observed to be significant during the initial relaxation event in our computations, which begin from a paramagnetic pinch equilibrium, as can be seen in Fig. 2.2. These computations will be described more fully later, but the important point to note here is...
is that the sign of the MHD dynamo depends on the sign of the parallel current; when \( \lambda > 0 \),
the MHD dynamo is positive in the core, acting to reduce the parallel current there, and
negative in the edge, acting to drive parallel current. The converse is true when \( \lambda < 0 \),
and the net effect in both cases is a flattening of the parallel current profile.

Pioneering work in computational MHD [19, 52, 57, 58, 74, 106, 119] has led to a detailed
understanding of the MHD dynamo and the sustainment of the reversed-field state as a
nonlinear interaction of multiple tearing modes. The dominant interaction here is three-wave
coupling [53, 57, 101]. Core-resonant \( m = 1, n = n_0 \) and edge-resonant \( m = 0, n = 1 \)
nonlinearly couple to \( m = 1, n = n_0 \pm 1 \), broadening the \( m = 1 \) spectrum. Coupling between
\( m = 1 \) modes of different \( n \) allows energy to be transferred into the stable \( m = 2 \) modes,
allowing nonlinear saturation of the spontaneously excited \( m = 1 \) mode and providing an
outlet for the free energy. Energy flows out of these stable modes through resistive and
viscous dissipation, and it may be deposited back into the axisymmetric fields at different
radial locations. This flattens the parallel current profile and drives the plasma closer towards
the relaxed state predicted by Taylor theory. In the process, poloidal flux is converted into
toroidal flux which helps to sustain the reversed-field configuration.

Two-fluid effects in the generalized Ohm’s law give rise to additional dynamo mechanisms
through the Hall term, and the Hall dynamo [88] is given by

\[
\mathcal{E}_{\text{Hall}} \equiv \frac{\langle \tilde{J} \times \tilde{B} \rangle}{\langle n \rangle e}.
\]  

(2.11)

The Hall dynamo is the dominant piece of the Hall term in the generalized Ohm’s law,
Eq. (2.9), which may be expanded as

\[
\frac{J \times B}{ne} = \frac{J \times B}{\langle n \rangle e (1 + \tilde{n}/\langle n \rangle)} = \frac{J \times B}{\langle n \rangle e} \left[ 1 - \frac{\tilde{n}}{\langle n \rangle} + \mathcal{O} \left( \left( \frac{\tilde{n}}{\langle n \rangle} \right)^2 \right) \right],
\]  

(2.12)

where we have assumed small density fluctuations, \( \tilde{n}/\langle n \rangle \ll 1 \). Ignoring the higher order
density perturbations, and taking the axisymmetric part of this, we find

\[
\langle J \times B \rangle \approx \frac{\langle J \rangle \times \langle B \rangle}{\langle n \rangle e} + \frac{\langle \tilde{J} \times \tilde{B} \rangle}{\langle n \rangle e} - \frac{\langle J \rangle \times \langle \tilde{n} \tilde{B} \rangle}{\langle n \rangle e} - \frac{\langle \tilde{n} \tilde{J} \rangle \times \langle B \rangle}{\langle n \rangle e} - \frac{\langle \tilde{n} \tilde{J} \times \tilde{B} \rangle}{\langle n \rangle e}.
\]  

(2.13)

The first term and second to last term have no projection parallel to \( \langle B \rangle \). We may neglect the
third to last term for similar reasons if the plasma is nearly force-free, \( \langle J \rangle \approx \lambda \langle B \rangle \). Finally,
the last term is third order in the fluctuations and may be neglected. The only significant
contribution to the parallel electric field then comes from the Hall dynamo term, Eq. (2.11).
The Hall dynamo is measured to be significant in our nonlinear computations, as can be seen in Fig. 2.2. During the initial relaxation event, both the MHD and Hall dynamo cooperate to relax the parallel current in the core of the plasma and increase it in the edge of the plasma, driving the RFP closer to the flat $\lambda$ Taylor state prediction. This cooperation is also observed in the initial relaxation event of previous computations at $S = 80,000$ [70], as can be seen in Fig. 2.3.

### 2.2.1 Experimental Measurements

The vast suite of plasma diagnostics available on MST is able to provide experimental evidence for the fluid dynamo model. Chordal spectroscopic measurements of velocity fluctuations in the plasma core are correlated with edge measurements of magnetic field fluctuations, assuming a phase relationship between core and edge $\hat{B}$, to yield estimates for the MHD dynamo in the plasma core [30]. In the plasma edge, insertable optical probes [40] are used to infer the radial and toroidal velocity fluctuations of impurity helium ions, which are assumed to match the bulk flow velocity. These are correlated with measurements of the fluctuating radial and toroidal magnetic field, measured via a multicoil magnetic probe at the same toroidal angle but separated poloidally by 18°, to estimate the MHD dynamo in the edge region [45]. The MHD dynamo is found to account for the change in toroidal flux near the edge of the plasma in MST discharges, as can be seen in Fig. 2.4. However, these same measurements of fluctuating flow and magnetic field are unable to account for the measured change in flux just inside the reversal surface.

Measurements of the Hall dynamo near the edge, $r/a \approx 0.92$, where the MHD dynamo
accounts for the anomalous electric field, found that the Hall dynamo was small there [109]. However, Kuritsyn et al. [73] measure the Hall dynamo contribution near the plasma reversal surface, \( r/a \approx 0.83 \), using insertable magnetic probes and find that it is able to account for the anomalous electric field where the MHD dynamo does not, as can be seen in Fig. 2.5. Flux conversion in the plasma edge region then occurs through both the MHD dynamo and the Hall dynamo in MST, although the exact balance of these two dynamo terms remains uncertain. An improved magnetic probe that may be inserted deeper into the plasma is expected to provide much more detailed information about the radial structure of the Hall dynamo [123].

Prior to Kuritsyn’s work, measurements of the Hall dynamo from the core-resonant \( m = 1, n = 6 \) tearing mode were made in the core of MST plasmas using laser Faraday rotation [35, 36]. They show a contribution of the \( m = 1, |n| = 6 \) mode to the Hall dynamo that is localized around the \( q = 1/6 \) surface. It is of sufficient magnitude to account for the parallel electric field there, as can be seen in Fig. 2.6. Contributions to the Hall dynamo from other modes are measured to be small at that radial location and are neglected in their
analysis. Their measurements of the Hall dynamo term are strongly correlated with the edge $m = 0$ mode amplitudes; when the $m = 0$ resonant surface is removed ($F = 0$ discharges), the measured Hall dynamo term vanishes. They argue that nonlinear three-wave coupling is needed to drive significant Hall dynamo. Further, based on the measured balance of Hall dynamo and electric field, Ding et al. [35] infer that the MHD dynamo must be small at the $q = 1/6$ surface.

As we will see, our nonlinear computations indicate that there are significant contributions from terms omitted in the analysis of Ding et al. [35]. First, we find that many modes contribute to the Hall dynamo at the location of the $q = 1/6$ surface, not just the $m = 1, n = 6$ mode. Second, we find that the MHD dynamo contributions to the parallel electric field are significant there as well. Lastly, as detailed in King [71] (see Sec. 18.2 and Appendix G in
that reference), the measurements made in Ding et al. [35] only consider one piece of the total contribution of the $m = 1, n = 6$ mode to the Hall dynamo. The nonlinear numerical computations of King [71] find that terms neglected in Ding et al. [35] are comparable to the measured term.

### 2.2.2 Discrepancies

Fontana et al. [45] show a parallel electric field near the edge that is positive, and the corresponding measured MHD dynamo is also positive. With $\lambda < 0$, this suggests an increase of parallel current in the edge region, consistent with the current relaxation process and our numerical computations. However, the results reported in Kuritsyn et al. [73] show a parallel electric field at the edge of the plasma that is negative, and a corresponding negative Hall dynamo term, $\langle \mathbf{J} \times \mathbf{B} \rangle_{\parallel} / \langle n \rangle e < 0$. With $\lambda < 0$, this would suggest a reduction of edge parallel current, not an increase.

Similarly, Ding et al. [35] show a contribution to $\langle \mathbf{J} \times \mathbf{B} \rangle_{\parallel} / \langle n \rangle e$ that is positive in the core. With $\lambda < 0$, this implies that the Hall dynamo compensates some of the resistive $\eta J_{\parallel} < 0$ and acts to increase the magnitude of the current density in that region. However, this directly contradicts their statement that “the large Hall electromotive force is comparable to the induced electric field...and acts to suppress equilibrium current during plasma relaxation.”

Based on discussion with MST experimentalists [private comm.], we conclude that Hall dynamo orientations reported in Ding et al. [35] and Kuritsyn et al. [73] are merely a convention of the measurements. That is, they have re-normalized their measurements such that a positive dynamo electric field corresponds to an anti-dynamo term, a reduction in the parallel current, and vice versa. In our computational results presented here and later in Chapters 4-7, we always specify the sign of $\lambda$. The dynamo or anti-dynamo nature of the electric field is then determined using the conventions in Table 2.1.

### 2.2.3 The Sawtooth Cycle

In principle, the activity that maintains the plasma in the reversed-field state may be either continuous or periodic. Measurements on experimental devices [30, 126] show that, for sufficiently large values of current drive, the toroidal flux is sustained in discrete bursts of magnetic activity, as the nonlinear plasma dynamics convert some of the supplied poloidal flux into toroidal flux. These events are observed to occur quasi-periodically in a typical discharge; this is the RFP sawtooth cycle. Measurements of the sawtooth cycle on MST are shown in Fig. 2.7. The toroidal flux increases by around $7 - 10\%$ at each sawtooth event [30].
The sawtooth cycle may be understood qualitatively as follows. Resistive diffusion and the inductive electric field drive the plasma towards its Ohmic steady-state, the paramagnetic pinch. This results in a peaking of the parallel current profile in the core of the plasma, and the resulting gradient provides free energy for the core-resonant $m = 1$ tearing modes. After an initial period of linear growth, these modes nonlinearly couple to stable modes in the system [58] and reduce their own drive through profile modification [57, 93]. The plasma is driven closer to the flat $\lambda$ state predicted by Taylor relaxation, but it does not reach it. Resistive diffusion and the inductive drive rebuild the current gradient, driving the plasma back towards the Ohmic steady-state, and the whole relaxation process may then be repeated quasi-periodically.

Nonlinear computations of RFP dynamics are consistent with this picture of the sawtooth cycle [17]. When the resistivity is large, the dynamo electric field is nearly continuous, but quasi-periodic oscillations of dynamo activity are observed when the resistivity is sufficiently small. This behavior is also observed in single-fluid MHD modeling using the cylindrical DEBS code [98, 100, 104]. However, DEBS uses an artificial viscosity that increases as needed to reduce grid-scale fluctuations during relaxation events, and it is not clear what effect this has on the computed evolution. In contrast to the previous DEBS computations, our computations have a fixed viscosity, and they also exhibit quasi-periodic relaxation events that are consistent with the previous picture of the RFP sawtooth cycle, as we will see in Chapter 4.

2.3 Flow in the RFP

In addition to the current relaxation dynamics, large changes in plasma flow are observed during sawteeth in the MST RFP [31, 53, 73]. Kuritsyn et al. [73] used Rutherford scattering
measurements to infer the bulk poloidal flow in the core, while the core toroidal flow was inferred from measurements of the mode rotation frequency using signals from a toroidal magnetic array. This was combined with Mach probe measurements of the flow near the edge to reconstruct the parallel flow profile in MST over a sawtooth cycle. In particular, the parallel flow profile is observed to flatten across the plasma radius over a relaxation event, as can be seen in Fig. 2.8. Prior to the relaxation event, the parallel flow is strongly sheared, being negative in the core and increasing to a positive value near the edge. At the crash, however, the flow flattens across the radius, suggesting a rapid equilibration of parallel momentum.

This coupling of current and flow relaxation may be readily explained within the extended MHD framework. Correlated fluctuations of current density and magnetic field arise in the axisymmetric component of the plasma momentum equation, Eq. (1.41), as a Lorentz force density

\[ \mathbf{F}_{\text{Lorentz}} \equiv \langle \mathbf{J} \times \mathbf{B} \rangle. \]  

(2.14)

If the Hall dynamo associated with the current relaxation dynamics is large, as it is in both our computations and experimental measurements on MST, then the Lorentz force density is also significant, and the current relaxation dynamics are naturally coupled to the flow evolution.

Previous extended MHD computations show significant changes in parallel flow associated with the relaxation event [70], as can be seen in Fig. 2.9(a). Similar changes are not observed with a single-fluid Ohm’s law, and the conclusion is that two-fluid effects are needed in the
model for substantial changes in plasma flow to occur during current relaxation.

For a magnetically confined plasma, the magnetic energy is much greater than the kinetic energy, and we may expect that the back reaction of the flow onto the current relaxation dynamics is negligible; the Lorentz force density may be viewed as a simple source term in the momentum balance. This large source term will drive fluid turbulence, and the resulting Reynolds force density, \( m_i \langle \nabla \cdot \mathbf{v} \rangle \), is observed to nearly balance in these computations, as can be seen in Fig. 2.9(b). Note, however, that the change in plasma flow is in the direction of the Lorentz force density, consistent with the magnetic relaxation dynamics determining the evolution of the flow. Similar results are also observed in our computations, as will be seen later and in Chapter 4. Changes in parallel flow are driven by the Lorentz force density, and this is balanced by the fluctuation-induced Reynolds force density.

### 2.3.1 Experimental Measurements

The balance between the fluctuation induced Lorentz force density and the Reynolds force density is also observed in measurements at the edge of MST [73], as can be seen in Fig. 2.10. Note that the change in parallel flow at the edge is negative, and it is in the direction of the Lorentz force density, as observed in both our computations and those of King et al. [70]. This is consistent with the magnetics setting the directionality of the flow relaxation. Like the Hall dynamo measurements of Ding et al. [35], Kuritsyn et al. [73] find that the momentum transport is significantly reduced when the \( m = 0 \) resonance is removed from the plasma. Correlated fluctuations of \( \mathbf{J} \) and \( \mathbf{B} \) evidently require significant nonlinear interaction. Our
nonlinear computations further bolster this argument. We find that $\langle \mathbf{J} \times \mathbf{B} \rangle$ is significant primarily during nonlinear relaxation events in our computation, and it is small otherwise.

Kuritsyn et al. [73] compare the previous measurements of the Hall dynamo from Ding et al. [35] to the changes in plasma flow that they observe in the core, shown here in Fig. 2.11. This figure suggests that the change in plasma flow is opposite to the direction of the Lorentz force density, becoming more negative over the relaxation event, $\rho \frac{\partial}{\partial t} \langle v \rangle < 0$ at $r/a \approx 0.35$. However, this directly contradicts a previous figure in their paper, reproduced here in Fig. 2.8(a), which shows that the parallel flow in the core increases at the crash, $\rho \frac{\partial}{\partial t} \langle v \rangle > 0$ at $r/a \approx 0.3$. Following discussions with MST experimentalists [private comm.], we conclude that the sign of $\rho \frac{\partial}{\partial t} \langle v \rangle$ is incorrect in the second plot; it should be in the same direction as the Lorentz force density there.
Figure 2.12: Fluctuation-induced force densities, $\langle \tilde{J} \times \tilde{B} \rangle_\parallel$ and $-\langle m_i n v \cdot \nabla v \rangle_\parallel$, and $\Delta \langle v \rangle_\parallel$ during the initial relaxation event for two-fluid computations with $S = 20,000$, with (a) $\lambda > 0$ and (b) $\lambda < 0$. The dashed lines indicate $\pm$ one standard deviation over the relaxation event.

2.3.2 Orientation of $J$ and $B$

Despite the apparent qualitative agreement between King’s computations and the measured flow changes on MST, there are important discrepancies. In King’s computations, the parallel current density is aligned with the magnetic field, $\lambda > 0$, while in typical MST operation, $\lambda < 0$. If the sign of $\lambda$ is changed, then the sign of the dynamo terms needed for current relaxation changes, as we have already seen in our nonlinear computations in Fig. 2.2. As a result, the $\Delta \langle v \rangle_\parallel$ at the first simulated relaxation event depends on the sign of $\lambda$, as can be seen in Fig. 2.12. When $\lambda > 0$ and the Hall dynamo acts to relax the parallel current in the core, then $\langle \tilde{J} \times \tilde{B} \rangle_\parallel > 0$ and the parallel flow in the core increases. In the opposite case, $\lambda < 0$, the core $\Delta \langle v \rangle_\parallel$ is negative, opposite to King’s previous computations, although consistent with our expectations. In both cases, however, the flow changes are in the direction of the dominant Lorentz force density.

Dr. Almagri and Dr. Mirnov performed experiments on MST to investigate how the orientation of the current and magnetic field affects the changes in plasma flow at the discrete current relaxation events [private comm. with Drs. Almagri and Mirnov]. The mode phase velocities, as measured from the toroidal magnetic array and ensemble-averaged over many relaxation events, are used as proxy measurements for toroidal plasma flow, as in past experiments [53, 73]. Because the magnetic field is dominantly toroidal in the core, the toroidal phase velocity of the core resonant $1/6$ mode may be used to infer the direction of the parallel flow there. In a completely isolated system, only two orientations, $J$ parallel to $B$ and $J$ anti-parallel to $B$, should be needed, but all four orientations were tested here to eliminate the possibility of systemic external effects.

Their first important finding is that the core flow velocity before a relaxation event is always in the same direction as the plasma current, as can be seen in Fig. 2.13. Changing the direction of the plasma current changes the direction of the core plasma flow, regardless of
Figure 2.13: Mode phase velocity measurements on MST for different orientations of plasma current and magnetic field. Plots courtesy of Dr. Almagri and Dr. Mirnov.

Figure 2.14: Changes in plasma current and flow during relaxation events in MST for (a) parallel current and (b) anti-parallel current.

the direction of $B$. This flow appears to be related to transport effects that are outside the scope of our extended MHD modeling. Nevertheless, the strong current relaxation dynamics are described by our extended MHD model, and we expect that changes in plasma flow at the relaxation are at least qualitatively correct.

The second important point is that the changes in the core plasma flow at the relaxation event are also independent of the direction of $B$. The magnitude of the core plasma flow is always reduced, relative to its initial value, by the relaxation dynamics. That is, $\Delta \langle J \rangle_\parallel$ and $\Delta \langle v \rangle_\parallel$ are always in the same direction in the plasma core, as illustrated in Fig. 2.14. This is in contrast to the initial relaxation events in our computations and those of King et al. [70] where $\Delta \langle v \rangle_\parallel$ and $\Delta \langle J \rangle_\parallel$ are in opposite directions.

Although the kinetic energy remains much smaller than the magnetic energy in MST, it is conceivable that the background flow profile that exists between the sawtooth relaxation events in the experiment may affect the current relaxation dynamics To test this, we included an ad...
Figure 2.15: Force densities and $\langle v\rangle_\parallel$ during the initial relaxation event for two-fluid computations with $S = 20,000$ with an ad hoc parallel flow profile, $v_\parallel/V_A = 0.1 (r/a)^2 - 0.03585$. (a) $\lambda > 0$ and (b) $\lambda < 0$. The dashed lines indicate ± one standard deviation over the relaxation event.

hoc parallel flow profile in our computations similar to that shown in Fig. 2.8(a) prior to the sawtooth event. We estimated the initial value of the parallel flow at the three radial locations to be: $v_\parallel (r/a = 0.3) \approx -30$ km/s, $v_\parallel (r/a = 0.5) \approx -5$ km/s, and $v_\parallel (r/a = 0.66) \approx 5$ km/s. A least-squares fit of a quadratic to the data points gives $v_\parallel/V_A = 0.1 (r/a)^2 - 0.03585$, where we have used $V_A \approx 10^3$ km/s for the low current discharges analyzed by Kuritsyn et al. [73].

The presence of this background flow profile does not alter the current relaxation dynamics in the initial relaxation event in our computations, regardless of the sign of $\lambda$, as can be seen in Fig. 2.15. The Lorentz force density, $\langle \mathbf{J} \times \mathbf{B}\rangle_\parallel$, is in the same direction as our previous computations without background flow (see Fig. 2.12), and the Hall dynamo cooperates with the MHD dynamo to relax the parallel current profile. The change in plasma flow, $\Delta \langle v\rangle_\parallel$, is also in the same direction as our previous computations.

2.3.3 Discussion

Based on these observations and the discussion in Sec. 2.2, it is clear that $\langle \mathbf{J} \times \mathbf{B}\rangle_\parallel$ can not simultaneously relax the parallel current density through the Hall dynamo and relax the axisymmetric flow through the Lorentz force density. Current relaxation has $\Delta \langle J\rangle_\parallel$ opposite to $\lambda$. If this occurs through the Hall dynamo, then $\langle \mathbf{J} \times \mathbf{B}\rangle_\parallel / \langle n \rangle e$ and $\rho e_\parallel \langle v\rangle_\parallel \sim \langle \mathbf{J} \times \mathbf{B}\rangle_\parallel$ must have the same sign as $\lambda$. But $\Delta \langle J\rangle_\parallel$ and $\Delta \langle v\rangle_\parallel$ are always observed to have the opposite sign of $\lambda$ in the core.

This apparent predicament can be resolved as follows. The Lorentz force density acts to reduce the core flow (flow relaxation), and the corresponding Hall dynamo acts to increase the parallel current density there (current drive). Current relaxation may still occur if the other substantial contribution to the dynamo electric field, the MHD dynamo, is larger than the Hall dynamo in magnitude and opposite in direction.
This behavior is exactly what is observed in many subsequent relaxation events in our numerical computations, as we will see in Chapter 4. The MHD dynamo acts to relax the parallel current in the core, and it is opposed by a Hall dynamo that is smaller in magnitude. This results in a correlated Lorentz force density that reduces the parallel plasma flow in the core. Both current and flow may relax in this case, consistent with the measurements on MST.

The competition of the dynamo terms was not observed in the initial relaxation event described in King et al. [70] and it is not observed to occur in the first relaxation event in our computations shown previously. As we will see in Chapter 4, this is attributed to our initialization. Prior to the first event, the plasma in our computations exists in a saturated single-helicity state as a result of the paramagnetic pinch initial conditions and the rapid linear growth of the core-resonant instability. Because the plasma current is so far from being fully relaxed, the MHD dynamo and Hall dynamo act in unison to drive the plasma closer to its relaxed state. These conditions differ significantly from the multi-helicity states in MST just prior to an individual sawtooth in typical discharges. Subsequent events in our computations occur from a more relaxed configuration with many more modes contributing to the dynamics, and these are believed to be more representative of typical RFP sawteeth.
3 NIMROD MODEL

The NIMROD code [111, 112] (Non-Ideal Magnetohydrodynamics, with Rotation - Open Discussion) is an initial value solver for the extended MHD system of equations introduced previously, and it is used to simulate nonlinear plasma evolution in a wide variety of configurations. We will use it here to simulate the nonlinear relaxation process that occurs quasi-periodically in the RFP. Describing the code in its entirety is well beyond the scope of this presentation, so instead we will restrict our discussion to only a few facets of it that are directly relevant for our results. We will first introduce the equations used in NIMROD, highlighting the additional numerical terms that are included in our computations. We will also discuss the separation of the fields in NIMROD into a steady-state piece and an evolving piece, and we will emphasize how this factors into our choice of initial conditions. Next, we will describe the spatial representation used in NIMROD including some of the possible grid decompositions. Lastly, we will discuss the initial and boundary conditions used in our nonlinear computations, which constitute the bulk of the thesis, along with the parameters of the model equations.

3.1 The Computational Model

3.1.1 Model Equations

We use NIMROD to solve the extended MHD system of equations

\[
\frac{\partial n}{\partial t} = - \nabla \cdot (n\mathbf{v}) + \nabla \cdot (D_n \nabla n) \tag{3.1}
\]

\[
m_i n \frac{\partial \mathbf{v}}{\partial t} = - m_i n \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{F} \tag{3.2}
\]

\[
n \frac{\partial T}{\partial t} = - n \mathbf{v} \cdot \nabla T - nT (\Gamma - 1) \nabla \cdot \mathbf{v} + \nabla \cdot (\chi n \nabla T) \tag{3.3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times \mathbf{E} + \kappa \nabla (\nabla \cdot \mathbf{B}) \tag{3.4}
\]

for our nonlinear computations of relaxation in the reversed-field pinch, presented in Chapters 4-7. A small artificial number density diffusion term is included in the density evolution for numerical stability, but, as we will see, it does not enter directly into the evolution of the global magnetic energy or helicities, the primary quantities of interest in relaxation theories. (In one of our computations, we also use a density hyperdiffusion, \(- \nabla \cdot (D_h \nabla \nabla^2 n)\), on the right-hand side of Eq. (3.1).) Detailed thermal transport dynamics are not considered.
here, and a single temperature is used \((p_{i,e} = p/2)\) with isotropic thermal conduction to provide some diffusion. In addition, viscous and Ohmic heating effects are neglected in the temperature equation, as the relaxation dynamics under investigation occur much faster than the transport time scales associated with those terms. Both of these result in a lack of energy conservation in our model, as shown in Appendix B, but we find that this does not sufficiently affect our results. Lastly, the NIMROD representation of fields does not satisfy \(\nabla \cdot \mathbf{B} = 0\) identically, so a divergence cleaning term is used with high-order spatial representation [111]. The influence of this numerical term on the evolution of magnetic energy, magnetic helicity, and cross helicity is quantified and discussed in Chapter 7.

The electric field is described by the generalized Ohm’s law

\[
\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} + \Lambda_e \left[ \frac{\mathbf{J} \times \mathbf{B}}{n_e} - \nabla p_e - \frac{m_e}{n_e^2} \frac{\partial \mathbf{J}}{\partial t} \right].
\]

(3.5)

The marker \(\Lambda_e\) is inserted to indicate where two-fluid effects enter the generalized Ohm’s law; \(\Lambda_e = 0\) for single-fluid MHD while \(\Lambda_e = 1\) for two-fluid models. As we saw previously, the first two terms on the right-hand side of Eq. (3.5) represent the standard MHD response, but MST measurements indicate that these terms alone are inadequate for a complete description of the physics of interest [35, 45, 73]. We include the electron inertia in the model equations to provide a resonance condition for the whistler wave at the electron cyclotron frequency in order to limit the range of temporal scales as spatial resolution is increased [112]. However, it is measured to have negligible effect on the evolution of the quantities of interest in the computations that are presented.

The center-of-mass force density is

\[
\mathbf{F} = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi}_{iso} - \Lambda_i \nabla \cdot \mathbf{\Pi}_{gyr}
\]

(3.6)

where the marker \(\Lambda_i\) is used to indicate where ion FLR effects enter the system; \(\Lambda_i = 0\) when they are excluded and \(\Lambda_i = 1\) when they are included. If the ions are not cold, these effects enter at the same order as the drift effects in Ohm’s law, and they can be included to first order via the Braginskii ion gyroviscous stress tensor [13], given by

\[
\mathbf{\Pi}_{gyr} = \frac{m_i p_i}{4e |\mathbf{B}|} [ \mathbf{\hat{b}} \times \mathbf{W} \cdot (\mathbf{I} + 3\mathbf{\hat{b}} \mathbf{\hat{b}}) - (\mathbf{I} + 3\mathbf{\hat{b}} \mathbf{\hat{b}}) \cdot \mathbf{W} \times \mathbf{\hat{b}} ]
\]

(3.7)

where \(\mathbf{W} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3} \mathbf{I} (\nabla \cdot \mathbf{v})\) and \(\mathbf{\hat{b}} \equiv \mathbf{B}/|\mathbf{B}|\). The gyroviscous stress tensor is non-dissipative and represents gyro-orbit frequency shifts and ellipticity resulting from \(\nabla \mathbf{E}\) [65]. We also include a simple collisional viscosity in the model, \(\mathbf{\Pi}_{iso} = \nu m_i n \mathbf{W}\).
3.1.2 Separating the Equilibrium and Perturbed Fields

In NIMROD, fields are separated into an evolving part and a steady-state, or equilibrium, part:

\[ f = f_{eq} + \tilde{f} \]

The equilibrium pieces of all equations are assumed to be in balance and are explicitly subtracted from the equations. If the equilibrium terms are not exactly in balance, this implies the existence of a source term in each of the equations that does provide this balance.

A simple example will suffice to make this clear. Consider this decomposition in the density equation, Eq. 3.1,

\[ \frac{\partial n}{\partial t} + \frac{\partial \tilde{n}}{\partial t} = - \nabla \cdot [n_{eq}v_{eq} + n_{eq}\tilde{v} + \tilde{n}\tilde{v}] + \nabla \cdot [D_n \nabla (n_{eq} + \tilde{n})]. \quad (3.8) \]

NIMROD assumes that the equilibrium pieces are steady-state and balanced by effects outside of the model; it solves only

\[ \frac{\partial \tilde{n}}{\partial t} = - \nabla \cdot [n_{eq}\tilde{v} + \tilde{n}v_{eq} + \tilde{v}\tilde{n}] + \nabla \cdot (D_n \nabla \tilde{n}). \quad (3.9) \]

If \( \nabla \cdot (D_n \nabla n_{eq} - n_{eq}v_{eq}) \neq 0 \), then this is equivalent to solving the complete density evolution, Eq. (3.1), provided that we include an anomalous source term on the right-hand side. Applying a similar analysis to Eqs. (3.2)-(3.4) we find

\[ \frac{\partial n}{\partial t} = - \nabla \cdot (n v) + \nabla \cdot (D_n \nabla n) + N_{eq} \quad (3.10) \]

\[ m_i n \frac{\partial v}{\partial t} = -m_i n v \cdot \nabla v + J \times B - \nabla p - \nabla \cdot \Pi_{eq} + F_{eq} \quad (3.11) \]

\[ n \frac{\partial T}{\partial t} = -n v \cdot \nabla T - n T (\Gamma - 1) \nabla \cdot v + \nabla \cdot (\chi n \nabla T) + Q_{eq} \quad (3.12) \]

\[ \frac{\partial B}{\partial t} = - \nabla \times E + \nabla \times \mathbf{E}_{eq} \quad (3.13) \]

where the anomalous terms are given by

\[ N_{eq} \equiv \nabla \cdot (n_{eq}v_{eq}) - \nabla \cdot (D_n \nabla n_{eq}) \quad (3.14) \]

\[ F_{eq} \equiv m_i n_{eq} v_{eq} \cdot \nabla v_{eq} + \nabla \cdot \Pi_{eq} \quad (3.15) \]

\[ Q_{eq} \equiv n_{eq} v_{eq} \cdot \nabla T_{eq} + n_{eq} T_{eq} (\Gamma - 1) \nabla \cdot v_{eq} - \nabla \cdot (\chi n_{eq} \nabla T_{eq}) \quad (3.16) \]

\[ \mathbf{E}_{eq} \equiv -v_{eq} \times B_{eq} + \eta J_{eq} + \frac{T_{eq,i}}{n_{eq} e} \nabla n_{eq} \quad (3.17) \]
and we have assumed that equilibrium force balance, $J_{eq} \times B_{eq} = \nabla p_{eq}$, is satisfied. Clearly, the choice of equilibrium greatly influences the solution computed by NIMROD.

### 3.2 NIMROD Spatial Grids

The NIMROD spatial representation makes use of a right-handed $(R, Z, \phi)$ system, where $R$ and $Z$ span what is called the “poloidal plane” in NIMROD (which is not necessarily a true poloidal plane for the magnetic topology), and $\phi$ spans the third direction, which is periodic. A spectral element representation [12, 32, 64] is used for the two directions in this poloidal plane, and a Fourier series representation is used for the periodic angle. The poloidal plane may be either circular or rectangular, and the periodic direction may be either linear or toroidal, yielding four possible combinations for the spatial grid, which are summarized in Table 3.1 and shown in Fig. 3.1. For a linear geometry, the periodic length is a direct input in NIMROD. Grid shapes other than circular and rectangular are also possible but are not used in the results presented here.

Slab computations utilize grid (a), and this provides a Cartesian $(x, y, z)$ coordinate system with $R \rightarrow x$, $Z \rightarrow y$, and $\phi \rightarrow z$. In this configuration, periodic boundary conditions may also be imposed for the $x$ and $y$ directions, if desired. This grid is used for our linear resistive drift computations in Chapter 9.

A cylindrical geometry may be achieved using either grid (b) or grid (c) provided that the poloidal plane extends over $0 \leq R \leq a$ in the latter case, where $a$ is the radius of the cylinder. In the first configuration, the “poloidal plane” is truly a poloidal plane of the magnetic configuration, and poloidal ($m$) harmonics are represented in terms of the spectral elements in the plane. The Fourier representation is used for the axial (or toroidal, $n$) direction of the cylinder here. In contrast, in grid (c) the “poloidal plane” represents a slice at fixed poloidal angle. Here, the axial ($n$) harmonics are represented in the spectral elements while the Fourier series representation is used for the poloidal $m$ harmonics.

It is advantageous to use grid (c) for nonlinear RFP computations because the reversed-field pinch magnetic geometry has many low $m$, high $n$ resonances, and it is dominated by low $m$ activity. In grid (c), fewer Fourier components are required to accurately resolve the

<table>
<thead>
<tr>
<th>Label</th>
<th>grid_shape=rect</th>
<th>grid_shape=circ</th>
</tr>
</thead>
<tbody>
<tr>
<td>geom=lin</td>
<td>(a) Slab</td>
<td>(b) Cylinder</td>
</tr>
<tr>
<td>geom=tor</td>
<td>(c) Annulus or Cylinder</td>
<td>(d) Torus</td>
</tr>
</tbody>
</table>

Table 3.1: Basic spatial grids in NIMROD.
Figure 3.1: Possible grid shapes in NIMROD, along with the NIMROD \((R, Z, \phi)\) coordinate system. (a) geom='lin', grid_shape='rect', (b) geom='lin', grid_shape='circ', (c) geom='tor', grid_shape='rect', and (d) geom='tor', grid_shape='circ'. The grid in each plot represents the poloidal plane, while the red lines represent the periodic direction.

A brief comment on the handedness of the coordinate system is in order when using grid (c) for cylindrical computations in NIMROD’s \((R, Z, \phi)\) coordinate system. The periodic angle, \(\phi\), here is opposite to the azimuthal angle, \(\theta\), in a usual cylindrical coordinate system, that is, \(\phi = -\theta\). Our nonlinear results in Chapters 4-7 correctly account for this behavior, and results labeled as \(\theta\) are with respect to the cylindrical \((r, \theta, z)\) right-handed system. Further, computations with \(\mathbf{J} \cdot \mathbf{B} > 0\) have \(q = \mathbf{B} \cdot \nabla \zeta / \mathbf{B} \cdot \nabla \theta > 0\), so that a Fourier decomposition \(\tilde{f} \sim \tilde{f}_{mn} \exp [im\theta + in\zeta]\) is resonant where \(q = -m/n\); that is, we expect that resonant modes have \(m > 0\) and \(n < 0\). The converse holds when \(\mathbf{J} \cdot \mathbf{B} < 0\).
The final configuration, grid (d), is the only truly toroidal configuration, and it is necessary if a realistic representation of the experimental geometry is desired. Our linear computations in toroidal geometry, presented in Chapter 8, use this grid. In principle, we could use either grid (b) or grid (c) for our comparison computations in cylindrical geometry. For simplicity, however, we choose grid (b) for these linear computations here because it allows a more direct comparison of the poloidal cross section of the eigenmode between cylindrical and toroidal geometries.

Lastly, we note that the four grids described previously are not the only grids that NIMROD is capable of using. It was primarily designed to be applicable to a broad range of toroidal devices, and it can handle arbitrarily shaped grids in the poloidal plane. The only restriction, at present, on NIMROD’s spatial representation is that the cross section must be axisymmetric. A limited class of stellarator relevant computations are performed with NIMROD in spite of this restriction [55, 105].

### 3.3 Model Parameters

#### 3.3.1 Initial and Boundary Conditions

We model the RFP dynamics in periodic cylinder geometry with minor radius \( a \) and axial length \( L = 2\pi R_0 \), where \( R_0 \) is the major radius, not to be confused with the NIMROD coordinate. The aspect ratio is \( R_0/a = 3 \), which is chosen to roughly match MST. The initial condition for the computations is a force-free paramagnetic pinch (described more fully in Chapter 1) with \( J \times B|_{t=0} = \nabla p|_{t=0} = 0 \) and finite plasma pressure. It is an Ohmic steady state that is sustained self-consistently by an externally applied electric field, \( E|_{t=0} = -v \times B|_{t=0} + \eta J|_{t=0} \), with \( \nabla \times E|_{t=0} = 0 \) [99]. This sustains the plasma current against resistive decay and injects both magnetic energy and magnetic helicity into the system. It is directed along the axis of the cylinder and is constant in time, and the normal component of magnetic field is held fixed, \( \frac{\partial}{\partial t} B \cdot \hat{n}|_{r=a} = 0 \), with \( B \cdot \hat{n}|_{r=a,t=0} = 0 \). Parallel current is preferentially driven in the plasma core, where the magnetic field is mostly aligned with \( E \), leading to a peaked parallel current profile that provides free energy for plasma instabilities.

The electric field drives a small radially inward pinch flow which is directly proportional to the plasma resistivity, given by Eq. (1.62), but there are no other flows in the initial state. No-slip boundary conditions are imposed on the tangential components of plasma flow. The temperature is held fixed at the boundary, \( \frac{\partial}{\partial t} T|_{r=a} = 0 \), and thermal energy may flow into and out of the system, although this rate is limited by the small isotropic thermal conductivity. With the artificial density diffusion, it is mathematically acceptable to utilize
Table 3.2: The normalizations used in our computations and corresponding experimental values.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Normalization</th>
<th>Units</th>
<th>200 kA</th>
<th>400 kA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>Minor Radius</td>
<td>a</td>
<td>0.52 m</td>
<td>0.52 m</td>
</tr>
<tr>
<td>Time</td>
<td>Alfvén Time</td>
<td>$\tau_A$</td>
<td>0.377 $\mu$s</td>
<td>0.188 $\mu$s</td>
</tr>
<tr>
<td>Velocity</td>
<td>Alfvén Velocity</td>
<td>$V_A$</td>
<td>$1.38 \cdot 10^3$ km/s</td>
<td>$2.76 \cdot 10^3$ km/s</td>
</tr>
<tr>
<td>Magnetic Field</td>
<td>On-axis field</td>
<td>$B_0$</td>
<td>0.20 T</td>
<td>0.40 T</td>
</tr>
<tr>
<td>Density</td>
<td>On-axis density</td>
<td>$n_0$</td>
<td>$1 \cdot 10^{19}$ m$^{-3}$</td>
<td>$1 \cdot 10^{19}$ m$^{-3}$</td>
</tr>
<tr>
<td>Electric Field</td>
<td>$V_A B_0$</td>
<td>2.76 $\cdot 10^5$ V/m</td>
<td>1.10 $\cdot 10^6$ V/m</td>
<td></td>
</tr>
<tr>
<td>Force density</td>
<td>$B_0^2 / (\mu_0 a)$</td>
<td>6.12 $\cdot 10^4$ N/m$^3$</td>
<td>2.45 $\cdot 10^5$ N/m$^3$</td>
<td></td>
</tr>
</tbody>
</table>

similar boundary conditions for the number density, $\frac{\partial n}{\partial t}|_{r=a} = 0$. As a result, particles may enter or exit the computational domain, but this also occurs on a much longer time-scale than the relaxation dynamics of interest.

The choice of the paramagnetic pinch initial condition eliminates the anomalous electric field in Eq. 3.13 ($\nabla \times \mathbf{E}_{eq} = 0$), and the evolution of magnetic energy in our computations is self-consistent. Additionally, the force-free condition, $\nabla p_{eq} = 0$, eliminates many of the other anomalous terms in $N_{eq}$, $F_{eq}$, and $Q_{eq}$. The pinch flow results in anomalous contributions to $N_{eq}$, $F_{eq}$, and $Q_{eq}$, but these terms are verified to be small in the computations. The paramagnetic pinch represents a logical set of initial conditions from which to begin nonlinear computations of plasma relaxation dynamics.

### 3.3.2 Parameters

The system of Eqs. (3.1)-(3.6) may be normalized for convenience. Lengths are normalized to the minor radius, $a$, times are normalized to the Alfvén time, $\tau_A = a/V_A$, speeds are normalized to the Alfvén speed, $V_A = B_0/\sqrt{\mu_0 m_i n_0}$, magnetic field values are normalized to the on-axis field, $B_0$, and the density is normalized to its on-axis value, $n_0$. Electric fields are then given in units of $V_A B_0$, and force densities are given in units of $B_0^2 / (\mu_0 a)$. The density in MST discharges is typically $n_0 \approx 0.7 - 1 \cdot 10^{19}$ m$^{-3}$. The MST discharges of Den Hartog et al. [30], Fontana et al. [45], Kuritsyn et al. [73] are at low plasma current, $I_p = 200 - 250$ kA, with modest on-axis magnetic field, $B_0 \approx 0.2$ T. The higher current discharges analyzed by Ding et al. [35] have $I_p = 350 - 400$ kA and a higher on-axis magnetic field, $B_0 \approx 0.4$ T. The normalizations are summarized in Table 3.2.

Independent of this normalization, there are seven dimensionless physical parameters and two numerical parameters in our model equations. The seven physical parameters are (1) the Lundquist number, the ratio of the Alfvén time to the resistive diffusion time, $S = \tau_R/\tau_A$,
where \( \tau_A = a\sqrt{\mu_0 m_i n}/B_0 \) and \( \tau_R = \mu_0 a^2/\eta; \) (2) the magnetic Prandtl number, the ratio of resistive to viscous times, \( P_m = \tau_R/\tau_\nu, \) where \( \tau_\nu = a^2/\nu; \) (3) the ratio of the viscous to thermal conduction times, \( \tau_\nu/\tau_\chi = \chi/\nu; \) (4) the plasma-\( \beta, \) the ratio of fluid pressure to magnetic pressure, \( \beta = 2\mu_0 p/B^2; \) (5) the magnetic Prandtl number, the ratio of resistive to viscous times, \( P_m = \tau_R/\tau_\nu, \) where \( \tau_\nu = a^2/\nu; \) (6) the plasma-\( \beta, \) the ratio of fluid pressure to magnetic pressure, \( \beta = 2\mu_0 p/B^2; \) (7) the plasma-\( \beta, \) the ratio of fluid pressure to magnetic pressure, \( \beta = 2\mu_0 p/B^2; \) (8) the normalized ion skin depth, \( d_i/a = c/\omega_{pi}a = \sqrt{m_i/\mu_0 n e^2/a}; \) and (7) the normalized electron skin depth, \( d_e/a = c/\omega_{pe}a = \sqrt{m_e/m_i d_i/a}, \) which can be varied artificially by changing the mass ratio \( m_e/m_i. \) The two numerical parameters are (1) the ratio of viscous to density diffusion times, \( \tau_\nu/\tau_D = D_n/\nu; \) and (2) the ratio of divergence cleaning to resistive diffusion times, \( \tau_{\nabla B}/\tau_R = \eta/\mu_0 \kappa. \)

All of the nonlinear computations presented here have \( \tau_\nu/\tau_\chi = 0.1, \) \( \beta = 0.10 \) on axis, \( d_i/a \approx 0.17, f_{Ti} = 0.5 \) and \( m_e/m_i \approx 2.72 \cdot 10^{-3}. \) The on-axis plasma-\( \beta \) is slightly higher than in the MST experiment, but the ion skin depth is comparable to estimates based on experimental values. The electron mass is artificially increased by a factor of 5 for numerical convenience. Most of our computations have \( P_m = 1.0, \) but one is run at \( P_m = 0.1. \) A shaped resistivity, \( \eta (r) = F(r) \eta (0), \) and viscosity, \( \nu (r) = F(r) \nu (0), \) profile are used that increase sharply near the plasma edge,

\[
F(r) \equiv \left[ 1 + \sqrt{19 \left( \frac{r}{a} \right)^{20}} \right]^2,
\]

in order to mitigate the formation of sharp boundary layers there. The values of Lundquist number and Prandtl number are the on-axis values.

Five distinct models under different physical conditions, summarized in Table 3.3, are analyzed. The models A, B, C, and D represent new results, while model E consists of the previous results of King et al. [70]. Model E is analyzed here only in the context of our discussion on helicity and relaxation in Chapter 7. The first model (A) uses the standard magnetohydrodynamic Ohm’s law \( (\Lambda_e = 0) \) and excludes the ion gyroviscous stress tensor \( (\Lambda_i = 0). \) The remaining models all incorporate two-fluid effects in the generalized Ohm’s law \( (\Lambda_e = 1). \) The second model (B) does not include the gyroviscous stress tensor \( (\Lambda_i = 0), \)
while the third, fourth, and fifth (C, D, and E) do ($\Lambda_i = 1$). Models A, B, C, and D are run with Lundquist number $S = 20,000$ and model E is run with $S = 80,000$. These values are roughly 2 orders of magnitude smaller than in the experiment, as limited by computational practicalities.

At $t = 0$, the dimensionless parallel current density, $a\lambda = a\mu_0 J \cdot B / |B|^2$, has an on-axis value of $a\lambda(0) = -3.88$ in models A-C and $a\lambda(0) = 3.88$ for models D and E. Consequently, the magnetic helicity is negative in models A-C and positive in models D and E, which will be important when considering the normalized evolution in Chapter 7.

In the cylindrical geometry under consideration here, the spectral elements are used for the radial ($r$) and axial ($z$) directions, as described in Sec. 3.2. The azimuthal (poloidal) angle ($\theta$) is represented by finite Fourier series, and the computations here use 6 harmonics, $0 \leq m \leq 5$, where $m$ is the poloidal harmonic number. Periodic boundary conditions are imposed in the axial direction to yield a topologically toroidal domain. Regularity conditions are enforced at $r = 0$, and the boundary conditions described in Sec. 3.3.1 are enforced at $r = a$. The $r - z$ plane is discretized with a rectangular grid consisting of uniformly spaced elements in the radial and axial directions, with basis functions of fixed polynomial degree within each element.

Computations for models A, B, and C use 120 radial and 64 axial finite elements. Convergence is tested for these three models using basis functions of polynomial degree 3 and polynomial degree 5 within each element; the polynomial degree 5 results are presented in the main text. The numerical divergence cleaning for models A, B, and C is $\tau \nabla \cdot B / \tau_R = 5 \cdot 10^{-5}$. Model D uses 180 radial and 128 axial finite elements with polynomial degree 3 within each element, and $\tau \nabla \cdot B / \tau_R = 1 \cdot 10^{-5}$. The $S = 80,000$ model E computation uses 240 radial and 60 axial finite elements with polynomial degree 4 and $\tau \nabla \cdot B / \tau_R = 1.25 \cdot 10^{-5}$. All models use $\tau_\nu / \tau_{D_n} = 0.1$. The resolutions used in the models are summarized in Table 3.4.

There is unfortunately no simple test to determine the exact resolution requirements for a nonlinear computation. The resolution used here is sufficient to be converged on the linear growth rates of the initial instabilities, but this is no guarantee that the subsequent nonlinear evolution won’t exhibit finer scale features. Based on the spectral decomposition of the energies, the computations appear relatively well-converged, however. As we will see, in some of our two-fluid computations there are fluctuations at the limit of radial resolution, but this is attributed to density fluctuations that do not appear to adversely affect the current relaxation dynamics of interest.

Models A, B, C, and D were run at NERSC (National Energy Research Scientific Computing Center) on the Edison supercomputer, a Cray XC30. Each of these models used 6 computational nodes comprising 144 total CPUs (24 CPUs per node) with 64 GB DDR3
RAM. A rough estimate of the total computational expense for these models is also listed in Table 3.4. Details of the model E computation are in King [71]; because we do not have all of the outputs from that computation, we are only able to make an estimate of the computational expense incurred.

Table 3.4: Numerical parameters used in our computations and computational cost.

<table>
<thead>
<tr>
<th>Model</th>
<th>mx</th>
<th>my</th>
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<th>Fourier Comps.</th>
<th>CPU Hours</th>
<th>Sim. Time</th>
</tr>
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<tr>
<td>A</td>
<td>120</td>
<td>64</td>
<td>5</td>
<td>6</td>
<td>48,488</td>
<td>8479 $\tau_A$</td>
</tr>
<tr>
<td>B</td>
<td>120</td>
<td>64</td>
<td>5</td>
<td>6</td>
<td>155,246</td>
<td>9668 $\tau_A$</td>
</tr>
<tr>
<td>C</td>
<td>120</td>
<td>64</td>
<td>5</td>
<td>6</td>
<td>201,713</td>
<td>13100 $\tau_A$</td>
</tr>
<tr>
<td>D</td>
<td>180</td>
<td>128</td>
<td>3</td>
<td>6</td>
<td>285,787</td>
<td>10000 $\tau_A$</td>
</tr>
<tr>
<td>E</td>
<td>240</td>
<td>60</td>
<td>4</td>
<td>6</td>
<td>*483,320</td>
<td>8081 $\tau_A$</td>
</tr>
</tbody>
</table>

(*estimate based on incomplete data)
Part II

Nonlinear Relaxation Dynamics
4 CURRENT RELAXATION AND DYNAMO DRIVE

We will begin with an overview of the nonlinear computations we have performed, focusing on global parameters such as the field reversal and the spectral magnetic energies. We will see that there are crucial differences between the first relaxation event and subsequent events. We will also discuss the similarities and differences between our single-fluid computations and the two-fluid cases with and without the ion gyroviscosity.

These computations are all nonlinear, but we estimate the linear growth rate of the dominant core-resonant tearing modes in these computations by examining the early evolution of the spectral magnetic energy for individual components. These results are shown in Table 4.1. Two-fluid effects in the generalized Ohm’s law increase the growth rate in the absence of the ion gyroviscosity, in agreement with predictions from linear theory [81] and numerical computations [69]. Here, the ion gyroviscosity is found to exert a stabilizing influence on the linear growth of these modes, indicating that these computations are in the intermediate drift-regime where $\rho_s$ is smaller than the resistive skin-depth and the electron-ion dynamics are only partially decoupled.

### Table 4.1: Growth rates estimated from early phases of nonlinear computations.

| $|m|/|n|$ = | 1/6 | 1/7 | 1/8 | 1/9 | 1/10 |
|------------|------|------|------|------|------|
| Model A: $\gamma \tau_A$ | 0.0264 | 0.0134 | 0.0100 | 0.0078 | 0.0059 |
| Model B: $\gamma \tau_A$ | 0.0304 | 0.0175 | 0.0142 | 0.0121 | 0.0100 |
| Model C: $\gamma \tau_A$ | 0.0181 | 0.0108 | 0.0086 | 0.0069 | 0.0051 |
| Model D: $\gamma \tau_A$ | 0.0181 | 0.0107 | 0.0085 | 0.0069 | 0.0051 |

4.1 Single-Fluid MHD

The evolution of the field reversal parameter $F$ and the magnetic and kinetic energies in the dominant spectral components of the fluctuations for our single-fluid MHD computation ($\Lambda_e = 0$, $\Lambda_i = 0$, model A) are shown in Fig. 4.1. The energy plots are divided in two regions, separated by a vertical black line. The region to the left uses the vertical axis on the left, while the region on the right uses the axis on the right. This is chosen to highlight the initial linear growth of the most unstable mode, as well as the nonlinear effects associated with other modes in the system.

Initially, the plasma evolves into a single-helicity state [43], characterized by a concentration of energy in the helical $|m/n| = 1/6$ modes (i.e. $|m/n| = 1/6, 2/12, 3/18...$). The single-helicity state may be considered as the nonlinear saturated state of a resistive kink mode
Figure 4.1: Field reversal parameter $F$ and magnetic and kinetic energy in dominant modes. The shaded regions indicate relaxation events, and there is a break in the mode energy plots to highlight the linear growth and single-helicity phase. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_c = 0$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)

[39, 76]. Here, it is a direct result of our initial conditions; the plasma begins from a linearly unstable state, with little energy in the magnetic fluctuations, and the single helicity state results from the much larger linear growth rate for the dominant $m = 1, n = 6$ mode compared to the other modes. Note that as the $m = 1, n = 6$ mode saturates nonlinearly, it affects the growth rates of the other modes of the system. In particular, note that the $m = 1, n = 5$ and $m = 1, n = 7$ modes begin growing much more slowly around $t/\tau_A \approx 2000$.

The single-helicity state persists only until the energy in the other unstable modes reaches a significant fraction of the dominant mode, which precipitates the first nonlinear relaxation event. We identify the onset of the first relaxation event as the time when the spectral activity [76]

$$N_s \equiv \left( \frac{\sum W_{B,1,n}}{\sum W_{B,1,n}^2} \right)^2$$

increases beyond 1.1. The spectral activity is a measure of the distribution of energy in the
$m = 1$ magnetic modes. For a pure single-helicity state, $N_s = 1$. When $N_s \sim 1.1$, the energy in all the other $m = 1$ modes is roughly 5% of the energy in the dominant $m = 1$ mode. The end of the relaxation event is the point of deepest field reversal $F$, and it appears strongly correlated with a peaking of the $m = 0, n = 1$ magnetic energy.

Following the first nonlinear relaxation event, there is some intermittent magnetic activity before the plasma undergoes two additional sawtooth-like events. The onset of the subsequent relaxation events is no longer identified based on the spectral activity index $N_s$ but rather on bursts of coherent magnetic activity, as identified in dynamo plots which will be shown later. The end of the event however, is still defined as the minimum in magnetic field reversal. Based on these three events, we find a sawtooth period that is roughly $1900 \, \tau_A$, about a tenth of a resistive diffusion time for $S = 20,000$.

The kinetic energy in our single-fluid MHD computation remains small relative to the energy in the magnetic field. For the $m = 1$ components, the kinetic energy is typically 1-2 orders of magnitude smaller than the corresponding magnetic energy. Additionally, the kinetic energy associated with the axisymmetric flow remains negligible next to the fluctuating components; there is little change in the axisymmetric flow in the single-fluid MHD system. Instead, the spectral kinetic energy from the $m = 0, n = 1$ component remains by far the largest contribution to the total kinetic energy following the first relaxation event.

### 4.1.1 Dynamo Drive

The components of the MHD dynamo and the correlated Lorentz force density parallel to the axisymmetric magnetic field for the single-fluid MHD case ($\Lambda_e = 0, \Lambda_i = 0$) are shown in the contour plots of Fig. 4.2. The horizontal axis represents time and the vertical axis in each plot is radial position. The dashed vertical lines indicate the beginning and end of a nonlinear relaxation event. The resonant surface locations for several low order modes are shown. The contour colors in each plot correspond to amplitude of the drive. While the beginning of the first event is identified by the increase of $N_s$ above 1.1, subsequent events are identified by the onset of significant dynamo activity, as diagnosed from these plots.

In all relaxation events, the MHD dynamo acts to flatten the parallel current in the core of the plasma and increase it near the edge. Note that with $a \lambda_0 < 0$ a negative dynamo contribution reduces the parallel current at that radial location, and the dynamo electric field increases it where it is positive. The MHD dynamo is largely negative inside the $|q| = 1/8$ surface and is positive outside that location, where it is redepositing the energy in all of the relaxation events. Based on quasilinear results [18], we expect that the contributions of each $m = 1$ mode to the MHD dynamo should be negative inside the mode’s rational surface and
positive outside of it. The $m = 1$ modes should each act to locally flatten the parallel current gradients around their own rational surfaces; the collective action of many modes then results in the redistribution of current from the core of the plasma out towards the edge. Note also that the dynamo is weaker by about a factor of 2 in the first relaxation event as compared to the subsequent events. As we will see next, there are significantly lower contributions to the dynamo from the $m = 1, n = 6$ mode during this first event.

We find that there are small correlated fluctuations of $\langle \tilde{J} \times \tilde{B} \rangle$ even with the absence of two-fluid effects ($\Lambda_e = 0$) in the generalized Ohm’s law, shown in the lower plot in Fig. 4.2. The linear phase relations of the single-fluid MHD model predict that the fluctuating current density $\tilde{J}$ and magnetic field $\tilde{B}$ are exactly out of phase [44, 80], so the quasilinear correlations $\langle \tilde{J} \times \tilde{B} \rangle$ vanish. The spatial correlations in our nonlinear results do not vanish, but there is little temporal correlation over a relaxation event. Consequently, the net effect of these fluctuations on the plasma flow in this model is minimal, as we will see in Chapter 5.

### 4.1.2 Modal Contributions to Correlated Fluctuations

The net effect that a given mode has on the axisymmetric fields may be found by decomposing that mode’s contribution to the total correlated fluctuations. The temporal average of this decomposition over each relaxation event is shown in Fig. 4.3. The total MHD dynamo appears similar between events, removing magnetic energy from the core and increasing it
Figure 4.3: Modal decomposition of the MHD dynamo and Lorentz force density parallel to the axisymmetric magnetic field averaged over the relaxation events. The shaded regions indicate the average plus or minus one standard deviation of the resonant surface location for each mode over the event. Note that the vertical axes differ. (Parameters: \( a\lambda_0 = -3.88 \), \( \Lambda_e = 0 \), \( \Lambda_i = 0 \), \( S = 20,000 \), \( P_m = 1.0 \))

in the edge region. However, as we noted previously, the magnitude differs by almost a factor of 2 in the core and 1.5 in the edge between the first and subsequent events. This appears to be largely a result of the \( m = 1, n = 6 \) behavior in the first event. As we will see in Chapter 6, the \( m = 1, n = 6 \) mode begins returning energy to the axisymmetric fields roughly halfway through the event. Averaging over the first event, the \( m = 1, n = 6 \) dynamo contribution is negative inside and positive outside of the \( q = \frac{1}{6} \) surface, but changes sign again near \( r/a \sim 0.55 \), reaching another local extremum at \( r/a \sim 0.65 \). The sign change further out from the resonant surface is similar to later events, but the magnitude of the dynamo contribution there is not. The weaker dynamo in the core and the large excursion at the edge are responsible for the reduced total dynamo relative to the subsequent events.

We see very little coherent structure among the modal contributions to \( \langle \mathbf{J} \times \mathbf{B} \rangle \parallel \) averaged...
over an event, as can be seen in the plots on the right of Fig. 4.3. The lack of coherent structure is reflected in the small total. There is slightly more coherent structure in the third event, but the exact reason for this remains unclear. We suspect it is due to pressure effects, as the density profile hollows out considerably just before the last event, dropping to about 75% of its value on the edge, which is held fixed in the simulations. This is in stark contrast to the density behavior in previous events, where the core value remains about 90 – 95% of the edge value. Recall that the flat pressure profile used in these simulations is not representative of experiment, which typically has a sharp pressure gradient in the edge. More detailed pressure modeling is needed to understand if this behavior is physically relevant.

Our later two-fluid computations will show correlated fluctuations of $\tilde{J}$ and $\tilde{B}$ in terms of the Hall dynamo, $\langle \tilde{J} \times \tilde{B} \rangle / \langle n \rangle e$, instead of as a Lorentz force density here, $\langle \tilde{J} \times \tilde{B} \rangle / \parallel$. For direct comparison to the correlated fluctuations of $\tilde{J}$ and $\tilde{B}$ in our later computations, a normalization is needed. Note that the force density normalization to $B_0^2/\mu_0 a$ can be expressed as $V_A B_0 \sqrt{m_i n_0 / \mu_0 a}$. With the factor of $1/ne \sim 1/n_0 e$, this becomes $(B_0^2/\mu_0 a)/ne \sim V_A B_0 d_i / a$. Multiplying the force densities shown here by $d_i/a \sim 0.173$ then allows direct comparison to the later Hall dynamo contributions.

4.2 Two-Fluid, No Ion Gyroviscosity

Our computation with two-fluid effects in the generalized Ohm’s law but without ion gyroviscosity (model B) shows more frequent relaxation events, as seen in Fig. 4.4. As in the single-fluid case, the plasma initially evolves into a single-helicity state, although the saturated $m = 1, n = 6$ magnetic energy is a bit larger here: $W_{B,1,6}/W_{B,0,0} (t = 0) \sim 0.0053$ while for the single-fluid case it was $W_{B,1,6}/W_{B,0,0} (t = 0) \sim 0.0038$. In addition, the first nonlinear relaxation event occurs much earlier as a result of the much higher growth rates in this system. Here, the first event is dominated by the $m = 1, n = 6, 8, 10$ modes and it ends with a peaking of the $m = 0, n = 2$ mode energy. Similar behavior is observed in the first events of our other two-fluid computations as well.

The sawtooth period is variable, with events occurring between $900 \tau_A$ to $1500 \tau_A$, however, the frequency of relaxation events remains greater than in the single-fluid case. This can be at least partially explained by the increased linear growth rates; after crossing a stability boundary, the mode amplitudes become appreciable much more quickly. In contrast to the

1 Note that here we turn on the artificial number density hyperdiffusion $(D_n = 1 \cdot 10^{-7} (a^4/\tau_A))$ at $t/\tau_A \sim 5240$, shortly into the third relaxation event, to help mitigate under-resolved high-$k$ behavior. However, there are no large transients observed in the subsequent evolution of the low $n$ modes shown here, and we conclude that the number density hyperdiffusion does not affect the evolution significantly.
single-fluid case, the relaxation events here have a much more regular structure. The field reversal parameter increases up to about $F = -0.02$ before dropping to $F = -0.10$ in most of the subsequent events. The only exception here is the final event, which occurs a bit prematurely, relative to the others, but the field still reverses to $F \approx -0.10$. The dominant modes in these later events are $m = 1, n = 6$, $m = 1, n = 7$ and $m = 0, n = 1$, as expected from a three-wave nonlinear interaction.

There is now a considerable amount of kinetic energy in the axisymmetric flow in our two-fluid computation. After the first relaxation event, it remains at considerable amplitude and is the largest contributor to the total kinetic energy for the duration of the computation. The other spectral components behave in a similar manner to the single-fluid MHD computation, however.

### 4.2.1 Dynamo Drive

From the comparison of the two dynamo drives, shown in Fig. 4.5, it is immediately evident that they do not always act constructively to relax the parallel current gradients. While the
MHD and Hall dynamos largely work in unison during the first event, there is clear opposition during some subsequent events. This is most evident in the fourth event, and there is some opposition in the second event and at the end of the third event as well. The onset of the fifth relaxation event, which occurs prematurely relative to the timing of the others, may be due to the strong opposition of the dynamo drives in the fourth event, which does not relax the parallel current profile as completely as at the end of the other events.

The MHD dynamo always acts to reduce current in the core and increase it out towards the edge, but the contributions from the Hall dynamo vary from event to event. We will see that this is a result of the differing modal contributions in subsequent events. The MHD dynamo always displays a very clear temporal signal, increasing at the start of an event, peaking, then falling back towards zero, but the Hall dynamo is much more intermittent with several closely spaced peaks between periods of smaller activity.

### 4.2.2 Modal Decomposition

The contributions from different modes, averaged over each event, are shown for the MHD and Hall dynamos in Fig. 4.6 Note that for the third event, the total Hall dynamo is very noisy near $r/a \sim 0.4$. The density hyperdiffusion was turned on partway through this event, and the results here are most likely showing some residual signs of the high-$k$ behavior that the hyperdiffusion is used to alleviate.
Figure 4.6: Modal decomposition of the MHD and Hall dynamos parallel to the axisymmetric magnetic field averaged over the relaxation events. The shaded regions indicate the average plus or minus one standard deviation of the resonant surface location for each mode over the event. Note that the vertical axes differ. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)
The amplitude of the total MHD dynamo here is comparable to that in the single-fluid MHD case. However, in the first event the secondary contributions come from the $m = 1, n = 8$ mode, instead of the $m = 1, n = 7$ mode, and the relaxation event ends with a peaking of the magnetic energy in the $m = 0, n = 2$ mode. The exact reason for this behavior is unknown, but it is also observed in our two-fluid cases with ion gyroviscosity.

The structure of the MHD dynamo is slightly more complicated here, as well. Each mode’s contribution is mostly negative inside the mode rational surface and positive outside of it, but the total is no longer roughly monotonic inside $r/a \sim 0.5$, and instead possesses local minima typically inside $r/a \sim 0.2$. The locations of these local minima roughly correspond with peaks in the Hall dynamo, giving the total dynamo here a shape that appears consistent with the MHD dynamo in computations without two-fluid effects ($\Lambda_e = 0$). In the first event, the local minima in the MHD dynamo occurs at $r/a \approx 0.05$, near the peak in the Hall dynamo. In the second and third events, this is near $r/a \approx 0.18$, again near a peak in Hall dynamo.

In contrast, the Hall dynamo has much more complex structure. It is typically positive at and just outside the mode rational surface and negative just inside of it. However, it changes sign going further away from the surface in either direction, becoming negative further out and positive further in, if the resonant surface location is sufficiently far from the magnetic axis. The width of the variation is on the order of $d_i$ here ($d_i/a \approx 0.17$), and it would be of interest to see how this changes with varying $d_i$.

The modal contributions to the Hall dynamo in the core of the plasma are primarily due to the $m = 1, n = 5, 6, 7$ and to a somewhat lesser extent the $n = 8$. However, their action may be either dynamo or anti-dynamo during a given event, and there is little apparent consistency between which modes do what during a given event. For example, in the second event, the $m = 1, n = 5$ mode is large and anti-dynamo (positive), and the $m = 1, n = 6$ is similarly large but dynamo, but this behavior is reversed in the third event. In the fourth event, which shows clear opposition between the MHD and Hall dynamos in the core, it appears that the large anti-dynamo behavior is a result of the $m = 1, n = 7$ as well as substantially muted $m = 1, n = 5$ and $m = 1, n = 6$ activity. The behavior of the total Hall dynamo is evidently the result of a complex nonlinear interaction of the modes.

### 4.3 Two-Fluid with Ion Gyroviscosity

Our two-fluid computations that include the ion gyroviscosity (model C) show greatly reduced magnetic activity following the initial relaxation event, as can be seen in Fig. 4.7. Like our previous computations, the initial relaxation event begins from a saturated single-helicity state as a result of the initial conditions. The significantly reduced magnetic activity following the
initial relaxation event is believed to be a result of the ion gyroviscosity. The ion gyroviscosity has a stabilizing effect on linear tearing modes in pinch configurations [69], and the results here are qualitatively consistent with a similar nonlinear computation at $S = 80,000$ [70, 71].

After the first event, the normalized magnetic energies range between $10^{-3}$ and $10^{-4}$, about 2-3 times lower than the peak energies in our single- and two-fluid computations without ion gyroviscosity. In particular, the core-resonant $m = 1, n = 6$ magnetic energy remains small during the subsequent evolution. We identify the few subsequent bursts of activity as relaxation events based on the change in field reversal, which remains modest relative to the initial drop, and a significant correlation of the fluctuations that results in dynamo drive.

The kinetic energy is again dominated by the contribution from the axisymmetric flow, and it remains at a similar level as our two-fluid computation without ion gyroviscosity, although it is much more quiescent. The only other spectral component of kinetic energy that appears significant is the $m = 0, n = 1$ component, which increases about an order of magnitude at each relaxation event before dropping down to the level of the other fluctuations.
4.3.1 Dynamo Drive

The components of the MHD and Hall dynamo parallel to the axisymmetric magnetic field for this computation are shown in the contour plots of Fig. 4.8. With ion gyroviscosity, the nonlinear activity after the first relaxation event is considerably reduced, as in a similar $S = 80,000$ simulation (model E) with the same effects included in the model [70]. The dynamo activity is much more sparse, but subsequent bursts of activity also show opposition between the MHD and Hall dynamo as in the case without the ion gyroviscosity. This is most pronounced in the second relaxation event of model C with dynamo amplitudes roughly matching the initial event, but is also prevalent in the third relaxation event, although the activity there is much more intermittent.

4.3.2 Modal Decomposition

In the first event, the structure of the dynamo contributions from individual modes are very similar to those in the two-fluid computation without ion gyroviscosity, as can be seen in Fig. 4.9. The first event is again dominated by the $m = 1, n = 6$ and $m = 1, n = 8$ modes, with some contributions from the $m = 1, n = 10$. However, the amplitude of the Hall dynamo contributions here are about 1.5 to 2 times larger than in the first event in model B, and

![Image of contour plots showing MHD and Hall dynamo contributions with colorbar axes differ and parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 1.0$.](image-url)
Figure 4.9: Modal decomposition of the MHD and Hall dynamos parallel to the axisymmetric magnetic field averaged over the relaxation events. The shaded regions indicate the average plus or minus one standard deviation of the resonant surface location for each mode over the event. Note that the vertical axes differ. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 1.0$)

this may be attributed to the ion gyroviscosity which inhibits the alignment of the current density and magnetic field fluctuations.

Subsequent events feature significant $m = 1, n = 5$ and $m = 1, n = 7$ activity, with modest contributions from $m = 1, n = 8$ as well. In contrast to cases without ion gyroviscosity, the Hall dynamo structure in the two subsequent events here is remarkably regular. Contributions from a particular mode in general are strongly positively peaked just outside the mode rational surface, strongly negatively peaked just inside, and have oscillatory behavior on the order of $d_i/a$ beyond that. The third relaxation event suffers from some higher $k$ activity, as evidenced by the noise just inside $r/a \sim 0.6$, similar to the third event in the computations without ion gyroviscosity, although no density hyperdiffusion is used here.

As in the case without ion gyroviscosity, the total dynamo electric field here (sum of
the MHD and Hall dynamos) is similar to just the MHD dynamo in our single-fluid MHD computation. The MHD and Hall dynamos, which cooperate during the first event, each account for roughly half of the total dynamo drive in the core. In the second event, however, the total MHD dynamo is significantly stronger in the core, where it is largely opposed by the Hall dynamo, and the total dynamo is only slightly smaller than in the first event. Similar behavior is observed for the third event, although the magnitudes are much smaller as a result of averaging over such intermittent activity.

4.4 Two-Fluid with Ion Gyroviscosity, $P_m = 0.1$

Our two-fluid computation with ion gyroviscosity at $P_m = 0.1$ and with parallel, as opposed to anti-parallel current (model D) shows a similar level of magnetic activity following the initial relaxation event, though there is significantly higher kinetic energy in the fluctuations and there are more relaxation events, as shown in Fig. 4.10. The most marked difference

![Figure 4.10](image-url)

**Figure 4.10**: Field reversal parameter $F$ and magnetic and kinetic energy in dominant modes. (Parameters: $a\lambda_0 = 3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 0.1$)

is the second relaxation event in this computation, which has a level of reversal that is
commensurate with the initial relaxation event. There is no significant change in the overall magnetic activity level as a result of the orientation of parallel current with respect to the magnetic field.

### 4.4.1 Dynamo Drive

Opposition between the two dynamo terms is observed in all of the subsequent events in this computation, as can be seen in Fig. 4.11. Recall that here \( a\lambda_0 > 0 \) so that a positive parallel dynamo electric field acts to reduce the parallel current density, while a negative dynamo electric field acts to increase the parallel current density. As in all of our previous computations, the MHD and Hall dynamos cooperate during the first event, acting to flatten the parallel current in the core and increase it near the edge. However, there is clear opposition in the subsequent events, which are much more pronounced with lower viscosity.

In particular, the second and fifth relaxation events here show a significant decrease in \( F \), as opposed to the slightly shallower events in this computation and all of the subsequent events in model C. These two events show strong opposition between the MHD and Hall dynamo terms, with the MHD dynamo acting to relax the parallel current in the plasma core and the Hall dynamo acting to increase it there. The MHD dynamo remains larger, however, and current relaxation occurs as expected.

![Figure 4.11: MHD and Hall dynamo parallel to the axisymmetric magnetic field. The relaxation events are indicated by the vertical dashed lines. Note the colorbar axes differ. (Parameters: \( a\lambda_0 = 3.88, \Lambda_e = 1, \Lambda_i = 1, S = 20,000, P_m = 0.1 \))](image-url)
4.4.2 Modal Decomposition

The MHD and Hall dynamo contributions from selected modes averaged over the relaxation events are shown in Fig. 4.12. The structure of the Hall dynamo contributions are again extremely consistent between the subsequent events in this computation, and they essentially mirror those of the previous model C computation with anti-parallel current. Again, the Hall dynamo opposition in the core here is driven largely by the $m = 1, n = -7$ mode. The $m = 1, n = -6$ mode’s contribution to the Hall dynamo is largely responsible for dynamo drive in the core, and outside the $|q| = 1/7$ surface.

4.5 Discussion

In all of our computations, it is clear that the initial relaxation event is substantially different from the subsequent relaxation events. The first event starts from a saturated single-helicity state that results from the initial conditions and the large growth of the core-most tearing mode. The spectral magnetic energy in the dominant core-resonant $m = 1, |n| = 6$ mode is generally between 2 and 5 times larger in the first event than it is in subsequent events. Further, the magnetic activity in the first event is largely confined to only a few modes, while subsequent events have a much broader spectrum of modes which contribute to the relaxation dynamics. Consequently, these first events are not expected to be representative of typical relaxation events in the RFP.

The first events in our computations always show cooperation between the MHD and Hall dynamos to relax the parallel current density in the core and increase it in the edge. As we have argued previously, in this case $\langle \tilde{J} \times \tilde{B} \rangle_\parallel$ can not simultaneously relax a parallel flow in the core that is in the same direction as the parallel current (as it is observed to be in MST). This contradicts the observed flattening of both parallel current and flow at the relaxation events in the experiment. However, in our subsequent events there are cases where strong opposition of the two dynamos is observed. The MHD dynamo acts to relax the parallel current in the core and increase it in the edge, as is expected for current relaxation, while the Hall dynamo has the opposite effect, albeit at lower amplitude. This would allow the parallel flow to relax via the correlated Lorentz force density, while the parallel current also relaxes via the MHD dynamo. As we will see in Chapter 5, the opposition of the dynamos allows for the changes in core parallel plasma flow to occur in the same direction as the changes in core parallel plasma current, consistent with the observations on MST.

Nevertheless, there are features that are common to all of the events in our computations and these confirm the model of the sawtooth cycle developed by previous computations [93].
Figure 4.12: Modal decomposition of the MHD and Hall dynamos parallel to the axisymmetric magnetic field averaged over the relaxation events. The shaded regions indicate the average plus or minus one standard deviation of the resonant surface location for each mode over the event. (Parameters: \(a\lambda_0 = 3.88\), \(\Lambda_e = 1\), \(\Lambda_i = 1\), \(S = 20,000\), \(P_m = 0.1\))
Nonlinear relaxation is generally preceded by linear growth of one of the core-resonant modes. In our single-fluid computations and our two-fluid computations without ion gyroviscosity this is almost always the $m = 1, |n| = 6$ mode, while in our two-fluid computations with ion gyroviscosity the $m = 1, |n| = 6$ mode may exist at a saturated level of fluctuations and the rapid growth of the $m = 1, |n| = 7$ mode then initiates the relaxation dynamics. The relaxation events generally end with a peaking of magnetic energy in one of the $m = 0$ modes; often this is $|n| = 1$, but in the first event in our two-fluid computations this is $|n| = 2$. 
We have seen that the correlated Lorentz force density is small in our single-fluid computations, but the parallel component of the Hall dynamo electric field, $\langle \tilde{J} \times \tilde{B} \rangle / (n) e$, is significant in our computations that include two-fluid effects in the generalized Ohm’s law. From this, we expect that there is a significant correlated Lorentz force density along the magnetic field, $\langle \tilde{J} \times \tilde{B} \rangle / n$, in our two-fluid computations. As we will see, this force density causes a rapid change in plasma flow along the magnetic field, although these changes are inhibited by other forces in the momentum balance equation, including the fluctuation-induced Reynolds force density and the viscous and gyroviscous stresses.

5.1 Parallel and Perpendicular Flows

Our single-fluid MHD computation shows very little coherent change in plasma flow as a result of the current relaxation events, as can be seen in Fig. 5.1, which shows the axisymmetric component of the flow parallel, $\langle \mathbf{v} \rangle \cdot \hat{b}$, and perpendicular, $\langle \mathbf{v} \rangle \cdot (\hat{b} \times \hat{r})$, to the axisymmetric magnetic field ($\hat{b} \equiv \langle \mathbf{B} \rangle / |\langle \mathbf{B} \rangle|$). The changes in parallel flow during a relaxation event largely mirror the correlated Lorentz force density (see Fig. 4.2); there is no significant temporal correlation over an event and no large scale flows are driven as a result. Both the parallel and perpendicular flow components show rapid changes during relaxation events.

Figure 5.1: Axisymmetric flow parallel and perpendicular to the axisymmetric magnetic field. The relaxation events are indicated by the vertical dashed lines. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 0$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)
and perpendicular components of axisymmetric flow remain less than a percent of the Alfvén velocity, and the kinetic energy associated with these axisymmetric flows remains much smaller than the energies associated with the fluctuations.

Our two-fluid computation without ion gyroviscosity shows much more substantial axisymmetric flows, as can be seen in Fig. 5.2. This is also evident in Fig. 4.4 where the kinetic energy associated with the axisymmetric flow is now substantially larger than the energies associated with the fluctuations. It is immediately evident that there is considerably more perpendicular flow than parallel flow following the initial relaxation event. The parallel flow, and changes in it, appear closely related to the current relaxation dynamics; significant changes in parallel flow are only observed around the relaxation events. In contrast, the perpendicular flow which arises after the first relaxation event persists for the duration of the computation, and it remains relatively unchanged during a relaxation event. Accurate modeling of the flow evolution between relaxation events requires effects that are outside the scope of our extended MHD model, and we do not expect that this flow is necessarily indicative of what might be expected to occur in the experiment.

The significance and persistence of the perpendicular component of flow is also evident in our two-fluid computations with ion gyroviscosity, seen in Fig. 5.3 and Fig. 5.4. The perpendicular flow in these computations is also concentrated largely between the \(|q| = 1/6\) surface and \(r/a \approx 0.6\), and it remains fairly regular throughout the evolution. The changes

Figure 5.2: Axisymmetric flow parallel and perpendicular to the axisymmetric magnetic field. The relaxation events are indicated by the vertical dashed lines. (Parameters: \(a\lambda_0 = -3.88\), \(\Lambda_e = 1\), \(\Lambda_i = 0\), \(S = 20,000\), \(P_m = 1.0\))
Figure 5.3: Axisymmetric flow parallel and perpendicular to the axisymmetric magnetic field. The relaxation events are indicated by the vertical dashed lines. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 1.0$)

Figure 5.4: Axisymmetric flow parallel and perpendicular to the axisymmetric magnetic field. The relaxation events are indicated by the vertical dashed lines. (Parameters: $a\lambda_0 = 3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 0.1$)
in parallel flow, however, are again strongly correlated with the current relaxation events. In
the first relaxation event in both of these computations, there are large changes in parallel
flow in the core driven by the Lorentz force density. The direction depends on the orientation
of the background current density, and the $\Delta \langle v \rangle_\parallel$ is in the opposite direction as the $\Delta \langle J \rangle_\parallel$
in this event. This is consistent with the relaxation event in the previous computations of
King et al. [70]. In subsequent events where there is now opposition of the MHD and Hall
dynamos, however, we observe that the $\Delta \langle v \rangle_\parallel$ in the core is also in the same direction as the
$\Delta \langle J \rangle_\parallel$ during relaxation. This is most evident in the second and fourth events of our model
D computation (Fig. 5.4), but it is also evident in the weaker second event in our model C
computation (Fig. 5.3).

5.1.1 Parallel Flow Evolution

The change in the axisymmetric component of plasma flow parallel to the axisymmetric
magnetic field, $\Delta \langle v \rangle_\parallel$, differs substantially between the first event and the subsequent events
in our computations. This is clearly evident in Fig. 5.5, which shows the difference between
the parallel flow profile at the beginning and end of each relaxation event. The strongest

![Figure 5.5: Change in parallel flow, $\Delta \langle v \rangle_\parallel$, over the relaxation events in (a) model A, (b) model B, (c) model C, and (d) model D. Note that vertical axes differ.](image)

opposition of the MHD and Hall dynamo drives in the core is observed in the fourth event in
model B and in the second events in both models C and D. Here, the flow changes in the core are very clearly opposite to the changes in the initial relaxation event.

We may compare the amplitudes of the flow changes in our computations to those observed experimentally in Kuritsyn et al. [73]. Their low current (∼ 200 kA) discharges have an Alfvén velocity of roughly 1000 km/s, and the flow changes observed there are between 5 – 20 km/s, roughly a percent or two of the Alfvén flow. These are on the same order as the changes in plasma flow that we observe in our computations, as was also reported in King et al. [70].

### 5.2 Momentum Evolution

There are also substantial changes in the total axial and angular plasma momentum in our two-fluid computations, as can be seen in Fig. 5.6. The single-fluid MHD case, in Fig. 5.6(a), shows very little correlated change in either the axial or angular plasma momentum. There are modest changes at the relaxation event, but the directionality changes between events and the total momentum remains small, relative to our two-fluid computations.

In all of our two-fluid computations, there are substantial changes in axial momentum, and the axial momentum tends to flow opposite to the axial plasma current in our computations.
When \( \lambda < 0 \), the axial momentum is positive, and when \( \lambda > 0 \), \( P_z < 0 \). There is a sharp increase in axial momentum at the first event in all of our two-fluid computations, but those with ion gyroviscosity show much less change over the subsequent events. In contrast, there are large increases in axial momentum at every event in our two-fluid computation without ion gyroviscosity, and the direction is the same regardless of cooperation or competition of the dynamo terms.

The angular momentum in our two-fluid computations appears to be independent of the plasma current; it is positive in both the parallel and anti-parallel current computations. However, it is larger by about a factor of 2 in our two-fluid computations that include the ion gyroviscosity suggesting that the angular momentum is at least partly related to flows associated with the ion gyroviscous stress tensor, rather than directly with the current relaxation dynamics.

While the changes in total momentum are strongly correlated with the relaxation events, we note that they can not be a direct result of the Lorentz force density, which is associated with local changes in plasma flow but does not affect the total momentum. As shown in Appendix A, the momentum density, \( p = m_i n v \), in our computational equations evolves as

\[
\frac{\partial p}{\partial t} = \nabla \cdot \mathbf{T} - m_i D_n \nabla (n - n_{eq}) \cdot \nabla v + m_i (v - v_{eq}) \nabla \cdot (n_{eq} v_{eq}) 
\]  

(5.1)

where

\[
\mathbf{T} \equiv -m_i n vv + m_i n_{eq} v_{eq} v_{eq} + \frac{BB}{\mu_0} \left( \frac{B^2}{2\mu_0} + p \right) - \left( \Pi_i - \Pi_{i,eq} \right) + m_i D_n \nabla (n - n_{eq}) v. 
\]  

(5.2)

The correlated Lorentz force density, \( \langle \mathbf{J} \times \mathbf{B} \rangle \| \), appears in the Maxwell stress tensor, \( BB/\mu_0 \), and the fluctuation-induced Reynolds force density takes the form of the Reynolds stress tensor, \(-m_i n vv\). With a perfectly conducting boundary and no-slip conditions on the plasma flow, these terms do not contribute to the evolution of the total axial or angular momentum of the plasma. The only physical terms that may affect the evolution of the total momentum are the viscous and gyroviscous coupling to the boundary.

There are two effects associated with numerical terms not present in the physical model that may also impact the momentum evolution in our computations. The first comes through the artificial density diffusion, which is measured to have little effect in our computations. The second comes from the anomalous sources required to balance the equilibrium pinch flow. Although this contribution is small, and scales as \( 1/S \), this term may still provide a spurious source or sink of momentum. In our computations, the divergence of the pinch flow
is negative so this term acts as a momentum sink, rather than a spurious source. As we will see, although it remains smaller than the physical terms during the current relaxation events, the contribution from this term is non-negligible in our computations.

### 5.2.1 Axial Momentum

Although the Maxwell stress does not change the total momentum directly, it generates the largest force density in the axial momentum evolution in our computations, as can be seen in Fig. 5.7 for model B, Fig. 5.8 for model C, and Fig. 5.9 for model D. These figures show the radial transport of axial momentum averaged over each of the relaxation events in our computations. At radial locations where the momentum flux is positive, the axial momentum increases within that radius, and when it is negative the axial momentum inside that radius decreases. There is strong regularity in the averaged stresses in subsequent relaxation events. The Maxwell stress is largely opposed by the Reynolds stress at all radial locations in our two-fluid computations without ion gyroviscosity (Fig. 5.7); when ion gyroviscosity is included, it also helps to balance the dominant Maxwell stress (Fig. 5.8 and Fig. 5.9). In all of the first events, the anomalous sink of momentum due to the pinch flow is small, but in subsequent events this term becomes more significant. Simulations that are run for longer transport time-scales will need to properly account for this behavior.

The significant changes in momentum are an indirect result of the Maxwell stress in our two-fluid computations; the direct result is viscous and gyroviscous coupling with the boundary. In our two-fluid computations without ion gyroviscosity, the large Maxwell stress in the edge region results in plasma flow which gets viscously coupled to the boundary, increasing the axial momentum during each relaxation event. When the ion gyroviscosity is included, a very thin boundary layer exists near the plasma edge where the viscous and gyroviscous stresses are large but nearly in balance. However, as the ion gyroviscosity scales directly with ion pressure, we do not expect this behavior is relevant to experimental measurements which typically have pressure profiles that drop off at the plasma edge. In addition, the perpendicular diffusivity scales with charged particle density, $n$, so a more realistic pressure is also expected to affect the viscous coupling. Much more detailed modeling is needed to resolve the momentum transport at the edge, but computations with more realistic pressure profiles are a logical next step towards more accurate modeling of RFP discharges.

The MHD and Hall dynamo opposition in the core during some of our subsequent relaxation events is also evident here in the Maxwell stress. In the first event and most of the subsequent events of our two-fluid computation without ion gyroviscosity, the Maxwell stress

\[1\text{The noise near } r/a \approx 0.6 \text{ in the third event stems from the ion gyroviscosity, and it is related to the under-resolved behavior in our previous dynamo plots.} \]
Figure 5.7: Terms in evolution of axial momentum averaged over the relaxation events. The dashed lines indicate ± one standard deviation over the event. Note that the axes differ. (Parameters: $a \lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)

Figure 5.8: Terms in evolution of axial momentum averaged over the relaxation events. The dashed lines indicate ± one standard deviation over the event. Note that the axes differ. (Parameters: $a \lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 1.0$)
Figure 5.9: Terms in evolution of axial momentum averaged over the relaxation events. The dashed lines indicate ± one standard deviation over the event. Note that the axes differ. (Parameters: $a\lambda_0 = 3.88$, $A_e = 1$, $A_i = 1$, $S = 20,000$, $P_m = 0.1$)

Figure 5.10: Terms in evolution of axial momentum averaged over relaxation events that show clear dynamo opposition: (a) model B, event 4, (b) model C, event 2, (c) model D, event 2.
is positive in the edge and negative inside of $r/a \approx 0.5$. In the one subsequent event with strong dynamo opposition here (event 4) the Maxwell stress is positive again inside $r/a \approx 0.2$, as can be seen in Fig. 5.10(a), which shows an enlarged view of that region. In the second relaxation events of our two-fluid computations with ion gyroviscosity (models C and D) the Maxwell stress switches sign a little further out near $r/a \approx 0.3$, as can be seen in Fig. 5.10(b) and (c) respectively.

5.2.2 Angular Momentum

The Maxwell stress is also the largest term in the angular momentum evolution of our two-fluid computation without ion gyroviscosity, as shown in Fig. 5.11. It is again largely opposed by the Reynolds stress and the viscosity, and viscous coupling to the boundary is responsible for the small changes in angular momentum observed at the relaxation events here. We can observe clear differences between the character of the first relaxation event and subsequent relaxation events in the Maxwell stress here as well, although there is little evidence of the parallel dynamo opposition in the core in the angular component. This is not surprising given that the magnetic field is mostly axial there.

In our two-fluid computations with ion gyroviscosity, the gyroviscous stress competes with and beats out the Maxwell stress in the angular momentum evolution during the first event, as can be seen in Fig. 5.12 and Fig. 5.13. However, as we noted previously, the significance of the gyroviscosity would drop off substantially near the edge with a realistic pressure profile, and we do not expect our results in the edge region to be indicative of experimental plasmas. The sharp boundary layer is also evident here as well, with competition between the viscous and gyroviscous stresses.

5.3 Discussion

Large changes in plasma flow are observed in our computations with two-fluid effects in the generalized Ohm’s law. The axisymmetric flow parallel to the axisymmetric magnetic field changes significantly at a relaxation event as a result of the correlated Lorentz force density, and this is indicative of the strong coupling between current and flow relaxation in two-fluid models. In subsequent relaxation events, the $\Delta \langle \nabla \rangle_\parallel$ over the event is in the same direction as the $\Delta \langle J \rangle_\parallel$ in the core when there is strong opposition in the dynamo fields, and this behavior is consistent with experimental observations on MST.

The Maxwell stress is largely opposed by the fluctuation-induced Reynolds stress, as in the previous computations of King et al. [70] and in experimental measurements on MST.
Figure 5.11: Terms in evolution of angular momentum averaged over the relaxation events. The dashed lines indicate ± one standard deviation over the event. Note that the axes differ. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)

Figure 5.12: Terms in evolution of angular momentum averaged over the relaxation events. The dashed lines indicate ± one standard deviation over the event. Note that the axes differ. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 1.0$)
The Maxwell stress alone may not alter the total momentum, but it drives strong flows near the edge which couple to the wall through viscosity and ion gyroviscosity. More detailed edge modeling, including the effects of a realistic pressure profile, is needed to determine if this behavior is realistic, however.
6 SPECTRAL POWER FLOW

6.1 Spectral Energy Transfer

Ho and Craddock [57] examine the spectral power transfer among components of the single-fluid MHD system in order to assess the nonlinear coupling present in the system. Following their work, we derive similar spectral transfer equations for the extended-MHD system (see Appendix D). Within the two-fluid model, ignoring contributions from the density evolution, the electron inertia term, and the divergence cleaning term, the magnetic and kinetic energy density associated with the axisymmetric fields evolve as:

\[
\frac{\partial}{\partial t} \left( \frac{m_i \langle n \rangle}{2} |\langle v \rangle|^2 \right) = \langle J \rangle \times \langle B \rangle \cdot \langle v \rangle + \langle \tilde{J} \times \tilde{B} \rangle \cdot \langle v \rangle - \langle m_i n \delta \cdot \langle v \rangle \rangle - \langle v \rangle \cdot \langle \nabla \langle h \rangle \rangle - \langle v \rangle \cdot \langle \nabla \cdot \Pi \rangle
\]  

\[
\frac{\partial}{\partial t} \left( \frac{1}{2\mu_0} |\langle B \rangle|^2 \right) = -\nabla \cdot \left( \frac{\langle E \rangle \times \langle B \rangle}{\mu_0} \right) - \eta |\langle J \rangle|^2 + \langle v \rangle \times \langle B \rangle \cdot \langle J \rangle + \langle \tilde{v} \times \tilde{B} \rangle \cdot \langle J \rangle - \Lambda_e \langle J \rangle (\langle n \rangle)^e \cdot \langle \tilde{J} \times \tilde{B} \rangle \cdot \langle B \rangle. \]  

Similarly, the magnetic and kinetic energy density associated with the fluctuating components evolve as:

\[
\frac{\partial}{\partial t} \left( \frac{m_i \langle n \rangle}{2} |\tilde{v}_{mn}|^2 \right) = \tilde{J}_{mn} \times \langle B \rangle \cdot \tilde{v}_{mn} + \langle J \rangle \times \tilde{B}_{mn} \cdot \tilde{v}_{mn} + \langle J \rangle \times \langle B \rangle_{mn} \cdot \tilde{v}_{mn} - \langle m_i n \delta \cdot \tilde{v}_{mn} \rangle - \langle \nabla \tilde{h} \rangle_{mn} - \langle \tilde{v}_{mn} \rangle \cdot (\nabla \rho)_{mn}
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2\mu_0} |\tilde{B}_{mn}|^2 \right) = -\nabla \cdot \left( \frac{\tilde{E}_{mn} \times \tilde{B}_{mn}}{\mu_0} \right) - \eta |\tilde{J}_{mn}|^2 + \tilde{J}_{mn} \cdot \left( \frac{1}{ne \nabla p_e} \right)_{mn}
\]

\[
+ \tilde{v}_{mn} \times \langle B \rangle \cdot \tilde{J}_{mn} + \langle v \rangle \times \tilde{B}_{mn} \cdot \tilde{J}_{mn} + \langle \tilde{v} \times \tilde{B} \rangle_{mn} \cdot \tilde{J}_{mn} + \Lambda_e \left( \frac{\langle J \rangle}{\langle n \rangle} \right)^e \cdot \tilde{J}_{mn} \cdot \tilde{B}_{mn}.
\]

The terms have been color-coded to highlight the power flow between helical harmonics. Terms in light green exchange energy between the magnetic and kinetic components of a given helical harmonic, and they vanish when considering the sum of the helical harmonic’s kinetic and magnetic energies, \( W_{mn} \). The terms in dark blue couple energy between the axisymmetric magnetic field to the fluctuating flows; this represents the action of the MHD dynamo. Taking \( \langle J \rangle \sim \lambda \langle B \rangle \), this term becomes \( \langle \tilde{v} \times \tilde{B} \rangle \cdot \langle J \rangle \sim \lambda \langle \tilde{v} \times \tilde{B} \rangle_{\parallel} |\langle B \rangle| \). If \( \lambda > 0 \), then the MHD dynamo acts to increase the energy associated with the axisymmetric magnetic
fields where $\langle \mathbf{v} \times \mathbf{B} \rangle_\parallel > 0$, and it acts to decrease it where $\langle \mathbf{v} \times \mathbf{B} \rangle_\parallel < 0$. The terms in dark green transfer energy between the fluctuating fields and the axisymmetric flows; they are present regardless of the form of the generalized Ohm’s law. In Eq. (6.1), the correlated Lorentz force density, $\langle \mathbf{J} \times \mathbf{B} \rangle$, may act to increase or decrease the energy associated with the axisymmetric flows, depending on the projection of this term onto $\langle \mathbf{v} \rangle$. The terms in red directly couple the energy associated with the axisymmetric magnetic field to energy associated with the fluctuating magnetic field, and are only present in the two-fluid ($\Lambda_e = 1$) system. This represents the effects of the Hall dynamo on the system. The remaining colored terms represent three-wave nonlinear coupling between fluctuations. From energy conservation, these terms will vanish when summed over all of the fluctuating components.

It is important to note that the $\langle \mathbf{v} \times \mathbf{B} \rangle \cdot \langle \mathbf{J} \rangle$ and $-\langle \mathbf{J} \times \mathbf{B} \rangle \cdot \langle \mathbf{J} \rangle / \langle n \rangle_e$ terms in Eq. (6.2) and the $\langle \mathbf{J} \times \mathbf{B} \rangle \cdot \langle \mathbf{v} \rangle$ term in Eq. (6.1) are exactly canceled by the corresponding terms in Eqs. (6.3) and (6.4), after summing over all the $m, n$ modes of the system. The electric field contributions from the MHD and Hall dynamos do not affect the total energy of the system, they merely redistribute and rearrange it.

If we sum Eq. (6.3) and Eq. (6.4) and integrate over the volume, we find the rate of change of mode magnetic and kinetic energy is given by:

$$\frac{\partial}{\partial t} W_{m,n} = \int \left\{ - \mathbf{v}_{mn} \times \mathbf{B}_{mn} \cdot \langle \mathbf{J} \rangle - \mathbf{J}_{mn} \times \mathbf{B}_{mn} \cdot \langle \mathbf{v} \rangle + \Lambda_e \frac{\langle \mathbf{J} \rangle}{\langle n \rangle_e} \cdot \mathbf{J}_{mn} \times \mathbf{B}_{mn} + \left( \mathbf{v} \times \mathbf{B} \right)_{mn} \cdot \mathbf{J}_{mn} + \left( \mathbf{J} \times \mathbf{B} \right)_{mn} \cdot \mathbf{v}_{mn} - \Lambda_e \frac{\mathbf{J}_{mn}}{\langle n \rangle_e} \cdot \left( \mathbf{J} \times \mathbf{B} \right)_{mn} - \eta |\mathbf{J}_{mn}|^2 - \dot{\mathbf{v}}_{mn} \cdot (\nabla \cdot \Pi_{iso})_{mn} - \Lambda_i \dot{\mathbf{v}}_{mn} \cdot (\nabla \cdot \Pi_{gyr})_{mn} - (m_i n \mathbf{v} \cdot \nabla \mathbf{v})_{mn} - \dot{\mathbf{v}}_{mn} \cdot (\nabla \tilde{p})_{mn} + \dot{\mathbf{J}}_{mn} \cdot \left( \frac{1}{n_e} \nabla p_e \right)_{mn} \right\} d^3 x. \tag{6.5}$$

The terms on the first line on the right-hand side of Eq. (6.5) represent exchange of energy between the spectral components and the axisymmetric fields. If they are positive, the mode extracts energy from the axisymmetric fields, while energy is deposited into the axisymmetric fields if they are negative. Ho and Craddock [57] refer to these as quasilinear terms, but we argue that this nomenclature is incorrect for our nonlinear computation. A true quasilinear calculation does not allow changes in the axisymmetric field so that $\langle \mathbf{J} \rangle$ and $\langle \mathbf{v} \rangle$ are constant, whereas in Eq. (6.5), and in our computations, these terms are allowed to change. We will instead refer to these terms via their mathematical definitions.

Terms on the second line represent nonlinear interactions of helical harmonics through three-wave coupling. Again, Ho and Craddock [57] refer to these as nonlinear terms, which they are, but we argue that this terminology masks the fact that the first three terms above
also include nonlinear effects through coupling with the axisymmetric fields. Instead, these terms will be again be referred to only through their mathematical definitions.

The first two terms on the third line of the RHS of Eq. (6.5) represent loss of mode energy through resistive and viscous decay respectively, and the third term represents FLR effects. The remaining terms represent other exchanges of energy. In the simulations presented here, the gyroviscous term and the terms on the fourth line are small compared to the highlighted terms and the dissipative effects.

6.2 The \( m = 1, |n| = 6 \) Mode

We now examine the evolution of energy in the \( m = 1, |n| = 6 \) component, as it is the core-resonant tearing mode in our computations. We will see that the power flow differs substantially between the first relaxation event and subsequent relaxation events in all of our computations. This provides further corroboration of our previous results. In these subsequent events, we will also note a difference in the character of the power flow that is correlated with the strong opposition of the parallel MHD and Hall dynamo electric fields in the core.

The power flow into the \( m = 1, n = 6 \) mode differs markedly from the first event to the subsequent events in our single-fluid MHD computations, as can be seen in Fig. 6.1. Prior to the first event, the \( m = 1, n = 6 \) power flow is nearly steady; dissipation balances

![Figure 6.1: Spectral transfer of energy for the \( m = 1, n = 6 \) component during relaxation events, which are highlighted. (Parameters: \( a\lambda_0 = -3.88, \Lambda_e = 0, \Lambda_i = 0, S = 20,000, P_m = 1.0 \) )](image-url)
the exchange of energy with the axisymmetric fields through \((-\tilde{v}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle\) in the saturated single-helicity state. At the start of the first event, there is a small increase in the \((-\tilde{v}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle\) term before a sudden change in the character of the drive occurs, and the \(m = 1, n = 6\) mode begins depositing energy back into the axisymmetric fields. At very nearly the same time, energy begins to flow into the \(m = 1, n = 6\) mode from other fluctuating components, although this rate is insufficient to balance the losses to the axisymmetric fields. In contrast, in the subsequent events the \(m = 1, n = 6\) mode primarily extracts energy from the axisymmetric fields for the duration of the relaxation event, and it loses energy to other fluctuating components through the \((\tilde{v} \times \tilde{B})_{mn} \cdot \tilde{J}_{mn}\) coupling term. Viscous and resistive losses have only a small contribution to the evolution of the \(m = 1, n = 6\) energy here. The only other term of some significance is the \((\tilde{J} \times \tilde{B})_{mn} \cdot \tilde{v}_{mn}\) coupling to other fluctuations, although this remains small relative to the dominant term.

The evolution of the \(m = 1, n = 6\) mode energy in the first event in our two-fluid computation without ion gyroviscosity is similar to the evolution in the single-fluid MHD computation, although now the additional two-fluid terms come into play. As seen in Fig. 6.2, there is balance initially in the single-helicity state, although shortly into the event this is altered and the \((-\tilde{v}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle\) term begins dumping energy back into the axisymmetric fields, while being driven by the \((\tilde{v} \times \tilde{B})_{mn} \cdot \tilde{J}_{mn}\) coupling to other modes. The two-fluid \((\tilde{J}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle / \langle n \rangle e\) term largely opposes the \((-\tilde{v}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle\) term in the first event; when one is positive, the other is negative. The radial profiles of these terms averaged over the relaxation event are similar to the parallel component of the MHD and Hall dynamo terms shown previously, and the negative \((\tilde{J}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle / \langle n \rangle e\) results from contributions at larger \(r/a\), not in the core region. The sign of these terms alone is insufficient to determine the local behavior of the dynamo electric fields. The two-fluid \((-\tilde{J} \times \tilde{B})_{mn} \cdot \tilde{J}_{mn} / \langle n \rangle e\) term that couples the fluctuating components very nearly balances the contribution from the two-fluid coupling to axisymmetric fields, and the two-fluid terms appear to act as a channel through which power may flow among individual helical harmonics.

The character of the drive differs substantially from the first event in the subsequent events here as well. In these latter events, the \((-\tilde{v}_{mn} \times \tilde{B}_{mn}) \cdot \langle J \rangle\) term remains mostly positive over the event, while the \((\tilde{v} \times \tilde{B})_{mn} \cdot \tilde{J}_{mn}\) coupling to the fluctuations remains mostly negative, consistent with the evolution in our single-fluid MHD computation. The only other substantial contributions to the mode energy evolution are the two-fluid terms, but again these nearly balance each other; energy is exchanged with the axisymmetric fields, and it is transferred between the fluctuating components.

As we saw in Chapter 4, in the fourth relaxation event of this computation there is strong opposition of the MHD and Hall dynamo electric fields in the core of the plasma. This appears
to be correlated with a notable difference in the power flow for the \( m = 1, n = 6 \) mode. Here, the \( m = 1, n = 6 \) energy remains largely flat during the event, with little net extraction of energy from the axisymmetric fields via the \( \left( \hat{\mathbf{v}} \times \hat{\mathbf{B}} \right)_{mn} \cdot \langle \mathbf{J} \rangle \) term and little net exchange with other fluctuating components via the \( \left( \hat{\mathbf{v}} \times \hat{\mathbf{B}} \right)_{mn} \cdot \mathbf{J}_{mn} \) coupling term. However, the large extraction of energy from the axisymmetric fields via the \( \left( \hat{\mathbf{J}}_{mn} \times \hat{\mathbf{B}}_{mn} \right) \cdot \langle \mathbf{J} \rangle / \langle n \rangle e \) term remains balanced by a correspondingly large deposition of energy into other fluctuating components via the \( \left( -\mathbf{J} \times \hat{\mathbf{B}} \right)_{mn} \cdot \mathbf{J}_{mn} / \langle n \rangle e \) coupling term. Although the total energy exchanged with the axisymmetric fields through the \( \left( -\hat{\mathbf{v}} \times \hat{\mathbf{B}} \right)_{mn} \cdot \langle \mathbf{J} \rangle \) term is small here, the \( m = 1, n = 6 \) modal contributions to the total MHD dynamo remain significant. The \( m = 1, n = 6 \) MHD dynamo contribution merely redistributes the energy in the axisymmetric fields, rather than extracting it.

The initial relaxation events in our two-fluid computations with ion gyroviscosity (model C and D) are similar to each other, but differ somewhat from the previous two-fluid computation without ion gyroviscosity, as can be seen in Fig. 6.3 (model C) and Fig. 6.4 (model D). In these, the transfer of energy back to the axisymmetric fields about halfway through
the event is mediated through the \( (\vec{J}_{mn} \times \vec{B}_{mn}) \cdot \langle \vec{J} \rangle / \langle n \rangle \) term rather than through the 
\( (\vec{v}_{mn} \times \vec{B}_{mn}) \cdot \langle \vec{J} \rangle \) term. In addition, there is substantial drive via the \( (\vec{v} \times \vec{B})_{mn} \cdot \vec{J}_{mn} / \langle n \rangle \) \( e \) coupling to other fluctuating components, while the \( (\vec{v} \times \vec{B})_{mn} \cdot \vec{J}_{mn} \) coupling is somewhat weaker.

The subsequent relaxation events in these computations also show opposition between the MHD and Hall dynamos in the core. We note that this is again reflected in the power flow for the \( m = 1, |n| = 6 \) mode. There is a large exchange of energy with the axisymmetric fields through the \( (\vec{J}_{mn} \times \vec{B}_{mn}) \cdot \langle \vec{J} \rangle / \langle n \rangle \) \( e \) term but this energy is largely funneled into the other fluctuating components. We note also that the total change in energy for the \( m = 1, |n| = 6 \) mode over these subsequent events is typically much smaller than in the first event.

### 6.3 The \( m = 0, |n| = 1 \) Mode

We next turn our attention to the \( m = 0, |n| = 1 \) magnetic and kinetic energy evolution. The \( m = 0, |n| = 1 \) mode plays a crucial role in mediating the nonlinear dynamics of the system, as it allows the \( m = 1 \) modes to couple with their nearest \( n \) neighbors. Among the previous studies to consider this component, King [71] identifies a subsequent relaxation event in two-fluid computations with ion gyroviscosity at \( S = 80,000 \) and concludes that the reduction in \( m = 0 \) magnetic energy is the result of a lack of nonlinear drive, rather than gyroviscous stabilization that is responsible. Choi et al. [25] measure the \( (\vec{v}_{mn} \times \vec{B}_{mn}) \cdot \langle \vec{J} \rangle \) term for
this component in MST, and they identify large and small reconnection events associated with the amplitude of that term. When this term is negative, they argue that the drive for \( m = 0 \) is predominantly nonlinear coupling to the core-resonant \( m = 1 \) modes, which are all excited before the \( m = 0 \) energy is increased. These are referred to as large reconnection events, and they represent typical RFP sawteeth. Conversely, when this term is positive, which is only observed in deeply reversed \( F \approx -0.5 \) plasmas, the temporal dynamics suggest that the \( m = 0, |n| = 1 \) mode is linearly unstable, while the core resonant modes grow only slightly. These are referred to as small reconnection events.

In our single-fluid MHD computation (model A), the power flow into the \( m = 0, n = 1 \) mode appears largely similar during the three relaxation events, as can be seen in Fig. 6.5. Note that all of these events have similar levels of reversal and \( m = 0, n = 1 \) magnetic energy following relaxation. In all cases, the dominant source of energy for this helical harmonic is exchange with the axisymmetric fields through the \( \left( -\vec{v}_{mn} \times \vec{B}_{mn} \right) \cdot \left( \vec{J} \right) \) term, although there are also significant contributions from the \( \left( \vec{v} \times \vec{B} \right)_{mn} \cdot \vec{J}_{mn} \) and \( \left( \vec{J} \times \vec{B} \right)_{mn} \cdot \vec{v}_{mn} \) coupling to other fluctuating components. The temporal dynamics suggest that the mode is first
excited by the coupling from the other fluctuations, consistent with it not being spontaneously unstable. The exchange of energy with the axisymmetric fields occurs then as a nonlinear response to the evolving profile.

In contrast to the $m = 1, n = 6$ mode, there are substantial losses of energy via resistive and viscous losses, as well as coupling to kinetic energy of other fluctuating components. This is likely due to the close proximity of the reversal surface to the plasma boundary and the sharp increase in resistivity and viscosity there. We do not necessarily expect that this behavior is unphysical, however, as the plasma temperature does fall off considerably in the edge region of typical RFP discharges. Nevertheless, more detailed modeling that includes a realistic pressure profile and a temperature-dependent resistivity instead of simply a shaped profile would be needed to confirm this.

As we noted previously, in our two-fluid computations the first relaxation event primarily couples the $m = 1, |n| = 6$ mode and the $m = 1, |n| = 8$ mode with the $m = 0, |n| = 2$ mode, and the relaxation event ends with a peaking of magnetic energy in the $m = 0, |n| = 2$ mode. Consequently, the drive for the $m = 0, |n| = 1$ remains small during the first event, as can be seen for our two-fluid computations without ion gyroviscosity in Fig. 6.6. Subsequent events feature much more energy in the $m = 0, |n| = 1$ mode, and the power flow in these cases has a much clearer interpretation. Shortly after the onset of the relaxation event, the $m = 0, |n| = 1$ mode is driven by the $\left(-\vec{v}_{mn} \times \vec{B}_{mn}\right) \cdot \langle \vec{J} \rangle$ and $\left(\vec{J}_{mn} \times \vec{B}_{mn}\right) \cdot \langle \vec{J} \rangle / \langle n \rangle e$ coupling to the axisymmetric fields along with the $\left(\vec{v} \times \vec{B}\right)_{mn} \cdot \vec{J}_{mn}$ coupling among other fluctuating components. The two-fluid $\left(-\vec{J} \times \vec{B}\right)_{mn} \cdot \langle \vec{J} \rangle_{mn} / \langle n \rangle e$ coupling to other fluctuations

Figure 6.5: Spectral transfer of energy for the $m = 0, n = 1$ component during relaxation events. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 0$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)
remains small or negative in most cases, and the $(\mathbf{J} \times \mathbf{B})_{mn} \cdot \mathbf{v}_{mn}$ coupling to other components of the fluctuating flow is also a sink of energy.

There are no clear differences in the power transfer for this mode between the fourth event, which shows the strong dynamo electric field opposition in the core, and the other events. This is unsurprising given that the opposition is observed in the plasma core, far from the $q = 0$ resonant surface. The regularity of the $m = 0, |n| = 1$ mode behavior in all of these subsequent events is likely responsible for the consistent depth of reversal in this computation.

Our two-fluid computations with ion gyroviscosity typically show much weaker drive of the $m = 0, |n| = 1$ mode with much shallower depths of reversal in the subsequent events, as seen in Fig. 6.7 (model C) and Fig. 6.8 (model D). The only notable exception here is the second event in our model D computation, which has a depth of reversal that is commensurate with the initial event. Here, the power flow for the $m = 0, |n| = 1$ mode is very similar to the subsequent events in our two-fluid computation without ion gyroviscosity.

The subsequent relaxation events reported in King et al. [70] also showed much less change in reversal and significantly less $m = 0$ drive. In King [71], this is attributed to
As mentioned in King [71], a pressure gradient in the vicinity of the reversal surface would without the ion gyroviscosity. Further modeling with a detailed pressure profile is needed.

dissipation inherent in the RFP sawtooth cycle.

Our computations show that the \( \left(-\mathbf{v}_{mn} \times \mathbf{B}_{mn}\right) \cdot \langle \mathbf{J} \rangle \) term (associated with the \( m = 0,|n|=1 \) extraction of energy from the axisymmetric fields) measured by Choi et al. [25] is positive across all of the relaxation events in our computations. This does not agree with their finding that this term is only positive in deeply reversed discharges, when the \( m = 0,|n|=1 \) mode is spontaneously excited. However, although this term is nearly always positive, we find that it is large only when the \( \left(\mathbf{v} \times \mathbf{B}\right)_{mn} \cdot \mathbf{J}_{mn} \) coupling to other fluctuating components is also large, and there is substantial increase in the \( m = 0,|n|=1 \) mode energy. That is, the \( m = 0,|n|=1 \) energy only tends to increase when it is driven nonlinearly by coupling to other fluctuating components, but when it is driven by the fluctuations, it also extracts energy from the axisymmetric fields.
The ability to track the transfer of energy among individual helical harmonics is an extremely useful tool for understanding the nonlinear evolution of a simulated plasma. It allows a detailed decomposition of the nonlinear coupling among modes that is, in principle, limited only by the level of detail that is desired. Here, we have analyzed this decomposition for the core-resonant $m = 1, |n| = 6$ mode and the edge-resonant $m = 0, |n| = 1$ mode only, and a more detailed study would certainly yield even more interesting results.

The $m = 1, |n| = 6$ power flow shows clear evidence that the first relaxation event is distinct from the subsequent events. The first event results from other modes upsetting a large $m = 1, |n| = 6$ saturated single-helicity state, and the $m = 1, |n| = 6$ mode begins putting energy back into the axisymmetric fields partway through the event. In the subsequent events, however, the $m = 1, |n| = 6$ mode activity is comparable to the behavior of the other modes, acting to relax the current profile throughout the event. There is also a clear correlation between which terms are dominant in the power flow for the events that show strong opposition between the MHD and Hall dynamo drive in the core. However, the terms

Figure 6.8: Spectral transfer of energy for the $m = 0, n = -1$ component during relaxation events. (Parameters: $a \lambda_0 = 3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 0.1$)

### 6.4 Discussion

The ability to track the transfer of energy among individual helical harmonics is an extremely useful tool for understanding the nonlinear evolution of a simulated plasma. It allows a detailed decomposition of the nonlinear coupling among modes that is, in principle, limited only by the level of detail that is desired. Here, we have analyzed this decomposition for the core-resonant $m = 1, |n| = 6$ mode and the edge-resonant $m = 0, |n| = 1$ mode only, and a more detailed study would certainly yield even more interesting results.

The $m = 1, |n| = 6$ power flow shows clear evidence that the first relaxation event is distinct from the subsequent events. The first event results from other modes upsetting a large $m = 1, |n| = 6$ saturated single-helicity state, and the $m = 1, |n| = 6$ mode begins putting energy back into the axisymmetric fields partway through the event. In the subsequent events, however, the $m = 1, |n| = 6$ mode activity is comparable to the behavior of the other modes, acting to relax the current profile throughout the event. There is also a clear correlation between which terms are dominant in the power flow for the events that show strong opposition between the MHD and Hall dynamo drive in the core. However, the terms
measured here stem from global integrals, so we can draw no definitive conclusions as to how this relates directly to the local electric field measurements.

There are no clear differences in the power flow for the edge-resonant $m = 0, |n| = 1$ mode between different relaxation events. The temporal dynamics suggest that it is driven initially by coupling from the other fluctuating helical harmonics, but it also extracts energy from the axisymmetric magnetic fields. This conflicts with measurements on MST, but more detailed edge modeling is needed to confirm if this behavior is representative of the experiment.
7 HELICITY EVOLUTION

We now turn our attention to the evolution of magnetic energy and generalized global helicities in our nonlinear computations of the relaxation dynamics [103]. Variational theories predict the outcome of relaxation events by minimizing free energy while maintaining the initial value of robust constraints. As we discussed in Chapter 2, the variational theories based on static magnetohydrodynamics are well known [120, 127]. More recent extensions consider plasma flow [41, 51] and two-fluid effects that are outside the scope of the MHD model [56, 75, 116, 124]. Further, ion FLR effects are not included in existing relaxation theories, and ion gyroviscosity has been suggested as one possible avenue towards obtaining better agreement with experimental measurements [117].

Our primary goal here is to understand the numerically predicted dynamics, as applied to conditions of magnetic confinement. We analyze the computed evolution of energy and components of helicity according to the different models and identify the contributions that cause deviation from ideal behavior. Our computations allow evaluation of individual terms in the temporal rate of change of these quantities, and the relative importance of each term in the evolution is ascertained. The coupling between various pieces of the hybrid helicity in two-fluid models is also analyzed.

As we will see, the magnetic helicity is well conserved relative to the magnetic energy over each event, which is short compared to the global resistive diffusion time. The magnetic energy decreases by roughly 1.5% of its initial value over a relaxation event, while the magnetic helicity changes by at most 0.2% of the initial value. The hybrid helicity is dominated by magnetic helicity in low-β pinch conditions and is also well conserved. Differences of less than 1% between magnetic helicity and hybrid helicity are observed with two-fluid modeling and result from cross helicity evolution. The cross helicity is found to change appreciably due to the first-order finite Larmor radius effects which have not been included in contemporary relaxation theories. The plasma current evolves towards the flat parallel current state predicted by Taylor relaxation theory but does not achieve it. Plasma flow develops significant structure for two-fluid models, and the flow perpendicular to the magnetic field is much more substantial than the flow along it.

7.1 Introduction

The invariants of ideal single-fluid MHD include the total energy, \( W = W_B + W_K + W_P \), and the global magnetic helicity, \( K \equiv \int A \cdot B \, d^3x \) [85, 127]. The physical justification for conserving helicity while minimizing energy in resistive conditions comes from recognizing
that turbulence in a plasma dissipates magnetic energy more readily than the global magnetic helicity [121]. A Fourier decomposition of the energy and helicity evolution yields the scalings

\[
\frac{\partial W_B}{\partial t} \sim \sum_k \eta k^2 B_k^2 \quad \frac{\partial K}{\partial t} \sim \sum_k \eta k B_k^2,
\]

the magnetic energy is dissipated by high-\(k\) fluctuations much more rapidly than the magnetic helicity. Conserving magnetic helicity and minimizing magnetic energy yields the force-free relaxed state (the Taylor state), \(\nabla \times B = \lambda_0 B\) with \(\lambda_0\) a global constant. For sufficiently large current, the Taylor model predicts magnetic field reversal, a hallmark of RFP experiments [10, 120].

The Taylor theory does not address plasma flows which are observed in experimental plasmas even in the absence of any external sources of momentum [29, 73]. However, we note that the kinetic energy is typically much smaller than the magnetic energy in magnetically confined plasmas, and the parallel flow in MST is a small fraction of the Alfvén velocity, \(V_\parallel/V_A \sim 10^{-2}\). Thus, the free energy available from gradients in parallel current density is proportionately larger than the energy available from gradients in plasma flow, and the magnetics are expected to dominate the relaxation dynamics.

Flow can be incorporated into a variational treatment through the cross helicity, \(X \equiv \int v \cdot B \, d^3x\) [24, 41, 110, 124]. The cross helicity is not conserved in general in single-fluid MHD, but it is conserved when certain restrictions are placed on the thermodynamic evolution [59]. Minimizing magnetic and kinetic energy while separately conserving the magnetic helicity and cross helicity yields a prediction for the relaxed state with both field-aligned currents, \(\nabla \times B = \lambda_0 B\), and flows, \(m_i n v = \lambda_1 B\) where \(\lambda_1\) is another global constant.

As we saw in Chapter 5, effects outside of the single-fluid MHD model, such as two-fluid terms in the generalized Ohm’s law, are needed to couple current and flow during relaxation. These effects become important when the ion and electron motions decouple, which occurs on scales below the ion-skin depth [8, 21, 80]. Extending the generalized Ohm’s law to include two-fluid effects alters the evolution of the cross helicity. It is not an ideal invariant in the same limits as in single-fluid MHD [89, 124]. An invariant of the incompressible Hall-MHD system of equations is the hybrid helicity \(H \equiv \int \Omega \cdot \nabla \times \Omega \, d^3x\) where \(\Omega \equiv A + \frac{m_e}{e} v\). The hybrid helicity measures the linking of canonical flux tubes and limits to the usual magnetic helicity as \(d_i \to 0\). Expanding the product with the appropriate boundary conditions, the hybrid helicity appears as a weighted sum, \(H = \mathcal{K} + 2 \frac{m_e}{e} \mathcal{X} + \left(\frac{m_e}{e}\right)^2 \mathcal{H}\), where \(\mathcal{H} \equiv \int v \cdot \omega \, d^3x\) is the kinetic helicity (with \(\omega \equiv \nabla \times v\), which measures the linkedness of fluid vorticity [85].

The magnetic helicity and hybrid helicity are the limiting values of the canonical species’ helicities (also called self helicities or generalized species’ helicities) \(\mathcal{K}_s \equiv \int A_s \cdot B_s \, d^3x\) for
\( \mathbf{A_s} = \mathbf{A} + \frac{m_s}{q_s} \mathbf{v_s} \) and \( \mathbf{B_s} = \nabla \times \mathbf{A_s} \) [56, 116, 129], where \( s = i, e \). In the \( m_e \to 0 \) limit, \( K_e \to K \) and \( K_i \to H \). The \( K_s \) are the ideal invariants of the more primitive two-fluid system of equations with a barotropic pressure for each species, \( p_s = p_s(n_s) \) [117]. Variational principles that minimize the total energy while conserving the \( K_{i,e} \) predict field-aligned currents and flows to lowest order in the species’ skin depths, \( d_{i,e}/a \) [56], similar to the previous single-fluid MHD results with cross helicity.

Two-fluid effects represent a singular perturbation of the single-fluid system, and these theories are singular in the \( d_s \to 0 \) limit. Variational theories with constraints that are more fragile than the target functional are mathematically ill-posed [67, 92, 128]. The kinetic helicity is a more fragile quantity than the kinetic energy, and relaxation theories that include kinetic helicity as a constraint while minimizing energy are not well posed. Different target functionals have been suggested [128], but their physical motivation is not evident [51]. Rather than focusing on the mathematical validity of various relaxation theories, we investigate how the quantities that are utilized in these theories evolve during simulated relaxation events with experimentally relevant two-fluid parameters.

### 7.2 Evolution in the NIMROD Model

Before turning to the computational results, we present a brief analysis of how the magnetic energy, magnetic helicity, and hybrid helicity evolve within the model Eqs. (3.1)-(3.6). Knowledge of how the various components of the hybrid helicity couple with each other and under what conditions they are expected to be invariant is useful in understanding our numerical results.

#### 7.2.1 Magnetic Energy

The global magnetic energy, \( W_B = \int |\mathbf{B}|^2/2\mu_0 \, d^3x \), is minimized in many relaxation theories [41, 117, 120, 124]. As shown in Appendix B, the magnetic energy in our dynamical system evolves as

\[
\frac{\partial W_B}{\partial t} = \int \left\{ -\nabla \cdot \left( \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) - \eta J^2 + \mathbf{v} \times \mathbf{B} \cdot \mathbf{J} \\
+ \Lambda_e \left[ \mathbf{J} \cdot \frac{\nabla p_e}{n_e} - \frac{m_e}{n_e^2} \frac{\partial \mathbf{J}}{\partial t} \cdot \mathbf{J} \right] - \frac{\kappa}{\mu_0} (\nabla \cdot \mathbf{B})^2 \right\} d^3x. \tag{7.2}
\]

The magnetic energy is not an ideal invariant alone; it couples to the kinetic energy through the MHD term \( \mathbf{v} \times \mathbf{B} \cdot \mathbf{J} \), and only the total energy, \( W_{\text{total}} = W_B + W_K + W_P \), is an ideal invariant. However, the magnetic energy is the dominant piece of the magnetofluid energy for
our low-\(\beta\) pinch conditions, \(W_{mf} \equiv W_B + W_K \approx W_B\). The Poynting flux on the right-hand side represents the injection of magnetic energy through the driving electric field on the boundary, and the resistive dissipation exactly balances this in the initial paramagnetic pinch state.

The two-fluid terms \((\Lambda_e = 1)\) couple to the electron thermal energy and the electron kinetic energy respectively. The use of a single temperature and the neglect of additional terms in the momentum equation renders these terms anomalous sources or sinks of magnetic energy [68]. However, these effects are small in our computations. We note also that the numerical divergence cleaning term can act as an anomalous loss of magnetic energy because the solenoidal and irrotational parts of the computed \(B\) are not completely independent. The separation improves with increasing spatial resolution, however [111].

### 7.2.2 Magnetic Helicity

The global magnetic helicity is well conserved relative to magnetic energy in a weakly dissipative plasma [121] and is conserved in many variational theories, for example Refs. [41, 120, 124]. We use a gauge-invariant relative magnetic helicity [129],

\[
\mathcal{K} = \int (A - A') \cdot (B + B') \, d^3x
\]

which evolves as (see Appendix C)

\[
\frac{\partial \mathcal{K}}{\partial t} = 2 \int \left\{ \nabla \cdot (A' \times E') - \eta J \cdot B + \Lambda_e \left[ B \cdot \frac{\nabla p_e}{n e} - \frac{m_e}{n e^2} \frac{\partial J}{\partial t} \cdot B \right] + \frac{\kappa}{2} A \cdot \nabla (\nabla \cdot B) \right\} d^3x.
\]

The reference magnetic field, \(B' = \nabla \times A'\), is chosen to be the vacuum field with \((A - A') \times \hat{n}|_{r=a} = 0\) and \((E - E') \times \hat{n}|_{r=a} = 0\). The reference field is steady, \(\frac{\partial}{\partial t} B' = -\nabla \times E' = 0\).

In a single-fluid MHD model, only the first two terms on the right remain, and the magnetic helicity is invariant in the ideal limit when both vanish. The first term represents injection of helicity from the boundary while the second represents resistive dissipation, and they exactly balance in the initial state in our computations.

In a two-fluid model \((\Lambda_e = 1, \kappa = 0)\), the magnetic helicity is not an ideal invariant, unless the electron mass is ignored and the electron pressure is barotropic, \(p_e = p_e(n)\) [117, 124]. The electron pressure term couples magnetic helicity to cross helicity only for models with two-fluid effects, as will be seen next. The numerical divergence cleaning term here allows unphysical changes in magnetic helicity, and it is not guaranteed to be a sink as it is for the magnetic energy.
7.2.3 Flow Invariants

Plasma flow is incorporated in variational theories by including the cross helicity, $\mathcal{X} = \int \mathbf{v} \cdot \mathbf{B} \, d^3x$, as a conserved quantity [41, 67, 124]. Including a weighting factor, $2\frac{m_e}{e}$, the weighted cross helicity ($\hat{\mathcal{X}} \equiv 2\frac{m_e}{e} \mathcal{X}$) evolves as

$$\frac{\partial \hat{\mathcal{X}}}{\partial t} = 2 \int \left\{ -\frac{m_i}{e} \mathbf{\omega} \cdot \left[ \eta \mathbf{J} + \Lambda_e \left( \frac{\mathbf{J} \times \mathbf{B}}{ne} - \nabla p_e \right) + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t} \right] \right\} \, d^3x.$$

The electron pressure parallel to the magnetic field is present regardless of the form of the generalized Ohm's law, but it couples to the magnetic helicity evolution only when two-fluid effects are included in the model.

Consider first a single-fluid MHD model ($\Lambda_e, \Lambda_i \rightarrow 0$). In the ideal ($\eta, \nu, \kappa \rightarrow 0$) limit, the cross helicity evolution only depends on the gradient of the total pressure along the magnetic field. It is only an ideal invariant for a single-fluid model when restrictions are placed on the thermodynamic evolution, such as negligible plasma pressure or a barotropic response. Note that an estimate of the resistive term here yields $\frac{\partial}{\partial t} \mathcal{X} \sim \eta \mathbf{J} \cdot \nabla \times \mathbf{v} \sim \sum_k \eta k^2 v_k B_k$ while a similar analysis [121] of the resistive term in the magnetic energy evolution results in $\frac{\partial}{\partial t} W_B \sim \eta J^2 \sim \sum_k \eta k^2 B_k$. A comparison of these terms shows that cross helicity is not conserved to a greater degree than the magnetic energy in a resistive plasma [92].

In models that include two-fluid effects ($\Lambda_e = 1$), the cross helicity is not an ideal invariant, even with restrictions on plasma pressure. Turner [124] recognized that a conserved quantity can be constructed for certain two-fluid models using the kinetic helicity, $\mathcal{H} = \int \mathbf{v} \cdot \mathbf{\omega} \, d^3x$ with $\mathbf{\omega} \equiv \nabla \times \mathbf{v}$, as part of a weighted sum. In our model equations, the weighted kinetic helicity ($\hat{\mathcal{H}} \equiv \left( \frac{m_e}{e} \right)^2 \mathcal{H}$) evolves as

$$\frac{\partial \hat{\mathcal{H}}}{\partial t} = 2 \int \left\{ \left[ \frac{\mathbf{J} \times \mathbf{B}}{ne} - \nabla p_e \right] - \frac{\nabla p_i}{ne} - \frac{\nabla \cdot \Pi_{\text{iso}}}{ne} - \Lambda_i \frac{\nabla \cdot \Pi_{\text{gyr}}}{ne} \right\} \cdot \frac{m_i}{e} \mathbf{\omega} \, d^3x. \tag{7.5}$$

The first two terms on the right hand side are equal and opposite to the first two $\Lambda_e$ terms in the cross helicity evolution, justifying the chosen weighting factors to effect a cancellation. The hybrid helicity is then constructed as a weighted sum of the magnetic, cross, and kinetic helicities: $H \equiv \mathcal{K} + 2\frac{m_e}{e} \mathcal{X} + \left( \frac{m_e}{e} \right)^2 \mathcal{H}$. Combining Eqs. (7.3)-(7.5) yields the evolution of the
The hybrid helicity is an ideal invariant in the two-fluid model considered by Turner ($\Lambda_e = 1$, $m_e = 0$, $p_i = 0$, $\Lambda_i = 0$, $\kappa = 0$). More generally, however, the hybrid helicity is not an invariant of the system of Eqs. (3.1)-(3.6), due to the FLR effects ($\Lambda_i = 1$), electron inertia ($m_e \neq 0$), and finite ion pressure ($p_i \neq 0$). The FLR effects on hybrid helicity evolution will be shown to be significant in our computations, and this provides additional motivation for the development of relaxation theories that include them [117].

### 7.2.4 Hybrid Helicity Scaling

The relative importance of each component of the hybrid helicity can be inferred from simple scaling estimates. For the magnetic helicity, the dominant contribution comes from the large axisymmetric magnetic field as opposed to the turbulent correlations, and it scales as $\mathbf{A} \cdot \mathbf{B} \sim a B_0^2$, where the minor radius $a$ is used as the characteristic length scale for $\mathbf{A}$. Flows are scaled by a characteristic flow velocity $V_0$, and the cross helicity contribution scales as $m_i e v \cdot \mathbf{B} \sim \left(\frac{m_i e B_0}{a}\right) \frac{V_0}{a} \sim (a B_0^2) (d_i/a) (V_0/V_A)$ where $V_A$ is the Alfvén velocity, $V_A \equiv B_0/\sqrt{\mu_0 m_i n}$. The fluid vorticity has no initial axisymmetric component, and it is expected to scale like $\mathbf{\omega} = \nabla \times \mathbf{v} \sim k V_0$, where $k$ represents the typical scale length of velocity fluctuations. Then the weighted kinetic helicity density is $\left(\frac{m_i e}{m_e}\right)^2 \mathbf{v} \cdot \mathbf{\omega} \sim (a B_0^2) \left(\frac{m_i e B_0}{a^2}\right) \frac{V_0^2}{a^2} (ka) \sim (a B_0^2) (d_i/a)^2 (V_0/V_A)^2 (ka)$.

Defining $\hat{\epsilon} \equiv (d_i/a) (V_0/V_A)$, the hybrid helicity can be written as

$$
\frac{H}{a B_0^2} = \int \hat{\mathbf{A}} \cdot \hat{\mathbf{B}} \, d^3 x + 2\hat{\epsilon} \int \hat{\mathbf{v}} \cdot \hat{\mathbf{B}} \, d^3 x + \hat{\epsilon}^2 (ka) \int \hat{\mathbf{v}} \cdot \hat{\mathbf{\omega}} \, d^3 x. \tag{7.7}
$$

If the characteristic flows are much less than an Alfvén speed, $V_0/V_A \ll 1$, and the ion skin depth is not large compared to a characteristic length, $d_i/a \ll 1$, then the hybrid helicity is dominated by the magnetic helicity. These conditions are typical for magnetic confinement. Also, the cross helicity contribution is an order of $\hat{\epsilon}^{-1}$ larger than the kinetic helicity contribution, assuming that $ka$ does not scale as $\hat{\epsilon}^{-1}$.
7.3 Helicity Evolution

All of our computations undergo at least one large discrete relaxation event as the nonlinear dynamics rearrange the magnetic topology and reverse the axial magnetic field at the boundary relative to the initial paramagnetic pinch state. This is evident in the evolution of the dimensionless field reversal parameter and the normalized magnetic energy, magnetic helicity, and hybrid helicity, shown for our computations in Fig. 7.1. The magnetic and hybrid helicities are well conserved relative to the magnetic energy over each relaxation event, which typically last between $200 \tau_A$ and $660 \tau_A$, a small fraction of a global resistive diffusion time for both the $S = 20,000$ and $S = 80,000$ computations. In the absence of two-fluid
effects (A), the magnetic helicity ($\mathcal{K}$) and hybrid helicity ($H$) are nearly indistinguishable. When two-fluid effects are included (B-E), the evolution of $\mathcal{K}$ and $H$ differ, a result that is discussed in greater detail in Sec. 7.4.

In all of our computations, the first event is qualitatively distinct from subsequent events as a result of the initial conditions. The magnetic energy decreases substantially over the first event in all computations, $\Delta W_B/W_B(0) \approx -1.4\%$ to $-1.9\%$, while the change in magnetic helicity is an order of magnitude smaller, $\Delta \mathcal{K}/\mathcal{K}(0) \approx 0.13 - 0.23\%$. In the single-fluid MHD computation (A), subsequent events occur roughly every $1900 \tau_A$, about a tenth of a resistive diffusion time for $S = 20,000$. The changes in magnetic energy and helicity are very similar to the first event. In our computation that includes two-fluid effects in the generalized Ohm’s law only (B), subsequent relaxation events occur more frequently, with an average interval of $1300 \tau_A$. However, the magnetic energy loss in these later events is less severe, typically between $0.5\%$ and $1.1\%$, with proportionately smaller changes in magnetic helicity of at most $0.05\%$ over the event. Our computations that include the ion gyroviscosity at $P_m = 1.0$ (C and E) show more muted magnetic activity following the first event, and the relaxation events that are observed subsequently are generally much smaller than the initial event. The ion gyroviscosity has a stabilizing effect on linear tearing modes in pinch configurations [69], and this may be responsible for the less dynamic behavior following the initial relaxation event. Consequently, we will limit our analysis of the evolution in model C and E to the first event only. Our computation that includes the ion gyroviscosity at smaller $P_m = 0.1$ (D) shows several additional large relaxation events, most notably the second event here. All of these events will be considered in the following analysis.

The recovery time scale, estimated by the time required for $W_B$ to return to its pre-crash value, is set by the injection of magnetic energy through the boundary electric field, which scales inversely with $S$. In our $S = 20,000$ computations (A, B, C, and D), the recovery time scale is only somewhat larger than the relaxation time scale, but the recovery time scale is significantly longer at larger $S$. In model E, the duration of the relaxation event is $461 \tau_A$, comparable to the $S = 20,000$ models, but the recovery process is $1582 \tau_A$, longer than in the lower Lundquist number models. The drop in magnetic energy is nearly the same as in computation C, but the changes in magnetic helicity and hybrid helicity are slightly reduced. As the Lundquist number approaches more realistic values, the difference between the relaxation time scale and the recovery time scale is expected to increase even further.

Magnetic energy is converted into kinetic energy during a relaxation event, and it is lost to resistive dissipation. This can be seen in Fig. 7.2, which shows the dominant terms in the magnetic energy evolution (Eq. (7.2)) for our computations. The electron pressure and electron inertia terms in our two-fluid computations (B-E) are negligible and are not
shown. Using $\int dt \int \mathbf{J} \cdot \nabla p_e/n_e \, d^3x$ as a gauge, where the temporal integration is over a relaxation event, we find that the neglect of separate temperatures for the electrons and ions does not substantially affect the evolution of magnetic energy. The associated $\Delta W_B/W_B(0)$ is typically $\lesssim 0.05\%$, while the total $\Delta W_B/W_B(0)$ is $1 - 2\%$. After the relaxation event, the magnetic energy recovers through the injection via the boundary electric field, which overcomes the resistive losses in the more relaxed profile. The coupling to kinetic energy has a more significant role in the recovery in subsequent relaxation events in our two-fluid computations (primarily in B and D).

The robustness of the magnetic helicity relative to the magnetic energy over a relaxation event is evident from Fig. 7.1. This property can also be seen by comparing the amplitudes in Fig. 7.3, which shows the dominant terms in the helicity evolution (Eq. (7.3)), to those in Fig. 7.2. The driving electric field injects magnetic helicity, and this balances the resistive dissipation across the plasma in the paramagnetic pinch initial conditions. However, as the plasma relaxes into a state of lower dissipation (when the red curves in Fig. 7.3 have positive values), the helicity within the plasma column increases on a resistive timescale.
two-fluid computations, the coupling to cross helicity through the parallel electron pressure gradient is small over the first event, although it becomes somewhat significant afterwards in computation B, peaking during the subsequent events. The electron inertia term is negligible and is not shown.

The magnetic energy and helicity evolution are well resolved in our computations A, B, and C. This conclusion is based on a comparison of the importance of physical terms in the evolution to the numerical divergence cleaning term. In these computations, the divergence cleaning accounts for at most 12.6% of the loss of magnetic energy over a relaxation event (which is about 1.5% of the total magnetic energy) and at most 1.5% of the roughly 0.2% net change in magnetic helicity. Applying a similar analysis to our computations D and E, we find that the magnetic helicity evolution is well resolved but the magnetic energy evolution is not. In computation D the divergence cleaning term accounts for 20.9% of the total loss of magnetic energy, $\Delta W_B/W_B(0) = -1.77\%$, and in computation E it accounts for 43.6% of the total loss of magnetic energy, $\Delta W_B/W_B(0) = -1.66\%$. However, in both of these the total change is comparable to computation C with $\Delta W_B/W_B(0) = -1.70\%$. In spite of the numerical losses, computations D and E behave similarly to our computation C with its more
easily resolved conditions.

7.4 Cross Helicity Evolution

The difference between hybrid helicity and magnetic helicity, \( H - \mathcal{K} = 2\frac{m_i}{e}\mathcal{X} + \left(\frac{m_i}{e}\right)^2 \mathcal{H} \), in our two-fluid computations is primarily due to the cross helicity. This can be seen in Fig. 7.4, which shows the evolution of the weighted cross and kinetic helicities. In all of our computations,

\[
\frac{2(m_i/e)\mathcal{X}(t)/\mathcal{K}(0)}{\left(\frac{m_i}{e}\right)^2 \mathcal{H}(t)/\mathcal{K}(0)}
\]

the weighted kinetic helicity is small, and this reflects the orderings of Eq. (7.7). Flows in these computations are sub-Alfvénic, \(|v|/V_A \leq 0.02\), and the normalized ion skin depth is \(d_i/a \approx 0.17\). Using the wavenumber for the dominant linear mode, \(ka = a(m/r_s + k_z) \approx 10\),

Figure 7.4: Normalized cross helicity \(2\frac{m_i}{e}\mathcal{X}(t)/\mathcal{K}(0)\), and kinetic helicity \(\left(\frac{m_i}{e}\right)^2 \mathcal{H}(t)/\mathcal{K}(0)\) for (a) model A, (b) model B, (c) model C, (d) model D, and (e) model E. The relaxation events are highlighted.
we obtain an estimate for the importance of each component: $\frac{m_i e}{c} \mathcal{X}/\mathcal{K} \approx 3.4 \cdot 10^{-3}$ and $(\frac{m_i}{e})^2 \mathcal{H}/\mathcal{K} \approx 1.2 \cdot 10^{-4}$. Consequently, the evolution of $\mathcal{H}$ is not considered here.

In model A, the weighted cross helicity is also small relative to the dominant magnetic helicity ($|\hat{\mathcal{X}}|/\mathcal{K}(0) \lesssim 0.06\%$), and the hybrid helicity remains approximately equal to the magnetic helicity throughout the evolution. Computations that include two-fluid effects display changes in cross helicity that are roughly an order of magnitude greater than in the single-fluid MHD computation. Comparing model B ($\Lambda_i = 0$) and model C ($\Lambda_i = 1$), it is clear that the ion gyroviscosity has a substantial effect on cross helicity evolution, and this change is largely independent of resistivity between $S = 20,000$ and $S = 80,000$ (models C and E). As mentioned in Chapter 3, the initial value of the magnetic helicity, $\mathcal{K}(0)$, is negative for models A-C and positive for models D and E; thus, the sign of $2\frac{m_i}{e} \mathcal{X}(t)/\mathcal{K}(0)$ is opposite to the cross helicity $\mathcal{X}(t)$ in models A-C. Although $\mathcal{X}$ has the opposite sign in model C from model D and E, the magnitude of the change at the first event is similar; $|\hat{\mathcal{X}}|$ increases to approximately 0.5% of $|\mathcal{K}|$. The cross helicity in model D remains at much larger amplitude than in any of the other two-fluid computations, a result which is attributed to the lower $P_m = 0.1$.

There is little apparent correlation between the changes in the cross helicity, which is a global measure of the parallel flow, and the opposition of the dynamo drives in the plasma core. Recall that strong opposition is observed in the fourth event in model B, and in all of the subsequent events in models C and D. There are no major distinguishing features in the cross helicity evolution in these events from the other events in those computations. In particular, the change in cross helicity in the second event in model D is in the same direction as in the first event despite the strong Hall dynamo opposition in that relaxation event.

Cross helicity is conserved well in our computations without two-fluid effects, even without thermodynamic restrictions on pressure. The small changes in cross helicity in our single-fluid MHD computation (model A) are primarily due to the viscosity, as seen in Fig. 7.5, which shows the dominant terms in the cross helicity evolution, Eq. (7.4), for our computations. Here, the viscosity becomes most significant at the relaxation events, although it changes sign about halfway through each event leading to little net change.

The two-fluid Hall term substantially affects cross helicity evolution in model B. During the first and second event, the viscosity and Hall effects generally reinforce each other, but the numerical divergence cleaning is also large in these events. Consequently, we can attach little physical significance to the cross helicity evolution here. However, in later events, which are better resolved, the two-fluid Hall term tends to drive the changes in cross helicity, and it is opposed by the viscosity. In addition, the pressure gradient along the magnetic field, which only couples cross helicity and magnetic helicity evolution for two-fluid models ($\Lambda_e = 1$), is
also moderately significant in these subsequent events in model B.

The ion gyroviscosity has a considerable effect on cross helicity evolution (models C, D, and E). The gyroviscous and viscous contributions both increase cross helicity in the direction of the parallel current during the relaxation, while the Hall term varies between reinforcement and opposition. The viscous effects are only slightly smaller in the first event in model D than in model C, despite the much smaller isotropic viscosity coefficient in those computations. Shortly after the first relaxation event in these computations, a balance is reached between the gyroviscosity and the remaining terms, and the cross helicity settles into a non-zero steady state. The divergence cleaning effects on cross helicity evolution in model D and E are comparable to the physical terms. However, the dynamics appear qualitatively consistent with the evolution in model C, which is well resolved. The ion gyroviscosity is largely responsible for changes in the cross helicity at the relaxation events and clearly breaks the invariance of the hybrid helicity when it is included in the model.
7.5 Profile Relaxation

7.5.1 Current Relaxation

Since the magnetic helicity is well conserved relative to magnetic energy over a timescale that is short compared to the global resistive diffusion time, it is appropriate to compare to the predictions of relaxation theories discussed in Sec. 7.1. Single-fluid theories [41, 120] and two-fluid extensions [56, 117, 124] predict a completely flat parallel current profile, \( \lambda = \mu_0 J \cdot B/|B|^2 = \text{constant} \), to lowest order in the species’ skin-depths, \( d_{i,e}/a \). In our computations, the normalized value of the magnetic helicity, \( |\hat{K}| = (a/R) (|K|/\psi_{tor}^2) = 5.3293 \), lies within the range \( 0 \leq |\hat{K}| \leq |\hat{K}|_{\text{crit}} \approx 8.21 \), where \( |\hat{K}|_{\text{crit}} \) is the value for which the minimum energy state ceases to be the axisymmetric one [96]. As a result, the predicted relaxed state for our computations is axisymmetric with the dimensionless parallel current density \( |\lambda a| = 2.916 \).

The parallel current gradient is reduced during relaxation, and the profile is driven closer to the fully relaxed state. The relaxation over the first event can be seen in Fig. 7.6; the parallel current profile at the end of subsequent events is similar. In the initial paramagnetic pinch state, the parallel current is strongly peaked and is far from the predicted flat profile. Relaxation reduces parallel current in the core of the plasma, where \( |\lambda a| \) exceeds the predicted relaxed state value, and increases it where \( |\lambda a| \) is below it. However, the fully relaxed current state is never achieved. A large current gradient persists in the edge region after relaxation, similar to experiment [14, 122]. The boundary electric field drive and resistive diffusion slowly rebuild the peaked profile between events, and they drive the plasma further from the fully relaxed state and closer to the paramagnetic pinch Ohmic steady-state. Plasma instabilities may then initiate another nonlinear relaxation event, and the whole process may repeat quasi-periodically.

![Figure 7.6: Dimensionless parallel current for the initial state, and at the end of the first relaxation event for models A, B, C, D, and E. The flat parallel current Taylor state prediction is also shown.](image-url)
7.5.2 Flow Relaxation

Single-fluid MHD theories that include the cross helicity as a conserved quantity [41] predict both flat parallel current and momentum density that is aligned with $B$, $m_i n v = \lambda_1 B$, where $\lambda_1$ is a global constant. Two-fluid theories that conserve the hybrid helicity [124], although mathematically ill-posed [128], predict relaxed states with more complicated flow structures. To lowest order in $d_{i,e}/a$, however, the flow is predicted to be parallel to the magnetic field [56].

The computations presented here begin with no parallel flow across the plasma, but relaxation gives rise to parallel flow with significant radial structure, as seen in Fig. 7.7. The
density fluctuations are small in our computations, and the structure of $n V_\parallel$ is very similar to that of $V_\parallel$. As we will see in Sec. 7.5.3, the energy in the perpendicular part of the flow exceeds that of the parallel component. Clearly, these predictions for flow are not realized following relaxation events in our computations.

The cross helicity is most robustly conserved in model A, and here the generated parallel flow varies between negative and positive across the plasma. However, the magnitude of the parallel flow remains small, $|V_\parallel|/V_A \lesssim 0.5\%$, in agreement with results reported in King et al. [70]. In our two-fluid computation without ion gyroviscosity (B), the cross helicity changes very little over the first event, and the magnitude of the parallel flow is comparable to model A. However, the physical terms in cross helicity evolution are offset by numerical effects over the first event, and the significance of the parallel flow profile here is questionable. Subsequent relaxation events in model B are better resolved, and the magnitude of $\Delta V_\parallel/V_A$ over these events can be comparable to models C and D. This can be seen in Fig. 7.8(a), which shows the parallel flow profile at the beginning and end of the subsequent relaxation events in model B. The parallel flow continues to exhibit significant radial variation following the subsequent relaxation events and does not approach a uniform flow state. However, in
these subsequent events, $\Delta V_\parallel/V_A > 0$ in the core, which contrasts with the behavior in the first event. Similar behavior is observed in the subsequent events in model D, as shown in Fig. 7.8(b). The parallel flow does not flatten after a relaxation event, although the $\Delta V_\parallel$ over the event often differs from the initial relaxation. This is especially evident in the core in Event 2.

Based on empirical observations from our computational results in model B, changes in the local parallel flow profile do not appear to be well correlated with the changes in cross helicity. In the third relaxation event the cross helicity roughly doubles in magnitude, $\Delta \hat{X}/K(0) \sim -0.134\%$, but the changes in the parallel flow profile are small. The change in cross helicity is much smaller over the fourth event, $\Delta \hat{X}/K(0) \sim -0.001\%$, but the parallel flow profile changes are much larger than in the third event.

In contrast to this, the changes in parallel flow and cross helicity appear well correlated in our computations that include both two-fluid effects and ion gyroviscosity (C and D). Both computations display large changes in cross helicity at the relaxation event, and the parallel flow profile after relaxation is strongly peaked in these computations. However, the direction of the generated parallel flow depends upon the initial parallel current density, which differs in models C and D, and this provides evidence that changes in parallel flow are intrinsically related to current relaxation.

### 7.5.3 Kinetic Energy Evolution

Our computations display a strong anisotropy in the axisymmetric flow parallel to the axisymmetric magnetic field, $\langle B \rangle \equiv \frac{1}{2\pi L} \int_0^{2\pi} d\theta \int_0^L dz B$, and perpendicular to it, where $\hat{e}_\perp \equiv \hat{e}_\parallel \times \hat{e}_r$. This can be seen in Fig. 7.9. In the single-fluid MHD computation (A), the parallel kinetic energy is much larger than the perpendicular kinetic energy during and after the first event, but equipartition is observed later. In addition, the kinetic energy remains much
smaller than in our two-fluid computations.

In our computations with two-fluid effects (B-E), the perpendicular energy rises to many times greater than the parallel energy during a relaxation event, and this persists after the event. This large imbalance is a direct result of the two-fluid effects in the generalized Ohm’s law, as it is evident in models with and without the ion gyroviscosity. As pointed out in Numata et al. [90] and King et al. [69], two-fluid effects are important in the generation of flows perpendicular to the magnetic field. The computation at smaller $P_m = 0.1$ (model D) shows the largest $W_{K,\parallel}$, but this remains significantly smaller than the perpendicular components.

### 7.6 Discussion

Our computations at low-$\beta$ pinch parameters with $\dot{\epsilon} \ll 1$ demonstrate that the magnetic helicity is well conserved relative to magnetic energy during relaxation, in agreement with
both theoretical predictions and experimental measurements on MST. This is true for single-fluid MHD and for two-fluid models with and without the ion gyroviscosity, even though the magnetic helicity alone is not an ideal invariant in our two-fluid model. The relaxation dynamics act to flatten the parallel current, and the plasma evolves towards the predicted relaxed state, although the relaxation remains incomplete.

The hybrid helicity is also well conserved relative to magnetic energy, because the hybrid helicity is dominated by the magnetic helicity at these pinch parameters relevant to magnetic confinement. In two-fluid models, differences between hybrid helicity and magnetic helicity arise at next order in \( \hat{\epsilon} \), primarily from changes to the cross helicity. Two-fluid computations without ion gyroviscosity demonstrate that the two-fluid Hall term has a substantial impact on cross helicity evolution, and it tends to offset viscous effects in the numerical evolution. When ion FLR effects are included, the respective contribution to cross helicity evolution has a substantial effect in our computations. The gyroviscous contribution increases or decreases the cross helicity depending on the direction of the parallel current, and the effect is largely independent of the plasma resistivity.

Changes in parallel plasma flow are small in our single-fluid MHD computation, and there is equipartition between the parallel and perpendicular components of kinetic energy, both of which remain small relative to our two-fluid computations. Substantial changes in plasma flow are observed in our two-fluid computations, and the relaxation results in a complex flow structure. The kinetic energy in the component of flow perpendicular to the magnetic field is several times larger than the energy in the flow along the field in our computations. The magnitude of parallel flow in our two-fluid computations is comparable to experimental observations, and the magnetic energy remains much larger than the kinetic energy. Free energy from gradients in plasma flow is insignificant compared to that available from gradients in the parallel current density, and the current relaxation dynamics are much more important than the flow relaxation dynamics in magnetically confined systems.
Part III

Linear Computations
Our nonlinear computations represent a significant step forward in understanding the
detailed dynamics of the RFP sawtooth cycle. However, there is always room for improvement,
and we now discuss several linear studies that we performed in an effort to improve our
understanding of the deficiencies of our model.

The first of these deficiencies concerns the geometry of the configuration; our computations
use a periodic cylinder geometry, but MST is toroidal. Recent measurements on MST have
emphasized that there are significant deviations from the predictions of cylindrical models,
and in Chapter 8 we examine the effects of this toroidicity on linear tearing modes in the
zero-$\beta$ single-fluid MHD system. We find that the effects of toroidicity can account for the
discrepancy between the experimental measurements and the cylindrical model predictions.

A second deficiency is related to the pressure profile. Our nonlinear computations here
have uniform equilibrium pressure in order to eliminate the effects of interchange drive
and retain only the current-driven dynamics. However, as we mentioned previously, the
gyroviscous response scales as $p_i$, and it is expected to be small near the plasma edge with
a realistic pressure profile. Because the dominant nonlinear coupling in the RFP occurs
through both core-resonant and edge-resonant modes, a modification of the gyroviscous
response for the edge-resonant modes may have important effects on the relaxation dynamics.
Pressure gradients result in diamagnetic flows, and our initial investigation into two-fluid
current-driven tearing modes in the presence of these flows revealed an additional instability.
We identify this as a resistive drift mode in Chapter 9, and we derive a simple analytic
dispersion relation that accurately captures the key features of the mode.
The straight periodic cylinder is often a good approximation for the RFP because the curvature is dominated by the poloidal component of the magnetic field, rather than the toroidal component, and the nonlinear coupling among different tearing modes tends to be significantly stronger than the poloidal coupling [114]. However, when making direct comparisons with magnetic probe signals on MST, there are small deviations from the values predicted by these cylindrical models. These measurements are made using a toroidal array of 64 magnetic probes spaced uniformly in toroidal angle, and each probe is located at the same poloidal angle \( \theta = -119^\circ \). The array is capable of measuring the amplitudes of the poloidal and toroidal perturbations, as well as the phase difference between them for the dominant \( n \) harmonics. The difference in phase between these components differs by \( 24 - 32^\circ \) from the cylindrical prediction, depending on the toroidal harmonic.

We use NIMROD to investigate the effects of the toroidal geometry on linear tearing modes within the zero-\( \beta \) resistive MHD model. Two different equilibrium profiles are considered here. The first profile is unstable to core-resonant, dominantly \( m = 1 \), tearing modes, while the second is unstable to edge-resonant, dominantly \( m = 0 \), modes. Our toroidal computations of the core-resonant instabilities show a phase difference between the poloidal and toroidal components of the linear eigenmode that is roughly \( 22 - 25^\circ \), and this agrees well with the experimental measurements. The toroidal computations of the edge-resonant instabilities do not agree well with the observations on MST, however.

### 8.1 Introduction

With some exceptions, the dominant instabilities in RFPs are tearing modes, resistive instabilities that bend and reconnect magnetic field-lines. Theoretical analysis finds that toroidal effects have a significant impact on linear tearing mode behavior in tokamak configurations [20, 26, 61] and in the RFP [131]. Our computations here provide additional evidence that these effects are important for MST-relevant configurations.

Tearing modes reconnect magnetic field-lines near resonant surfaces where the stabilizing line-bending response is minimized. These surfaces may be identified in a general toroidal coordinate system \((\rho, \theta, \zeta)\) as the locations where the parallel derivative acting on the fluctuations, \( B \cdot \nabla \tilde{f} \), is minimized. In these general toroidal coordinates, \( \rho \) is a flux-surface label, analogous to the minor radial coordinate in a cylindrical system, and the angle coordinates \( \theta \) and \( \zeta \) cover the two directions tangential to the surface and increase by \( 2\pi \) going around once in their respective directions. We decompose perturbations in terms
of poloidal and toroidal Fourier harmonics, \( \tilde{f} = \sum_{m,n} \tilde{f}_{m,n} (r) e^{im\theta + in\zeta} \). Then the parallel derivative is given by

\[
\mathbf{B} \cdot \nabla \tilde{f} = \sum_{m,n} \text{i} n \mathbf{B} \cdot \nabla \theta \left[ \frac{m}{n} + \hat{q} \right] \tilde{f}_{m,n} e^{im\theta + in\zeta},
\]

(8.1)

where \( \hat{q} \equiv \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} \), which need not be constant on a flux surface.

In an axisymmetric cylinder, the angle coordinates are ignorable and \( \mathbf{B} \cdot \nabla \theta \) and \( \hat{q} \) depend only on the radial coordinate, \( \rho \). Consequently, Eq. (8.1) is uncoupled for every \( m \) and \( n \); linear modes of the system are composed of a single \( m \) and a single \( n \) harmonic. The parallel derivative for a given \( m, n \) mode vanishes at the surface where \( \hat{q} (\rho) = -\frac{m}{n} \).

Toroidal effects appear in Eq. (8.1) in two ways. First, there is geometric coupling of different \( m \) harmonics through \( \nabla \zeta \sim \hat{e}_\zeta / R = \hat{e}_\zeta / [R_0 + r \cos \theta] \). A second coupling comes from the poloidal variation of the magnetic field. In toroidal geometry, \( \mathbf{B} \) depends explicitly on the poloidal angle, which is no longer an ignorable coordinate. Both of these effects couple different \( m \) harmonics together in Eq. (8.1), and each linear eigenmode has a broad poloidal spectrum. Eigenmodes in toroidal geometry may still be labeled by an \( m, n \) pair, where \( m \) is the dominant component of the poloidal spectrum.

### 8.1.1 Straight-Field Coordinates

In general, magnetic field-lines travel both toroidally and poloidally as they wind around the torus. In a toroidal increment \( d\zeta \), there is a poloidal increment, \( d\theta \), related via \( d\zeta = \hat{q} d\theta \). When \( \hat{q} = \hat{q} (\rho, \theta) \), the same fixed toroidal increment results in a different poloidal increment at different poloidal locations. However, this variation may be eliminated with a suitable coordinate transformation to what are called straight-field coordinates [34]. These coordinates, \( (\theta_f, \zeta_f) \), are defined such that \( \hat{q} = d\zeta_f / d\theta_f \) is constant over a flux surface.

Straight-field angles are useful from an analytic perspective as they map some of the poloidal variation of the magnetic field into the coordinate system itself. However, the poloidal dependence in the magnetic field is intrinsic to the toroidal topology and cannot be transformed away completely. Here, we use straight-field angles to help highlight the natural contours of the equilibrium magnetic field in toroidal geometry. The equilibria that we consider here are axisymmetric, so it is sensible to use \( \zeta_f = \phi \), where \( \phi \) is the usual geometric toroidal angle. The poloidal angle requires some additional consideration. In what follows, we will use \( \theta_g \) to refer to the poloidal angle measured geometrically with respect to the geometric axis of the device, while \( \theta_f \) is the straight-field angle. In cylindrical systems \( \theta_f = \theta_g \), while the construction of \( \theta_f \) from \( \theta_g \) in toroidal systems is described in Appendix G.
The remainder of this chapter is organized as follows. In Sec. 8.2 we introduce the model equations and the two equilibrium configurations used in this work. An analysis of the tearing stability factor, \( \Delta' \), is given for each configuration along with their straight-field angle transformations. The results of our numerical computations for the two configurations are presented in Sec. 8.3, and we make direct comparisons to experimental measurements. We close with a short summary of the key findings and possible extensions of the model.

### 8.2 Model System

#### 8.2.1 Equations

The NIMROD code\^[111, 112\] is used to solve the zero-\( \beta \) single-fluid resistive MHD equations,

\[
    m_i n_0 \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla \cdot (m_i n_0 \nu \mathbf{W}) \tag{8.2}
\]

\[
    \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left[ -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} \right], \tag{8.3}
\]

where \( \mathbf{W} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3} \mathbf{I} (\nabla \cdot \mathbf{v}) \) represents collisional viscosity, and the low-frequency Ampere’s law, \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \), is used, as is appropriate for MHD modes. The viscosity, \( \nu \), and resistivity, \( \eta \), are global constants here (the shaped profile is not used), as are the ion mass, \( m_i \), and the background density, \( n_0 \). The boundary is a perfect conductor, so \( \frac{\partial}{\partial t} \hat{n} \cdot \mathbf{B} \bigg|_{r=a} = 0 \) and \( \hat{n} \cdot \mathbf{B} \bigg|_{r=a,t=0} = 0 \). The boundary is impermeable and no-slip conditions are applied on tangential components of the flow, \( \mathbf{v} \bigg|_{r=a} = 0 \).

The simple resistive MHD system (Eqs. (8.2)-(8.3)) is characterized by two numbers; (1) the Lundquist number, which is the ratio of the Alfvén time to the resistive diffusion time, \( S = \tau_R/\tau_A \), where \( \tau_A = a a \sqrt{\mu_0 m_i n_0 / |\mathbf{B}|} \) and \( \tau_R = \mu_0 a^2 / \eta \); and (2) the magnetic Prandtl number, the ratio of resistive to viscous diffusion times, \( P_m = \tau_R / \tau_\nu \), where \( \tau_\nu = a^2 / \nu \). The computations presented here have \( S = 10^6 \) and \( P_m = 0.1 \).

The aspect ratio is chosen to roughly match the MST device with \( R_0 / a = 1.5/0.5 = 3 \). Each computation is run in two different geometries, a periodic cylinder with axial length \( L = 2\pi R_0 \) and a torus with major radius \( R_0 \). The spatial discretization in the NIMROD code uses a spectral-element expansion \^[111\] for two directions with a Fourier representation in the third periodic direction. Here, the spectral elements are used in the \( r - \theta \) plane to span a circular-polar grid, and the axial (toroidal) direction is represented by a finite Fourier series. Linear modes are properly decomposed by axial (toroidal) harmonic so computations are performed separately for each individual \( n \) mode.
The results presented here use a uniform mesh in the $r - \theta$ plane with 180 radial and 64 poloidal finite elements with basis functions of polynomial degree 4 within each element. This resolution is much larger than is typically necessary for a linear calculation as we do not align the computational grid with the flux surfaces, and we do not pack the grid in the domain. Convergence was tested for these computations using the same total number of finite elements, but with mesh packing around the location of the rational surfaces for each of these modes. There was no observable change in the mode structure, and the growth rate was unchanged. In addition, the mode growth rate is insensitive to changes in the divergence cleaning coefficient when scanned over 7 orders of magnitude.

8.2.2 Equilibrium Model

Axisymmetric toroidal equilibria are solutions to the Grad-Shafranov equation [48, 108]

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}$$

(8.4)

where $\psi$ is the poloidal flux, $p$ is the pressure, $F = RB_\phi$, and the differential operator is $\Delta^* = R \frac{\partial}{\partial R} \left( R^{-1} \frac{\partial}{\partial R} \right) + \frac{\partial^2}{\partial Z^2}$. These equilibria are characterized by two free profiles as functions of the poloidal flux, $\psi$. Common choices are the normalized parallel current density, $\lambda(\psi) = \mu_0 J \cdot B/|B|^2$, and the pressure, $p(\psi)$. The parallel current is related to $F$ and $p$ via $\lambda = -\frac{dF}{d\psi} - \mu_0 \frac{dp}{d\psi} F/B^2$. Our computations here neglect the plasma pressure, $p(\psi) = 0$, so the force-free relation, $dF/d\psi = -\lambda(\psi)$, holds.

We consider two equilibria, described below, and both are implemented in NIMEQ [60], the Grad-Shafranov solver built using the NIMROD framework. The first equilibrium is a non-reversed state with core-resonant (dominantly $|m| = 1$) modes being the most unstable. The second equilibrium has reversed toroidal field at the boundary, with nearly flat parallel current density in the core and a sharp gradient at the edge. The instabilities here are dominantly $m = 0$ and are resonant at the reversal surface.

8.2.2.1 Alpha Model

The alpha model for parallel current density is widely used in studies of the RFP [2]. A similar model is used here, where

$$\lambda(\hat{\psi}) = \lambda_0 \left( 1 - \hat{\psi}^{\alpha/2} \right),$$

(8.5)

with the normalized poloidal flux given by $\hat{\psi}(r, \theta) \equiv (\psi(r, \theta) - \psi_0)(\psi_W - \psi_0)$. The parameter $\lambda_0$ determines the on-axis parallel current density and $\alpha$ determines the flatness of the profile.
The function $F$ is then

$$F\left(\hat{\psi}\right) = F\left(\psi_0\right) - \lambda_0 \left(\frac{d\psi}{d\hat{\psi}}\right) \left[\hat{\psi} - \frac{\hat{\psi}^{1+\alpha/2}}{1 + \alpha/2}\right].$$  \hspace{1cm} (8.6)

We model a plasma near field reversal with $a\lambda_0 = -3.5$ and $\alpha = 4.1$. The reversal parameter in cylindrical geometry is $F = 0.052$, and the pinch parameter is $\Theta = -1.455$. In toroidal geometry, $F = 0.100$ and $\Theta = -1.420$.

The safety factor profile has many low order rational surfaces in the core of the plasma, as can be seen in the top plot of Fig. 8.1(a). The parallel current density gradient is large here, as can be seen in the bottom plot of Fig. 8.1(a). Note that the profiles are largely similar between cylindrical and toroidal geometry, with some modest deviation at the plasma edge.

The cylindrical geometry equilibrium was imported into RESTER [113], a numerical tool for calculating stability of cylindrical force-free profiles. This allows a calculation of the tearing stability parameter, $\Delta^t$, for the modes of interest. The dominant linear mode is the $m = 1, n = 6$ mode, although modes with $m = 1$ and $6 \leq n \leq 14$ are predicted to be resistively unstable for this profile, as can be seen in Fig. 8.1(b). Recall for the simple Harris sheet equilibrium in slab geometry, the expected behavior is $\Delta^t a = 2 (1/ka - ka)$ [9]. For the alpha model equilibrium, $\Delta^t$ falls off roughly like $1/n$, mirroring the predictions from the simple Harris sheet model. Finally, note that the reversal surface is not present in the plasma, so $m = 0$ modes are stable, as are all higher $m$ modes.

The flux surfaces for the toroidal equilibrium are nearly circular, as can be seen in Fig. 8.2. However, they are no longer concentric with the geometric axis as they are in the cylindrical case. The shift of the magnetic axis from the geometric axis is small but non-negligible: $\Delta R/a \approx 0.084$. The straight-field angle, $\theta_f$, differs appreciably from the geometric poloidal angle, $\theta_g$, and this increases towards the plasma boundary, as can be seen in Fig. 8.2(b).

Figure 8.1: (a) Safety factor and parallel current density for alpha model equilibrium. (b) Tearing mode stability parameter, $\Delta^t$, for alpha model equilibrium in cylindrical geometry.
8.2.2.2 \textit{tanh} Model

While reversed equilibria can be constructed with the alpha model, the tearing stability parameter, $\Delta'$, for modes resonant at the reversal surface remains small [2]. Consequently, unstable tearing modes resonant at $q = 0$ are not observed in numerical computations with the alpha model equilibrium. To determine the phase dependence of modes resonant at the reversal surface, we construct a hyperbolic tangent profile for the parallel current density

$$\lambda (\hat{\psi}) = \frac{\lambda_0}{2} \left[ 1 + \tanh \left( \frac{\psi_m - \hat{\psi}}{\delta_{\psi}} \right) \right].$$

This allows more direct control of the location of the largest gradient relative to a given surface. The parameter $\psi_m$ sets the location of the maximum current gradient, while the parameter $\delta_{\psi}$ determines the gradient scale length. For this model, the function $F$ is

$$F (\hat{\psi}) = F (\psi_0) - \frac{\lambda_0}{2} \left( \frac{d\psi}{d\hat{\psi}} \right) \left\{ \hat{\psi} - \delta_{\psi} \log \left[ \frac{\cosh \left( (\psi_m - \hat{\psi}) / \delta_{\psi} \right)}{\cosh (\psi_m / \delta_{\psi})} \right] \right\}.$$  

For both cylindrical and toroidal equilibria, the reversal surface is nearly coincident with the location of the largest gradient in parallel current density, as can be seen in Fig. 8.3(a). The parameters here are $a\lambda_0 = -3.1$, $\psi_m = 0.8$, and $\delta_{\psi} = 0.1$. In cylindrical geometry, the
reversal parameter is \( F = -0.208 \), and the pinch parameter is \( \Theta = -1.617 \). In toroidal geometry, \( F = -0.164 \) and \( \Theta = -1.598 \).

The \textit{tanh} equilibrium in our cylindrical geometry computation was also run through \textsc{RESTER} to calculate the stability for this profile. Modes with \( m = 0 \) and \( 1 \leq n \leq 31 \) are found to be resistively unstable. In marked contrast to the alpha model profile, the low \( n \) modes here all have nearly the same value of \( \Delta' \), as can be seen in Fig. 8.3(b). The stability factor decreases nearly monotonically with \( n \), instead of the expected \( 1/n \) behavior from the simple Harris sheet model. The relatively flat parallel current in the core of the \textit{tanh} equilibrium results in all \(|m|=1\) modes in cylindrical geometry being predicted to be stable.

The flux surfaces remain nearly circular for the \textit{tanh} profile in toroidal geometry, as can be seen in Fig. 8.4(a). The magnetic axis is shifted from the geometric axis by a comparable amount as in the alpha profile: \( \Delta R/a \approx 0.078 \). The lower current density on axis in the \textit{tanh} profile results in the core-resonant rational surfaces being displaced outwards relative to their location in the alpha profile. The reversal surface has considerable variation of straight-field angle, \( \theta_f \), with respect to geometric angle, \( \theta_g \), as can be seen in Fig. 8.4(b).

### 8.3 Results

#### 8.3.1 Core Resonant Tearing in the Alpha Model

The alpha model equilibrium is unstable to a range of \( n \) modes, with the dominant linear mode having \( n = 6 \). The growth rates of modes with \( n = 6 - 12 \) in the cylindrical and toroidal geometries are shown in Fig. 8.5. Toroidal effects enhance the growth rate of these modes, with the most pronounced increase in \( \gamma \tau_A \) occurring for the \( n = 6 \) mode. The stability parameter is quite large for \( n = 6 \), \( \Delta' a \approx 36 \), and the eigenmode appears close to the transition to resistive kink behavior. The constant-\( \psi \) approximation for tearing modes
Figure 8.4: tanh model equilibrium in toroidal geometry. (a) Lines of constant $\rho$ (rational surfaces in color) and lines of constant $\theta_f$ (dashed) and $\theta_g$ (solid), evenly spaced by $\pi/6$. (b) Difference $\theta_f - \theta_g$ at select surfaces.

Figure 8.5: (a) Tearing mode growth rates for the alpha model equilibrium in cylindrical and toroidal geometry. (b) The value of $\Delta'\delta$, where the layer width, $\delta$, is estimated from the eigenfunction and $\Delta'$ is taken from the cylindrical geometry equilibrium.
Figure 8.6: Magnetic perturbations for \( n = 6 - 12 \) eigenmodes at \( r = a \) in the alpha model equilibrium. (a) Ratio of \( |\tilde{B}_\phi| \) to \( |\tilde{B}_\theta| \) in toroidal (solid) and cylindrical (dashed) geometry. (b) Phase difference \( \delta_\phi - \delta_\theta \) in toroidal geometry. The poloidal location of the magnetic coil array is indicated in both figures.

is approaching the limits of validity for the \( n = 6 \) eigenmode in both our cylindrical and toroidal geometry computations since \( \Delta'\delta \sim O(1) \),[9] as shown in Fig. 8.5(b). For the higher \( n \) modes, however, the normalized layer width is considerably smaller and the asymptotic tearing behavior is expected here.

We now turn our attention to the structure of the eigenmodes. For direct comparison to experimental measurements, we will examine the toroidal (\( \tilde{B} \cdot \hat{e}_\phi \)) and poloidal (\( \tilde{B} \cdot \hat{e}_\theta \)) components with respect to the geometric angles, not the straight-field angles. Here, and in the remainder of this chapter, the angle \( \theta \) will be used interchangeably with the geometric angle, \( \theta_g \).

Assuming that \( \hat{n} \cdot \vec{J} \big|_{r=a} = 0 \) restricts the relative phase and amplitude of the poloidal and toroidal components of the linear eigenmode. In cylindrical geometry, it can be shown (see Appendix H) that the relationship between the amplitudes and the phases is

\[
\frac{|\tilde{B}_\phi|}{|\tilde{B}_\theta|} = \frac{n}{m} \frac{a}{R_0} \cos (\delta_\phi - \delta_\theta) \cos \left( \delta_\phi - \delta_\theta \right) = 0, \tag{8.9}
\]

where \( \delta_{(\theta,\phi)} = \arctan \left( \frac{\text{Im}[B_{(\theta,\phi)}]}{\text{Re}[B_{(\theta,\phi)}]} \right) \) is the mode phase. In other words, the toroidal and poloidal field perturbations are expected to be exactly in phase. NIMROD does not enforce \( \hat{n} \cdot \vec{J} \big|_{r=a} = 0 \), but Eqs. (8.9) and (8.10) are well satisfied in our cylindrical computations, as can be seen in Fig. 8.6(a), noting that \( (n/m) (a/R_0) \cos (\delta_\phi - \delta_\theta) = n/3 \) for our aspect ratio and \( m = 1 \) modes.

The relative mode phase between the poloidal and toroidal magnetic field components, \( \delta_\phi - \delta_\theta \), at \( r = a \) differs by at most \( 25^\circ \) on the boundary, seen in Fig. 8.6(b). There is a weak
dependence on toroidal mode number for the measured phase difference, with increasing \( n \) having a larger phase difference between components. Due to the toroidal symmetry, we may freely translate in \( \phi \) and only the phase of the perturbed fields will be changed. We may fix the phase by specifying that one of the components, say \( \tilde{B}_\phi \), is purely real at a given toroidal and poloidal angle. The phase difference between fields, however, is independent of this choice of phase.

### 8.3.1.1 Experimental Comparison

Experimental measurements of the amplitudes and relative phases of the magnetic perturbations decomposed by toroidal harmonic, \( n \), are taken in the window between 2 ms and 1 ms before a sawtooth relaxation event. Here, the mode amplitudes remain relatively low, and the nonlinear coupling among the individual tearing modes is believed to be small. The results of these measurements are shown in Fig. 8.7. The original analysis used to construct this data is not available, so we may only use these plots to infer the actual measurements for the individual modes. We will compare our computational results to estimates of the mode amplitude and phase taken from the band-pass filtered results. Note as well that there is a 180° offset in the measured phase difference in the experimental results and in our computations. This is a result of differences in the directions used in MST and NIMROD.
Figure 8.8: Magnetic perturbations for $n = 6 - 12$ eigenmodes at the toroidal array ($r = a$, $\theta_g = -119^\circ$) in the alpha model equilibrium. (a) Ratio of $|\tilde{B}_\phi|$ to $|\tilde{B}_\theta|$, and (b) Phase difference $\delta_\phi - \delta_\theta$ (experimental results shifted by +180°).

(NIMROD’s $\phi$ direction is opposite to MST’s $\phi$ direction), and it is easily corrected by including a +180° shift in our inferences of the experimental measurements.

The ratio of $|\tilde{B}_\phi|$ to $|\tilde{B}_\theta|$ at the location of the toroidal array increases linearly with $n$ in our computations, although it remains slightly smaller than the experimental measurements on MST, as can be seen in Fig. 8.8(a). The phase difference at the location of the toroidal array in our toroidal geometry computations also increases as the toroidal wavenumber $n$ increases, but it remains smaller than the experimental measurements, especially at higher $n$, shown in Fig. 8.8(b). It should be noted that our computations here are strictly linear, and we may not rule out the possibility that nonlinear phenomena are affecting the experimental measurements. Fully nonlinear toroidal computations are a logical next step for comparison to experiment.

8.3.1.2 $n = 6$ Eigenmode Structure

From Fig. (8.6), it is immediately clear that the structure of the magnetic perturbations at the boundary does not vary significantly for $n = 6 - 12$. As a result, we now turn our attention to the detailed structure of the $n = 6$ tearing mode. We first note that the poloidal symmetry present in the cylindrical geometry computation (see Fig. 8.9) is not present in the comparable results for the toroidal computation, shown in Fig. 8.10. In toroidal geometry, the magnetic field perturbation is shifted slightly inwards towards the major axis of the configuration, similar to the shift in straight-field angle, $\theta_f$.

This shift is also evident when examining the eigenmode magnetic field components at the boundary ($r = a$). As before, we project the eigenmode onto the geometric poloidal and toroidal angles, $\mathbf{B} \cdot \hat{e}_{\theta_g}$ and $\mathbf{B} \cdot \hat{e}_\phi$, for more direct comparison to experimental measurements. In the toroidal computation, there are small deviations between the outboard ($\theta = 0$) and
Figure 8.9: The $n = 6$ eigenmode in cylindrical geometry for the alpha profile equilibrium. Lines of constant $\rho$ are marked ($q = 1/6$ in red) along with evenly spaced lines in $\theta_g$.

Figure 8.10: The $n = 6$ eigenmode in toroidal geometry for the alpha profile equilibrium. The top plots show $\mathbf{B} \cdot \hat{r}$, $\mathbf{B} \cdot \hat{e}_\theta$, and $\mathbf{B} \cdot \hat{\phi}$, while the bottom show $\mathbf{B} \cdot \nabla \psi / |\nabla \psi|$, $\mathbf{B} \cdot \mathbf{b} \times \nabla \psi / |\nabla \psi|$, and $\mathbf{B} \cdot \mathbf{b}$. Lines of constant $\rho$ are marked ($q = 1/6$ in red) along with evenly spaced lines in $\theta_g$ and $\theta_f$. 
Figure 8.11: Magnetic perturbations for the $n = 6$ eigenmode at $r = a$ in the alpha model equilibrium. (a) The eigenmode in configuration space in toroidal (solid colored lines) and cylindrical (dashed black lines) geometry. (b) Fourier decomposition of the eigenmode with respect to geometric poloidal angle, $\theta_g$.

inboard sides ($\theta = \pi$), as can be seen in Fig. 8.11(a). The perturbed fields broadly match an $m = 1$ harmonic, although there are important contributions from different harmonics. (There is an overall phase shift as a result of a difference in the normalization for the plots shown here and the contour plots shown previously.) The contributions from different $m$ harmonics are made apparent by examining the Fourier transform in the geometric poloidal angle, $\theta_g$, at the surface. In Fig. 8.11(b), we see a clear broadening of the poloidal spectrum for both the toroidal and poloidal components of the magnetic field.

8.3.2 Tearing Modes in the tanh Model

The reversal surface is distinct from other rational surfaces in that every $n$ harmonic is resonant there. The general expectation for tearing modes is that the most unstable mode is the one with the longest wavelength as the tearing stability parameter, $\Delta'$, generally increases as $k$ decreases. The fastest growing mode resonant at the reversal surface is not $n = 1$, as can be seen in Fig. 8.12(a). At low $n$, the growth rate increases with $n$, reaching a maximum around $n = 10$, and then begins falling off with increasing $n$. This is similar to previous linear results for edge resonant modes [84].

We attribute the non-monotonic behavior of the growth rate with $k$ to a breakdown in the constant-$\psi$ approximation, which occurs when the layer width, $\delta$, becomes large enough such that $\Delta' \delta \sim \mathcal{O}(1)$ [9]. This can be seen in Fig. 8.12(b) which shows the layer width, $\delta$, estimated from the radial extent of the toroidal current perturbation, times the mode stability parameter, $\Delta'$, taken from the cylindrical geometry equilibrium. The estimated layer width is largely identical for the eigenmode in our cylindrical and toroidal geometry computations.

Recall that the condition $\hat{n} \cdot \mathbf{J}_{r=a} = 0$ in cylindrical geometry restricts the amplitude and
(a) Tearing mode growth rates for the tanh model equilibrium in cylindrical and toroidal geometry. (b) The value of $\Delta'/\delta$, where the layer width, $\delta$, is estimated from the eigenfunction and $\Delta'$ is taken from the cylindrical geometry equilibrium.

Figure 8.13: Magnetic perturbations for $n = 1 - 7$ eigenmodes at $r = a$ in the tanh model equilibrium for the toroidal geometry computations. (a) Ratio of $|\tilde{B}_\theta|$ to $|\tilde{B}_\phi|$. (b) Phase difference $\delta_\phi - \delta_\theta$.

phase of the perturbations. Inverting Eq. (8.9), we find

$$\frac{\tilde{B}_\theta}{\tilde{B}_\phi} = \frac{m R_0}{n a} \frac{1}{\cos(\delta_\phi - \delta_\theta)}. \quad (8.11)$$

For $m = 0$ modes in cylindrical geometry, there is no perturbed poloidal magnetic field at the boundary. This is well-satisfied in our cylindrical computations, but in our toroidal computations the weight of $|\tilde{B}_\theta|$ relative to $|\tilde{B}_\phi|$ increases with increasing $n$, seen in Fig. 8.13(a), although it remains at most a few percent of it. The measured phase difference, $|\delta_\phi - \delta_\theta|$, increases substantially with increasing $n$, as can be seen in Fig. 8.13(b). For the $n = 1$ eigenmode, the phase difference in toroidal geometry is usually around $\pm 15^\circ$ from the cylindrical geometry expectation (either $0^\circ$ or $\pm 180^\circ$). However, for larger $n$ the phase difference is much closer to $\delta_\phi - \delta_\theta = \pm 90^\circ$. These effects are quite pronounced at the location of the toroidal array, $\theta_g = -119^\circ$, seen in Fig. 8.14.
Figure 8.14: Magnetic perturbations for \( n = 1 - 7 \) eigenmodes at the toroidal array \((r = a, \theta_g = -119^\circ)\) in the tanh model equilibrium. (a) Ratio of \(|\tilde{B}_\theta|\) to \(|\tilde{B}_\phi|\). (b) Phase difference \(\delta_\phi - \delta_\theta\).

8.3.2.1 \( n = 1 \) Eigenmode Structure

In contrast to the dominantly \( m = 1 \) core-resonant tearing modes which all have largely similar eigenmodes, the edge-resonant modes here show quite different behavior for the different \( n \). We will examine the structure of only the \( n = 1 \) mode, which plays an important part in the nonlinear interaction of the core-resonant modes with \( n \) and \( n' = n \pm 1 \).

In cylindrical geometry, the \( n = 1 \) eigenmode shows no poloidal variation, and the expected MHD phase relationship where \( \tilde{B}_r \) is exactly out of phase with \( \tilde{B}_\theta \) and \( \tilde{B}_\phi \) is valid. This can be seen in Fig. 8.15. The expected MHD phase relationship is a result of \( \nabla \cdot \tilde{B} = 0 \) and the

\[
0 = \nabla \cdot \tilde{B} = \sum_{m,n} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{B}_{r,m,n} \right) + \frac{im}{r} \tilde{B}_{\theta,m,n} + \frac{in}{R} \tilde{B}_{\phi,m,n} \right].
\]

(8.12)
The toroidal analog of this is
\[
0 = \nabla \cdot \tilde{\mathbf{B}} = \sum_{m,n} \left[ \frac{1}{R} \frac{\partial}{\partial R} \left( R \tilde{B}_{R,m,n} \right) + \frac{\partial}{\partial Z} \tilde{B}_{Z,m,n} + i \frac{n}{R} \tilde{B}_{\phi,m,n} \right].
\] (8.13)

The geometric radial and poloidal perturbations are related to \(\tilde{B}_r\) and \(\tilde{B}_\theta\) through \(\tilde{B}_r = \tilde{B}_R \cos(\theta) + \tilde{B}_Z \sin(\theta)\) and \(\tilde{B}_\theta = \tilde{B}_Z \cos(\theta) - \tilde{B}_R \sin(\theta)\); coupling among real and imaginary components of \(\tilde{B}_r\), \(\tilde{B}_\theta\), and \(\tilde{B}_\phi\) is evident. The comparable \(n = 1\) eigenmode in toroidal geometry has significant poloidal variation, seen in Fig. 8.16. This is most evident in the components that are expected to vanish in the cylindrical MHD system (here \(Re\tilde{B}_r\), \(Im\tilde{B}_\theta\), and \(Im\tilde{B}_\phi\)), which display a strong \(|m| = 1\) character.

8.3.2.2 Experimental Comparison

It should be noted that the low-\(n\) modes (\(n \sim 1 - 4\)) are expected to be linearly stable in experimentally relevant profiles. They are often observed to be significant only following the nonlinear sawtooth relaxation events in experiment. As such, comparisons between experimental data for these modes and the strictly linear results for growing modes of the tanh profile should be taken with caution. A more relevant comparison would again require fully nonlinear toroidal computations.
Experimental measurements of the amplitude of the low-$n$ fluctuations at the plasma edge averaged over a sawtooth event are shown in Fig. 8.17 (at the time of this writing, detailed measurements of the phase were not available). The amplitude of $\tilde{B}_\theta$ here is expressed as a percentage of the background magnetic field, and it varies significantly between the standard discharges and the enhanced confinement discharges that use pulsed poloidal current drive (PPCD) [102, 118] to reduce the tearing activity. Again, the raw experimental data is not readily available, so we infer the experimental amplitude of $\tilde{B}_\theta$ based on these plots. We estimate that $\tilde{B}_\theta$ for the $n = 1$ mode is roughly 0.375% $B_0$.

We may use our linear numerical results to estimate the corresponding amplitude of $\tilde{B}_\phi$ in terms of $B_0$. Our numerical computations predict that $\tilde{B}_\theta/\tilde{B}_\phi \approx 0.02$, so the predicted amplitude of $\tilde{B}_\phi$ based on the standard experimental measurements would be roughly 18.75% of the background magnetic field, an unrealistically large value. We conclude that our linear results for the low-$n$ modes do not agree well with experimental measurements. Further investigation into this discrepancy is needed.

8.4 Discussion

Our linear computations of core-resonant tearing modes agree well with experimental measurements of the core-resonant modes in MST shortly before the sawtooth crash, when nonlinear interaction among modes is expected to be minimal. The ratio of the amplitudes is found to increase nearly linearly with $n$, and they match the estimates of the experimental measurements well. The phase difference at the location of the toroidal array in our computations also agrees well with the experimental estimates. The phase difference is on the same order as
the experimental observations, differing only between $2 - 8^\circ$ from the estimates, with larger disagreement occurring for higher $n$.

However, the edge-resonant mode measurements do not agree well with our linear computations and warrant additional investigation. Specifically, the measured $\tilde{B}_\theta$ for these modes is much larger than the expectations for a perturbation with a dominantly $m = 0$ component, and this is not remedied by toroidal coupling. Fully nonlinear toroidal computations of the sawtooth relaxation cycle in the reversed-field pinch are a clear logical next step in this investigation. Another step would be to investigate extensions beyond the simple zero-$\beta$ single-fluid MHD model used here. Including two-fluid effects and/or finite pressure in linear computations in toroidal geometry is a relatively straightforward extension.
Our investigation of two-fluid current-driven tearing modes in the presence of a pressure gradient revealed an additional instability that is destabilized by diamagnetic drifts even in the absence of magnetic shear [83]. We identify this mode as a resistive drift wave, a drift wave driven unstable by the electron collisionality. While basic drift waves are stable, they can become destabilized if the electron response is non-adiabatic; an electron density that is slightly out of phase with the perturbed electrostatic potential allows the drift wave to extract free energy from thermodynamic gradients. The resistive drift wave was first described analytically in the electrostatic limit by Moiseev and Sagdeev [87]. Mikhailovskii [78] derived a comprehensive dispersion relation for the resistive drift mode that included electromagnetic effects, and this was later extended to fully kinetic calculations [79].

Observations of unstable drift modes were not reported in Refs. [49, 50], despite the similarity of our models. However, later nonlinear computations using the same model as Refs. [49, 50] reported two different types of saturated states [94]. The first is characteristic of a saturated tearing mode, while the second has features that they identify with drift waves and indicates a potential coupling of the drift-tearing modes and drift modes, a possibility pointed out earlier by Waelbroeck et al. [125].

Here, we present a linear study of the resistive drift wave, which is a logical first step towards understanding the evolution in the nonlinear system. In Sec. 9.1, we introduce the equilibrium configuration used in this chapter. It can support pressure gradients with and without magnetic shear, and it is verified to support both linear drift-tearing modes and resistive drift modes in the appropriate limits. The model equations used here are introduced in Sec. 9.2, and a brief derivation of the resistive drift mode dispersion relation is provided. In Sec. 9.3, we compare this dispersion relation to the results of numerical computations with NIMROD, and we find good agreement between them. In addition, we are able to examine how the resistive drift mode behaves in the presence of magnetic shear in our bounded domains. Shear is found to strongly stabilize the drift mode, consistent with the predictions for the unbounded domain.

9.1 Model Equilibrium

We consider a doubly-periodic equilibrium in slab geometry with perfectly conducting boundaries at $x = \pm a$. The magnetic field can support nontrivial pressure profiles with and without magnetic shear, making it well-suited for studies of the resistive drift mode, and is
given by
\[
B_0(x) = B_0(x) \left[ \sqrt{1 - f(x)^2} \hat{z} + f(x) \hat{y} \right].
\] (9.1)
The function \( f \) controls the orientation and shear of the magnetic field through
\[
f(x) = \sin \left\{ \arcsin \left[ \epsilon \sin \left( x L_s^{-1} \right) \right] + \theta_b \right\}.
\] (9.2)
The parameter \( \epsilon \) controls the strength of the sheared magnetic field, and the parameter \( L_s^{-1} \) sets the shear length scale. When \( \epsilon = 0 \) or \( L_s^{-1} = 0 \) there is no sheared magnetic field, and the parameter \( \theta_b \) allows one to set the values of \( k_\parallel \) and \( k_\perp \) at \( x = 0 \).

The standard tearing mode analysis takes \( k = k_\parallel \hat{y} \) and \( \theta_b = 0 \) so that the resonant surface, \( k_\parallel = 0 \), is located at \( x = 0 \). With \( \theta_b \neq 0 \) however, the location of the resonant surface is shifted. For this equilibrium, \( k_\parallel (x) = 0 \) at \( x = x_c \) where
\[
x_c \equiv L_s \arcsin \left[ \frac{k_\parallel (0)}{\epsilon |k|} \right].
\] (9.3)
If \( \epsilon < k_\parallel (0) / |k| \), then the parallel wavevector never vanishes. If this is not the case, then \( k_\parallel \) vanishes at a finite radial location; whether this is within the plasma depends on \( L_s \).

The equilibrium parallel current density is directly controlled by \( f \), and it is given by
\[
\frac{\mathbf{J}_0 \cdot \mathbf{B}_0}{B_0^2} = \frac{1}{\sqrt{1 - f^2}} \frac{df}{dx}.
\] (9.4)

Clearly if \( \frac{df}{dx} = 0 \), there are no parallel currents and hence no current-driven tearing instability. The equilibrium can still support a pressure gradient, however. Force balance, \( \mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0 \), gives an expression for \( B_0(x) \) in terms of the pressure,
\[
B_0(x) = B_0(0) \left[ (1 + 2\beta) - 2 \mu_0 p_0(x) / B_0(0)^2 \right]^{1/2}
\] (9.5)
where we have used the definition \( \beta \equiv \mu_0 p_0(0) / B_0(0)^2 \), which differs from the usual definition by a factor of 2.

The pressure profile is specified by
\[
p(x) = p_0 \left[ 1 + \delta_p \tanh \left( x / \Delta_p \right) \right].
\] (9.6)
The temperature is taken to be isothermal, so all of the pressure variation is in the density.
The inverse gradient length scale is then

\[ L_p^{-1} = \frac{1}{\bar{p}} \frac{dp}{dx} \left( \frac{\delta_p}{\Delta_p} \right) \frac{1 - \tanh^2 \left( \frac{x}{\Delta_p} \right)}{1 + \delta_p \tanh \left( \frac{x}{\Delta_p} \right)} = \frac{L_n^{-1}}{n}. \]  

\[ \text{(9.7)} \]

### 9.1.1 NIMROD Implementation

It is common in linear studies to take the magnetic field to be oriented mainly in the \( \hat{z} \) direction, \( \mathbf{B} \approx B_0 \hat{z} \), with periodic variation allowed only in the \( \hat{y} \) direction (i.e. \( \frac{\partial}{\partial z} = 0 \)). The derivation in Sec. 9.2 uses this approach. However, the implementation of this in our NIMROD computations requires further discussion.

In general, the wavevector, \( \mathbf{k} \), can have projections in both of the periodic directions in NIMROD. Setting the magnitude of the wavevector in NIMROD's Fourier-represented direction is trivial, but the other periodic direction, which uses a finite element representation, is slightly more complicated. There, we may only specify the minimum value the wavenumber may take using the length of that direction, but higher wavenumbers are also represented in the system, up to the limit of grid resolution.

For tearing modes, the longest wavelength modes are generally the most unstable so simply setting a minimum wavenumber in that direction is not problematic. While tearing modes with \( k = 2k_0, 3k_0, ... \) and so on will also be present in the computational system, we expect them to be much more slowly growing if they are not damped away completely. However, resistive drift modes have a maximum growth rate at finite values of \( k_\parallel \) and \( k_\perp \), as we will see presently, and this makes it more difficult to converge on drift modes with a specific value of \( \mathbf{k} \). A straightforward workaround is to reduce the height in the periodic direction represented by finite elements so that only very high wavelength modes are represented in that direction. If this is done, we can effectively eliminate the variation in that direction and the parallel and perpendicular wavenumbers can be set directly with \( \theta_b \) and \( |\mathbf{k}| \). At \( x = 0 \), this gives

\[ k_\parallel,0 = |\mathbf{k}| \cos (\theta_b), \quad |\mathbf{k}_\perp,0| = |\mathbf{k}| \sin (\theta_b). \]  

\[ \text{(9.8)} \]

This equilibrium configuration is shown in Fig. 9.1. The strong guide field, \( \mathbf{B}_g \), is largely oriented along \( \hat{z} \) (NIMROD's \( \hat{Z} \)) while the sheared field, \( \mathbf{B}_s \), is primarily oriented along \(-\hat{y}\), (NIMROD's \( \hat{\phi} \)). The parallel and perpendicular components of the wavevector, \( \mathbf{k} \), are shown for the location \( x = 0 \), but are offset for visualization. When \( \mathbf{B}_s = 0 \), the wavevector is the same at all locations.
9.2 Analytic Dispersion Relation

In what follows, we will assume that thermal conduction is much more rapid than other dynamics in the system, so that the temperature may be treated as a constant. Particle motion is largely unconstrained along the magnetic field, so that any gradients in that direction are rapidly eliminated and the isothermal assumption is quite good. However, particle motion across the magnetic field is much less rapid, as particles only drift across the field at much smaller velocities. Here, the isothermal assumption is a poor choice, but for the analysis that follows, it will suffice to provide a starting point. We will also ignore the plasma fluid viscosity, which typically only tends to damp away the dynamics of interest here. Lastly, we will neglect the electron inertia \((m_e \to 0)\) and the ion gyroviscosity (i.e. we take the cold ion limit \(p_i \to 0\)).

We consider the system of equations

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0 \tag{9.9} \\
m_i n \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) &= \mathbf{J} \times \mathbf{B} - \nabla p \tag{9.10} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \left[ -\mathbf{v} \times \mathbf{B} + \Lambda_e \frac{1}{ne} (\mathbf{J} \times \mathbf{B} - \nabla p_e) + \eta \mathbf{J} \right], \tag{9.11}
\end{align*}
\]

with \(\mu_0 \mathbf{J} = \nabla \times \mathbf{B}\). We will linearize these about the slab geometry equilibrium introduced previously. The linearization follows from the discussion in Appendix I, but now we must deal with the complete linearized system, Eqs. \((I.16)-(I.21)\).
Following the derivation given in Appendix J, we find the equation for $\tilde{B}_x$ is

$$
(\omega^2 \tilde{B}_x + \omega k_y B_{0y} \tilde{v}_x) = i \omega \frac{\eta}{\mu} \nabla^2 \tilde{B}_x + \Lambda_e \left( i \omega \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \right) \tilde{B}_x - \Lambda_e \left( \omega \frac{\beta k_y B_{0z} L^{-1}_n}{\mu_0 n_0 e} \right) \tilde{B}_x, \quad (9.12)
$$

the equation for $\tilde{B}_z$ is

$$
\left\{ \left( \omega^2 - \beta \left[ \frac{k_y^2 B_{0y}^2}{\mu_0 m_z n_0} + \omega \frac{\eta}{\mu_0} \nabla^2 \right] \right) \Lambda_e i \omega \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \right\} \tilde{B}_z = - \omega^2 \beta \Lambda_e^2 \left( \frac{k_y B_{0y}}{\mu_0 n_0 e} \right)^2 \left( \nabla^2 - \frac{B_{0y}''}{B_{0y}} \right) \tilde{B}_x - \left( \frac{\beta k_y B_{0z} L^{-1}_n}{\mu_0 n_0 e} \right) \omega \Lambda_e \left\{ (\omega^2 \tilde{B}_x + \omega k_y B_{0y} \tilde{v}_x) - \left[ \omega^2 - \frac{\beta k_y^2 B_{0y}^2}{\mu_0 m_z n_0} \right] \tilde{B}_x \right\}, \quad (9.13)
$$

and the equation for $\tilde{v}_x$ is

$$
- i \omega \mu_0 m_z n_0 \nabla^2 \tilde{v}_x = i k_y B_{0y} \left[ \nabla^2 - \frac{B_{0y}''}{B_{0y}} \right] \tilde{B}_x - i \omega \mu_0 m_z n_0 \left[ \left( L^{-1}_n + \frac{\partial}{\partial x} \right) (\nabla \cdot \tilde{v}) - L^{-1}_n \frac{\partial \tilde{v}_x}{\partial x} \right]. \quad (9.14)
$$

This system of three second order differential equations is simplified as follows. As we will see, the resistive drift mode grows fastest at larger $k_{\perp} a \sim 20$, contrary to the tearing mode behavior where fastest growth generally occurs for small $k_{\perp} a$. Consequently, we expect that the dominant balance in Eq. (9.14) occurs between the $\nabla^2 \tilde{v}_x \sim -k_{\perp}^2 \tilde{v}_x$ and $\nabla^2 \tilde{B}_x \sim -k_{\perp}^2 \tilde{B}_x$ terms, and we will take

$$
\tilde{v}_x = - \frac{k_y B_{0y}}{\omega \mu_0 m_z n_0} \tilde{B}_x. \quad (9.15)
$$

Comparing the resistive diffusion of $\tilde{B}_z$ in Eq. (9.13) to the $\omega^2$ contribution originating from the flow compressibility, we find

$$
\frac{i \beta \omega \frac{\eta}{\mu_0} \nabla^2}{\omega^2} \sim \frac{i \beta k_{\perp}^2 a^2}{\omega \tau_A} \frac{\eta}{\mu_0 a^2} \sim \frac{\beta k_{\perp}^2 a^2}{\omega \tau_A S}. \quad (9.16)
$$

The resistive diffusion of $\tilde{B}_z$ may be neglected in the limit that $S \gg \beta k_{\perp}^2 a^2 / \omega \tau_A$. Neglecting this term simplifies the algebra substantially, and we may then combine Eqs. (9.12), (9.13), and (9.15) to yield a second order ordinary differential equation for $\tilde{B}_x (x)$, where the coefficients depend on $x$. Ignoring the $x$-dependence of the coefficients, we assume that solutions take the form of a standing wave, $\tilde{B}_x = C \cos (n \pi x / 2a)$, where $n = 2m + 1$, in order to satisfy the boundary conditions $\tilde{B}_x (x = \pm a) = 0$. 


A simplified dispersion relation then results as

$$
\left( \Omega^2 - k^2 \right) \left[ \Omega^2 - \Omega \Omega^{(n)}_e - \beta k^2 \right] = \Omega^2 k^2 k^2 \beta d_i^2 - \frac{i \Omega k^2}{S} \left( \Omega^2 - \beta k^2 \right)
$$

(9.17)

where $\Omega \equiv \omega \tau_A$, all lengths have been normalized to $a$, $\Omega^{(n)}_e \equiv \beta k d_i L_n^{-1}$, and $k^2 = k^2 + \pi^2 n^2 / 4$. Mirnov et al. [82] worked out a more complete dispersion relation that includes the resistive diffusion of $B_z$, and this is given by:

$$
\left[ \Omega - \frac{k^2}{\Omega} + \frac{i \beta k^2}{S} \right] \left[ \Omega^2 - \beta k^2 + \frac{i \beta k^2 \Omega}{S} - \Omega \Omega^{(n)}_e \right] = \Omega \beta k^2 k^2 d_i^2 - \frac{i k^2}{S} (1 - \beta) \left[ \Omega^2 - \beta k^2 + \frac{i \beta k^2 \Omega}{S} \right].
$$

(9.18)

Both of these dispersion relations are fourth order polynomials in $\Omega$, and they possess four roots corresponding to different modes in the system.

Two-fluid effects enter these dispersion relations in two places. First, they appear in the diamagnetic drift frequency on the left hand side. Second, they appear in the first term on the right hand side, which is associated with ion polarization drifts and finite Larmor radius stabilization. This term is responsible for the enhanced two-fluid tearing through KAW effects, as described in Mirnov et al. [81]. The electrostatic limit of these dispersion relations is found from taking $\Omega^2 \ll k^2$ [87]. In this limit, Eq. (9.17) reduces to

$$
\left[ \Omega^2 - \Omega \Omega^{(n)}_e - \beta k^2 \right] = -\Omega^2 k^2 \beta d_i^2 + \frac{i \Omega k^2}{k^2} \left( \frac{\Omega^2}{k^2} - \beta \right)
$$

(9.19)

Here, the two-fluid effects on the right hand side are found to decrease the growth of the resistive drift mode, in contrast to their effects on the tearing behavior.

### 9.3 Computational Results

Our computations with $\beta = 0.10$ and $d_i/a = 0.10$ and no magnetic shear ($\epsilon = 0$) show an unstable resistive drift mode that grows preferentially with small $k^2 a$ and large $k^2 a$, as can be seen in Fig. 9.2(a). This compares very favorably with the numerical solution of the dispersion relation, Eq. (9.18), shown in Fig. 9.2(b). The fastest growing mode has $k^2 a \approx 20$ and $k^2 a \approx 0.07$ in both.

The fastest growth rate and the preferred orientation of the wavevector do not change
Figure 9.2: Growth rate of the resistive drift mode as a function of $k_\parallel a$ and $k_\perp a$ in the cos equilibrium with $\beta = 0.10$, $d_i/a = 0.10$, $S = 10^4$, $P_m = 0.001$, and $aL_p^{-1} = aL_n^{-1} = 0.5$. (a) NIMROD results and (b) NIMROD results (solid lines) plotted with numerical solution of the dispersion relation, Eq. (9.18) (dashed lines).

Figure 9.3: Growth rate for resistive drift modes in the cos equilibrium with $S = 10^4$, $P_m = 0.001$, $aL_p^{-1} = aL_n^{-1} = 0.5$, and (a) $\beta = 0.05$, $d_i/a = 0.10$, and (e) $\beta = 0.05$, $d_i/a = 0.05$.

monotonically with the two-fluid parameter, $\rho_s$, in our computations. Here, using our definition of plasma-$\beta$ and in the isothermal limit with $\Gamma = 1$, $\rho_s \equiv \sqrt{\beta}d_i$. When $\beta$ is decreased from 0.10 to 0.05 at fixed $d_i/a = 0.10$, the maximum of the growth rate increases to $\gamma \tau_A = 6.829 \cdot 10^{-3}$, which occurs for $k_\perp a = 30$ and $k_\parallel a = 0.06$. When the ion skin depth is lowered from $d_i/a = 0.1$ to $d_i/a = 0.05$, (keeping $\beta = 0.05$) the maximum growth rate decreases to $\gamma \tau_A = 4.843 \cdot 10^{-3}$, at $k_\perp a = 36$ and $k_\parallel a = 0.07$. The growth rates as functions of $k_\parallel a$ and $k_\perp a$ for both of these parameter choices are shown in Fig. 9.3.

The resistive drift mode frequency is typically an order of magnitude larger than the mode growth rate in our computations, and it varies strongly with $k_\parallel a$ and weakly with $k_\perp a$, as can be seen in Fig. 9.4. The frequency is monotonic with the two-fluid parameter, $\rho_s$; it decreases with both $\beta$ and $d_i/a$. However, the behavior of the frequency as a function of $k_\parallel a$ and $k_\perp a$ does not change substantially between these cases.
9.3.1 Scaling with Resistivity

To examine the resistive drift mode’s dependence on the plasma resistivity, computations with $\beta = 0.10$, and $d_i/a = 0.10$ are run at different values of $S$ and $k_\parallel a$, for fixed $k_\perp a$. For $S \geq 5 \cdot 10^3$, these scans are at $k_\perp a = 22$, roughly near the mode maximum in $k_\perp a$ space, while for $S = 10^3$ the scan is performed at $k_\perp a = 8$ where the fastest growth rate is observed in this case. To mitigate the effects of the isotropic viscosity, it is held fixed between these computations; for $S = 10^5$, $P_m = 10^{-3}$, and smaller values of $S$ correspond to even smaller $P_m$.

The growth rate decreases as $S$ increases, as expected for a mode that depends on resistive effects, between $S = 5 \cdot 10^3$ and $S = 10^5$, as can be seen in Fig. 9.5. At $S = 10^3$, the growth rate again drops back down as the large resistivity begins to inhibit the growth of the mode. This behavior is missed when using the dispersion relation Eq. (9.17), which is only valid for $S \gg k^2 \beta/\Omega$. We see much more favorable agreement between our computational results and the numerical solution of the more accurate dispersion relation, Eq. (9.18), shown in the dashed lines in Fig. 9.5 for the same parameters, although the analytic solution still appears to be overestimating the growth rate at the lowest $S = 10^3$. 

Figure 9.4: Mode frequency for resistive drift modes in the cos equilibrium with $S = 10^4$, $P_m = 0.001$, $aL_p^{-1} = aL_n^{-1} = 0.5$, and (a) $\beta = 0.10$, $d_i/a = 0.10$, (b) $\beta = 0.05$, $d_i/a = 0.10$, and (c) $\beta = 0.05$, $d_i/a = 0.05$. The vertical axes differ.
Figure 9.5: Growth rate as a function of $k_{∥a}$ for $S = 10^3$ with $k_{⊥a} = 8$, and $S = 10^4$ and $S = 10^5$ with $k_{⊥a} = 22$ for NIMROD computations (solid lines) and the analytic solution of the dispersion relation (dashed lines).

Figure 9.6: Growth rates of the fastest growing resistive drift mode with $k_{⊥a} \in [0, 30]$ and $k_{∥a} \in [0, 0.06]$, as a function of the shear strength, $\epsilon$, and the shear length, $a_{L_s}^{-1}$, for the cosine equilibrium with $aL_p^{-1} = aL_n^{-1} = 0.5$, $S = 10^4$, $P_m = 0.001$, in case B ($\beta = 0.05$, $d_i/a = 0.1$).

### 9.3.2 Effects of Magnetic Shear

Previous work [3, 66] has shown that the resistive drift mode is stabilized by magnetic shear in unbounded domains. We analyze the effects of a sheared magnetic field in our bounded domain computations with $\beta = 0.05$ and $d_i/a = 0.10$ by now varying the shear field strength, $\epsilon$, and the shear length, $a_{L_s}^{-1}$. For each value of $\epsilon$ and $a_{L_s}^{-1}$, we scan wavenumbers in the range $k_{⊥0a} \in [0, 30]$ and $k_{∥0a} \in [0, 0.06]$, where $k_{⊥0a}$ and $k_{∥0a}$ are the values at $x = 0$. This range includes the fastest growing pair ($k_{⊥a} = 30$, $k_{∥a} = 0.06$) when there is no magnetic shear. We find that the resistive drift mode is strongly stabilized by magnetic shear, as can be seen in Fig. 9.6. For modest values of shear strength, $\epsilon \lesssim 0.025$, the mode is quickly stabilized as the inverse shear length increases.
9.4 Discussion

We have confirmed the existence of the resistive drift mode within the extended MHD system of equations that is solved by NIMROD. Our numerical computations show agreement with the analytic dispersion relation developed for the resistive drift mode here, and this represents a small but significant step forward in the verification of NIMROD for this class of problems. The dispersion relation is a natural extension of the work in Mirnov et al. [81] to include the effects of an equilibrium pressure gradient. In addition, the equilibrium developed for our computations is universally valid for both uniform and sheared magnetic field configurations. Our work here may be easily extended to include additional effects, such as temperature gradients and finite thermal conduction.
Part IV

Conclusions
10 CONCLUSIONS

Our extended MHD computations of plasma relaxation dynamics demonstrate many interesting features for plasma conditions relevant to the reversed-field pinch experiment, including multiple discrete relaxation events that are comparable to the RFP sawtooth cycle and changes in plasma flow with two-fluid models that are on the order of the observed changes in experiment. Our linear studies of tearing modes in toroidal geometry and the comparison with experiment support the notion that toroidal effects are significant when interpreting experimental magnetic probe measurements. In addition, we have verified the NIMROD computational model for the linear resistive drift mode in slab geometry.

10.1 Conclusions from Nonlinear Studies

One of the primary motivations for our nonlinear simulations was to understand the discrepancy between the changes in plasma flow, $\Delta \langle v \rangle_\parallel$, and plasma current, $\Delta \langle J \rangle_\parallel$, observed in the initial relaxation event in our computations and in the experimental measurements. Based on experimental observations, the changes in plasma flow and current are always in the same direction in the plasma core. In our computations, and in previous work [70], the $\Delta \langle v \rangle_\parallel$ and $\Delta \langle J \rangle_\parallel$ are in opposite directions in the first relaxation event. Here, both the MHD and Hall dynamo electric fields cooperate to relax the parallel current profile. The associated flow changes are in the direction of the large Lorentz force density, and this is consistent with the magnetic activity determining the flow profile evolution.

We have found that the first event in our computations differs substantially from subsequent relaxation events as a result of our choice of initial conditions. The paramagnetic pinch initial conditions in our computations are a self-consistent choice from which to begin nonlinear computations, but the early evolution into a saturated single-helicity state prior to the first event is not representative of the conditions preceding the relaxation events in a typical RFP discharge. We conclude that comparisons to experimental measurements over relaxation events in typical multi-helicity conditions should be restricted to the subsequent events in our computations.

A key result is that the MHD and Hall dynamo electric fields that alter the parallel current profile differ substantially between the first event and the subsequent events in our two-fluid computations. In the first, both dynamos cooperate to relax the parallel current in the core and drive the plasma closer to the flat $\lambda$ state. This results in a correlated Lorentz force density that generates parallel flows, $\Delta \langle v \rangle_\parallel$, that are in the opposite direction as the $\Delta \langle J \rangle_\parallel$ in the core. In contrast, in the subsequent events we see opposition of the dynamo
electric fields in the core. The MHD dynamo relaxes the current profile and the Hall dynamo tends to peak it, but the MHD dynamo wins the competition. The associated Lorentz force density generates a $\Delta \langle v \rangle_\parallel$ in the same direction as the core $\Delta \langle J \rangle_\parallel$. This behavior more closely matches experimental observations where the changes in parallel flow and parallel current are always observed to be in the same direction. From this, we speculate that in experimental relaxation events the MHD dynamo relaxes the current in the core and it is opposed by a Hall dynamo that is smaller in magnitude. The associated Lorentz force density is then responsible for the changes in the plasma flow at the relaxation.

The changes in plasma momentum that are observed in our computations result from viscous coupling to the boundary of flows generated by the large relaxation-induced Maxwell stress. The Maxwell stress transports momentum radially and is the primary driver of the change in plasma flow in our computations, but we do not expect that the changes in total momentum are representative of the experimental conditions. We lack a realistic pressure profile and detailed edge effects are not included here.

The global magnetic helicity and the appropriate two-fluid generalization of it, the hybrid helicity, are well conserved relative to the global magnetic energy over the relaxation events in our computations. There are substantial changes in cross helicity as a result of two-fluid effects in the generalized Ohm's law and ion gyroviscosity, but the hybrid helicity is dominated by the magnetic helicity at these low-$\beta$ pinch conditions. This lends support to the Taylor variational approach of minimizing the energy while conserving the helicities, but we reiterate that the two-fluid variational theories that conserve hybrid helicity while minimizing energy are mathematically ill-posed [67, 92, 128]. While the plasma current approaches the predicted relaxed state in our computations and in the experiment, it never achieves it. From an energy standpoint alone, we note that the Taylor minimum energy state is not self-consistent. There is no energy to drive fluctuations, but we have seen that fluctuations are needed to maintain the reversed-field state. With regards to the flow relaxation, there is a significant fraction of kinetic energy in the plasma flow perpendicular to the magnetic field, and this stands in direct contradiction to the variational theory prediction that the flow in the relaxed state should be parallel to the field. We conclude that while variational theories may provide a useful starting point for understanding the qualitative features of the relaxation, they should not be viewed as a definitive end state of the plasma.

10.2 Conclusions from Linear Studies

Our linear studies of the toroidal effects on tearing modes show a phase shift commensurate with experimental observations for the core-resonant modes. The experimental measurements
are taken prior to the sawtooth event when the nonlinear coupling among modes is believed to be small, and there is very good agreement between our linear results and the measurements. Comparisons between our linear computations of edge-resonant modes and the experimental measurements requires further analysis, however. Both sets of computations are limited, but missing important features of the edge region appears to be more important for the nonlinearly driven edge-resonant modes.

We have also confirmed the existence of the resistive drift wave in our linear two-fluid model. Comparisons of the mode growth rate to an analytic dispersion relation show quantitative agreement, and this represents a partial verification of the numerical model. The dispersion relation and equilibrium model may be readily extended to include the effects of temperature gradients and finite electron thermal conduction, allowing for additional verification opportunities for NIMROD. We note that the resistive drift mode is very strongly stabilized by magnetic shear, and we do not expect it be strongly unstable in realistic configurations, although it may have effects in the nonlinear system.

10.3 Recommendations for Future Work

Our nonlinear computations show that the MHD dynamo is substantial in the plasma core and it competes with and beats the Hall dynamo to relax the plasma current. From an experimental standpoint, more detailed measurements of the MHD dynamo in the plasma core would be extremely useful for comparison. Further, we have seen that many modes contribute to the core Hall dynamo in our computations, and the analysis in King [71] indicates that terms neglected in the measurements of Ding et al. [35] are important. Revisiting these experimental laser polarimetry measurements in light of these two findings would certainly be fruitful. It would also be of interest to examine experimental measurements of the dynamo electric field from relaxation events that begin from a quasi-single-helicity state as opposed to the more typical multi-helicity states in MST discharges. The former may be more directly applicable to the first event in our computations.

More detailed experimental measurements of the Hall dynamo at the edge of MST [123] also show considerable promise for direct comparison to our computations. However, there are clear deficiencies in our computational model, especially in the edge region, that likely must be addressed before these comparisons can be meaningful. Chief among these is the use of a realistic pressure profile, especially in our two-fluid computations. A pressure gradient would introduce diamagnetic flows into the equilibrium, and, when ion gyroviscosity is included, the gyroviscous stabilization of King et al. [69] will change significantly in the edge. Both of these may alter the nonlinear relaxation dynamics, which depend on coupling
between core-resonant and edge-resonant modes. A similar deficiency is our lack of detailed transport modeling. Here, we have restricted our attention to the changes in parallel flow observed at the relaxation events because of this. The evolution of the flow between events undoubtedly depends on transport effects that are not included here.

Experimental measurements of dynamo electric fields are ensemble-averaged over many relaxation events, and a more appropriate comparison may result from averaging our computations over the subsequent relaxation events. However, owing to the small number of subsequent relaxation events in our computations (at most 4 distinct subsequent events) this averaging is not performed here. Longer time scale computations that feature many more relaxation events are needed, but we note that our computations here are already run for about a half of a resistive diffusion time. Again, more detailed transport modeling is likely also required for these longer simulations to be more reflective of the evolution in the experiment.

The most obvious next step in our toroidal geometry studies is to include two-fluid effects and/or a pressure profile. Two-fluid effects alter the phase relationship among different components of the magnetic field perturbation and this may significantly impact the signal at the toroidal array. Similarly, a realistic pressure profile would result in interchange drive which would be especially large for the edge-resonant modes, and this may change the character of the most unstable linear mode in our computations. An investigation of the nonlinear effects is also warranted. Our nonlinear computations here are in cylindrical geometry, but it would be rather straightforward to extract the phase relations between different fluctuating components. This would provide information on the temporal behavior of the phases during nonlinear events, and it would shed light on whether the nonlinear drive of the edge-resonant modes significantly alters the phases of their magnetic components. The significance of the toroidal effects in these linear computations also suggests that they may be significant in nonlinear computations as well. Full nonlinear computations beginning with the simplest zero-$\beta$ resistive MHD model are a logical first step.

Lastly, the equilibrium model developed to study the resistive drift mode is deliberately chosen to be readily generalizable to include the effects of a temperature gradient as well. The behavior of the resistive drift mode in the presence of both a temperature and density gradient would certainly be an interesting avenue of investigation. Further, the dispersion relation of Mirnov et al. [83] includes finite electron thermal conduction, although the numerical computations presented there and in our work here do not. Including these additional effects in the model results in a much larger parameter space to consider, and there is clearly much more uncharted territory that remains to be explored.
Part V

Appendices
A  EVOLUTION OF MOMENTA IN THE NIMROD SYSTEM

The momentum is defined to be

$$P \equiv \int m_i n v d^3 x \quad \rightarrow \quad \frac{\partial P}{\partial t} = m_i v \frac{\partial n}{\partial t} + m_i n \frac{\partial v}{\partial t}$$  \hspace{1cm} (A.1)$$

In NIMROD, the density and momentum evolve according to

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n v) + \nabla \cdot (D_n \nabla n) + N$$  \hspace{1cm} (A.2)

$$m_i \frac{\partial v}{\partial t} = -m_i n v \cdot \nabla v + J \times B - \nabla p - \nabla \cdot \Pi + F = -m_i n v \cdot \nabla v + F + \mathcal{F}$$  \hspace{1cm} (A.3)

where the anomalous terms are given by:

$$N = \nabla \cdot (n_{eq} v_{eq}) - \nabla \cdot (D_n \nabla n_{eq})$$  \hspace{1cm} (A.4)

$$\mathcal{F} = m_i n_{eq} v_{eq} \cdot \nabla v_{eq} + \nabla \cdot \Pi_{i,eq}$$  \hspace{1cm} (A.5)

Then

$$\frac{\partial P}{\partial t} = m_i v \nabla \cdot [-(n v) + (n_{eq} v_{eq}) + D_n \nabla (n - n_{eq})]$$

$$-m_i n v \cdot \nabla v + m_i n_{eq} v_{eq} \cdot \nabla v_{eq} + F + \nabla \cdot \Pi_{i,eq}$$  \hspace{1cm} (A.6)

Recalling the vector identity \((\nabla \cdot A) B + A \cdot \nabla B = \nabla \cdot (AB)\) we can combine the first terms on the RHS of each line, and manipulate the density diffusion term

$$\frac{\partial P}{\partial t} = \nabla \cdot [-m_i n v v] + \nabla \cdot [m_i D_n \nabla (n - n_{eq}) v] - m_i D_n \nabla (n - n_{eq}) \cdot \nabla v$$

$$+ F + \nabla \cdot \Pi_{i,eq} + m_i v \nabla \cdot (n_{eq} v_{eq}) + m_i n_{eq} v_{eq} \cdot \nabla v_{eq}$$  \hspace{1cm} (A.7)

Recalling that

$$J \times B = \frac{1}{\mu_0} (\nabla \times B) \times B = \frac{1}{\mu_0} B \cdot \nabla B - \nabla \left( \frac{B^2}{2\mu_0} \right) = \nabla \cdot \left( \frac{BB}{\mu_0} \right) - \nabla \left( \frac{B^2}{2\mu_0} \right)$$  \hspace{1cm} (A.8)

Then

$$F = \nabla \cdot \left( \frac{BB}{\mu_0} \right) - \nabla \left( \frac{B^2}{2\mu_0} + p \right) - \nabla \cdot \Pi_i = \nabla \cdot \left[ \frac{BB}{\mu_0} - \left( \frac{B^2}{2\mu_0} + p \right) I - \Pi \right]$$  \hspace{1cm} (A.9)
Note that we can combine some of the equilibrium terms as

\[ m_i \mathbf{v} \nabla \cdot (n_{eq} \mathbf{v}_{eq}) + m_i n_{eq} \mathbf{v}_{eq} \cdot \nabla \mathbf{v}_{eq} = m_i (\mathbf{v} - \mathbf{v}_{eq}) \nabla \cdot (n_{eq} \mathbf{v}_{eq}) + m_i \mathbf{v}_{eq} \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \]
\[ + m_i n_{eq} \mathbf{v}_{eq} \cdot \nabla \mathbf{v}_{eq} \]
\[ = m_i (\mathbf{v} - \mathbf{v}_{eq}) \nabla \cdot (n_{eq} \mathbf{v}_{eq}) + \nabla \cdot (m_i n_{eq} \mathbf{v}_{eq} \mathbf{v}_{eq}) \]  

(A.10)

Using this in the momenta evolution

\[ \frac{\partial \mathbf{p}}{\partial t} = \nabla \cdot \left\{ -m_i n \mathbf{v} \mathbf{v} + m_i n_{eq} \mathbf{v}_{eq} \mathbf{v}_{eq} + \frac{\mathbf{B} \mathbf{B}}{\mu_0} - \left( \frac{B^2}{2 \mu_0} + p \right) \mathbf{I} - \left( \mathbf{\Pi}_i - \mathbf{\Pi}_i,_{eq} \right) \right\} \]
\[ + \nabla \cdot \left\{ m_i D_n \nabla (n - n_{eq}) \mathbf{v} \right\} - m_i D_n \nabla (n - n_{eq}) \cdot \nabla \mathbf{v} + m_i (\mathbf{v} - \mathbf{v}_{eq}) \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \]

(A.11)

Combining all of the tensor divergence terms into one factor

\[ \frac{\partial \mathbf{p}}{\partial t} = \nabla \cdot \mathbf{T} - m_i D_n \nabla (n - n_{eq}) \cdot \nabla \mathbf{v} + m_i (\mathbf{v} - \mathbf{v}_{eq}) \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \]  

(A.12)

where

\[ \mathbf{T} \equiv -m_i n \mathbf{v} \mathbf{v} + m_i n_{eq} \mathbf{v}_{eq} \mathbf{v}_{eq} + \frac{\mathbf{B} \mathbf{B}}{\mu_0} - \left( \frac{B^2}{2 \mu_0} + p \right) \mathbf{I} - \left( \mathbf{\Pi}_i - \mathbf{\Pi}_i,_{eq} \right) + m_i D_n \nabla (n - n_{eq}) \mathbf{v} \]  

(A.13)

The axial momentum is a linear momentum, so we can just dot this equation with \( \hat{\mathbf{z}} \) to find the time evolution. Recalling that \( \hat{\mathbf{z}} \) has no spatial dependence we have that

\[ \hat{\mathbf{z}} \cdot (\nabla \cdot \mathbf{T}) = \nabla \cdot (\mathbf{T} \cdot \hat{\mathbf{z}}) - \mathbf{T} : [\nabla \hat{\mathbf{z}}]^T = \nabla \cdot (\mathbf{T} \cdot \hat{\mathbf{z}}) \]  

(A.14)

Then the axial momentum density evolves according to

\[ \frac{\partial}{\partial t} (\hat{\mathbf{z}} \cdot \mathbf{p}) = \nabla \cdot \left\{ \left[ -m_i n \mathbf{v} \mathbf{v} + m_i n_{eq} \mathbf{v}_{eq} \mathbf{v}_{eq} + \frac{\mathbf{B} \mathbf{B}}{\mu_0} - \left( \frac{B^2}{2 \mu_0} + p \right) \mathbf{I} - \left( \mathbf{\Pi}_i - \mathbf{\Pi}_i,_{eq} \right) \right] \cdot \hat{\mathbf{z}} \right\} \]
\[ + \nabla \cdot \left\{ \left[ m_i D_n \nabla (n - n_{eq}) \mathbf{v} \right] \cdot \hat{\mathbf{z}} \right\} \]
\[ - \hat{\mathbf{z}} \cdot \left[ m_i D_n \nabla (n - n_{eq}) \cdot \nabla \mathbf{v} \right] + \hat{\mathbf{z}} \cdot \left[ m_i (\mathbf{v} - \mathbf{v}_{eq}) \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \right] \]

(A.15)

Then recalling that

\[ \int \nabla \cdot \mathbf{A} \, d^3 x = \int \mathbf{\hat{n}} \cdot \mathbf{A} \, dA \]  

(A.16)
this becomes

\[
\frac{\partial}{\partial t} (\hat{z} \cdot \mathbf{P}) = \int \nabla \cdot \left\{ -m_i n \mathbf{v} v + m_i n_{eq} \mathbf{v}_{eq} v_{eq} + \frac{\mathbf{BB}}{\mu_0} - \left( \frac{B^2}{2 \mu_0} + p \right) \mathbf{I} - \left( \Pi_i - \Pi_{i,eq} \right) \right\} \cdot \hat{z} \, dA \\
+ \int \nabla \cdot \left[ m_i D_n \nabla (n - n_{eq}) \mathbf{v} \right] \cdot \hat{z} \, dA \\
+ \int \left\{ -\hat{z} \cdot \left[ m_i D_n \nabla (n - n_{eq}) \right] \cdot \nabla \mathbf{v} + \hat{z} \cdot \left[ m_i \left( \mathbf{v} - \mathbf{v}_{eq} \right) \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \right] \right\} \, d^3x
\]

(A.17)

The angular momentum is not a simple linear momentum. Instead, we must take \( \hat{z} \cdot r \hat{r} \times \) with this equation. Note

\[
\hat{z} \cdot r \hat{r} \times \frac{\partial \mathbf{P}}{\partial t} = \hat{r} \left( \hat{r} \cdot \nabla T + \hat{r} \times \hat{z} T \right) = -r P_\theta
\]

(A.18)

for a right-handed \((r, z, \theta)\) system. The terms in the volume integral are similar: a factor of \(r\) is included, along with a negative sign for the \(\theta\) component.

Alternatively, simply take \(r \hat{\theta} \cdot \) this equation. Note there is now spatial dependence here so that

\[
r \hat{\theta} \cdot (\nabla \cdot \mathbf{T}) = \nabla \cdot \left[ \mathbf{T} \cdot (r \hat{\theta}) \right] - \mathbf{T} : \left[ \nabla (r \hat{\theta}) \right]^T
\]

(A.19)

The tensor \( \nabla (r \hat{\theta}) \) can be found by recalling that in cylindrical coordinates

\[
\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{z} \frac{\partial}{\partial z}
\]

(A.20)

and that the unit vectors have some \(\theta\) dependence:

\[
\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}
\]

(A.21)

This is true for the traditional cylindrical \((r, \theta, z)\) system and NIMROD’s \((r, z, \theta)\) system. Then

\[
\nabla (r \hat{\theta}) = \left[ \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{z} \frac{\partial}{\partial z} \right] (r \hat{\theta}) = \hat{r} \hat{\theta} - \hat{\theta} \hat{r} \quad \rightarrow \quad \left[ \nabla (r \hat{\theta}) \right]^T = -\hat{r} \hat{\theta} + \hat{\theta} \hat{r}
\]

(A.22)

Recalling that the double-dot product is defined to be

\[
\mathbf{P} : \mathbf{R} \equiv \sum_{i,j} P^{ij} R_{ji}
\]

(A.23)
we can write
\[ \mathbf{T} : [\nabla (r \hat{\theta})]^T = T_{r,\theta} - T_{\theta,r} \] (A.24)

Then
\[
\frac{\partial}{\partial t} (r \hat{\theta} \cdot \mathbf{p}) = r \hat{\theta} \cdot \left\{ \nabla \cdot \mathbf{T} - m_i D_n \nabla (n - n_{eq}) \cdot \nabla \mathbf{v} + m_i (\mathbf{v} - \mathbf{v}_{eq}) \cdot \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \right\}
\]
\[
= \nabla \cdot \left( \mathbf{T} \cdot r \hat{\theta} \right) + T_{\theta,r} - T_{r,\theta}
\]
\[
+ r \hat{\theta} \cdot \left[ -m_i D_n \nabla (n - n_{eq}) \cdot \nabla \mathbf{v} \right] + r \hat{\theta} \cdot \left[ m_i (\mathbf{v} - \mathbf{v}_{eq}) \cdot \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \right]
\] (A.25)

The symmetric parts of \( \mathbf{T} \) will not survive in the double-dot product term so that
\[
T_{\theta,r} - T_{r,\theta} = \left\{ [m_i D_n \nabla (n - n_{eq}) \mathbf{v}]_{\theta,r} - [m_i D_n \nabla (n - n_{eq}) \mathbf{v}]_{r,\theta} \right\}
\]
\[
- \left\{ [\Pi_i - \Pi_{i,eq}]_{\theta,r} - [\Pi_i - \Pi_{i,eq}]_{r,\theta} \right\}
\] (A.26)

One can also show that the isotropic viscous stress tensor and the gyroviscous stress tensor
\[
\Pi_{iso} \equiv \nu m_i n \mathbf{W} \quad \Pi_{gv} = \frac{m_i p_i}{4 e B} \left[ \hat{\mathbf{b}} \times \mathbf{W} \cdot \left( \mathbf{I} + 3 \hat{\mathbf{b}} \hat{\mathbf{b}} \right) - \left( \mathbf{I} + 3 \hat{\mathbf{b}} \hat{\mathbf{b}} \right) \cdot \mathbf{W} \times \hat{\mathbf{b}} \right]
\] (A.28)

where
\[
\mathbf{W} = \nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbf{I}
\] (A.29)

are both symmetric, so that \( \Pi_{r,\theta} - \Pi_{\theta,r} = \mathbf{0} \).

The angular momentum density evolves as
\[
\frac{\partial}{\partial t} (r \hat{\theta} \cdot \mathbf{p}) = \nabla \cdot \left\{ \left[ -m_i n \mathbf{v} \mathbf{v} + m_i n_{eq} \mathbf{v}_{eq} \mathbf{v}_{eq} + \frac{\mathbf{B} \mathbf{B}}{\mu_0} - \left( \frac{\mathbf{B}^2}{2 \mu_0} + p \right) \mathbf{I} - \left( \Pi_i - \Pi_{i,eq} \right) \right] \cdot r \hat{\theta} \right\}
\]
\[
+ \nabla \cdot \left\{ [m_i D_n \nabla (n - n_{eq}) \mathbf{v}] \cdot r \hat{\theta} \right\}
\]
\[
+ \left[ m_i D_n \nabla (n - n_{eq}) \mathbf{v} \right]_{\theta,r} - \left[ m_i D_n \nabla (n - n_{eq}) \mathbf{v} \right]_{r,\theta}
\]
\[
+ r \hat{\theta} \cdot \left[ -m_i D_n \nabla (n - n_{eq}) \cdot \nabla \mathbf{v} \right] + r \hat{\theta} \cdot \left[ m_i (\mathbf{v} - \mathbf{v}_{eq}) \cdot \nabla \cdot (n_{eq} \mathbf{v}_{eq}) \right]
\] (A.30)
Integrating over the volume, this becomes

\[
\frac{\partial}{\partial t} (r \hat{\theta} \cdot P) = \int \hat{n} \cdot \left\{ -m_i n v v + m_i n_{eq} v_{eq} v_{eq} + \frac{B B}{\mu_0} - \left( \frac{B^2}{2 \mu_0} + p \right) I - (\Pi_i - \Pi_{i,eq}) \right\} \cdot r \hat{\theta} \, dA
+ \int \hat{n} \cdot \left[ m_i D_n \nabla (n - n_{eq}) v \right] \cdot r \hat{\theta} \, dA
+ \int \left\{ [m_i D_n \nabla (n - n_{eq}) v]_{\theta,r} - [m_i D_n \nabla (n - n_{eq}) v]_{r,\theta} \right\} \, d^3x
+ \int \left\{ r \hat{\theta} \cdot [-m_i D_n \nabla (n - n_{eq}) \cdot \nabla v] + r \hat{\theta} \cdot [m_i (v - v_{eq}) \nabla \cdot (n_{eq} v_{eq})] \right\} \, d^3x
\]  
(A.31)
B.1 Magnetic Energy

The magnetic energy is defined as

\[ W_B \equiv \int \frac{\mathbf{B} \cdot \mathbf{B}}{2\mu_0} d^3x \rightarrow \frac{\partial w_B}{\partial t} = \frac{\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}}{\mu_0} \quad (B.1) \]

Recalling the induction equation used in NIMROD this becomes

\[ \frac{\partial w_B}{\partial t} = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\nabla \cdot \left( \mathbf{E} \times \mathbf{B} \frac{\mu_0}{\mu_0} - \mathbf{E} \cdot \mathbf{J} + \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} - \nabla \cdot \left( \frac{\kappa}{\mu_0} (\nabla \cdot \mathbf{B}) \right) \right) - \frac{\kappa}{\mu_0} (\nabla \cdot \mathbf{B})^2 \quad (B.2) \]

The total magnetic energy evolution is then (assuming the bounding volume is fixed in time, and that \( \mathbf{B} \cdot \hat{n} = 0 \) on the boundary)

\[ \frac{\partial W_B}{\partial t} = -\int \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \cdot \hat{n} dA - \int \left[ \mathbf{E} \cdot \mathbf{J} + \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} - \frac{\kappa}{\mu_0} (\nabla \cdot \mathbf{B})^2 \right] d^3x \quad (B.3) \]

With the two-fluid generalized Ohm’s Law, this becomes

\[ \frac{\partial W_B}{\partial t} = -\int \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \cdot \hat{n} dA + \int \left\{ -\eta J^2 - \mathbf{v} \cdot \mathbf{J} \times \mathbf{B} + \Lambda_e \left[ \frac{1}{ne} \mathbf{J} \cdot \nabla p_e + \frac{m_e}{ne^2} \mathbf{J} \cdot \frac{\partial \mathbf{J}}{\partial t} \right] + \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} - \frac{\kappa}{\mu_0} (\nabla \cdot \mathbf{B})^2 \right\} d^3x \quad (B.4) \]

B.2 Kinetic Energy

The kinetic energy density evolves according to

\[ W_K \equiv \frac{1}{2} \int m_i n \mathbf{v} \cdot \mathbf{v} d^3x \rightarrow \frac{\partial w_K}{\partial t} = \mathbf{v} \cdot \left( m_i n \frac{\partial \mathbf{v}}{\partial t} \right) + \left( \frac{1}{2} m_i \mathbf{v} \cdot \mathbf{v} \right) \frac{\partial n}{\partial t} \quad (B.5) \]
Using the vector identity $\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} \nabla \cdot (\mathbf{v} \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$ we can express the kinetic energy density evolution as

$$\frac{\partial w_K}{\partial t} = m_i n v \cdot \left[ -\frac{1}{2} \nabla \cdot (v^2) + \mathbf{v} \times (\nabla \times \mathbf{v}) \right] + \mathbf{v} \cdot \mathbf{F} + \mathbf{v} \cdot \mathbf{F}$$

(B.6)

The second term above is zero, and the first and fifth terms can be combined using the vector identity $f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f = \nabla \cdot (f \mathbf{A})$ to yield:

$$\frac{\partial w_K}{\partial t} = -\nabla \cdot \left[ \left(\frac{1}{2}m_i n v^2\right)v \right] + \nabla \cdot \left( \mathbf{F} \mathbf{F} \right) + \nabla \cdot (D_n \nabla n) + \mathbf{v} \cdot \mathbf{F} + \left(\frac{1}{2}m_i v^2\right) N \right)$$

(B.7)

Recall that the center of mass forcing is $\mathbf{F} = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi}$ and using

$$-\mathbf{v} \cdot (\nabla \cdot \mathbf{\Pi}) = -\nabla \cdot (\mathbf{\Pi} \cdot \mathbf{v}) + \mathbf{\Pi} : (\nabla \mathbf{v})^T$$

(B.8)

it follows that the kinetic energy density evolution becomes

$$\frac{\partial w_K}{\partial t} = -\nabla \cdot \left[ \left(\frac{1}{2}m_i n v^2\right)v \right] - \nabla \cdot (\mathbf{\Pi} \cdot \mathbf{v}) + \nabla \cdot (\mathbf{J} \times \mathbf{B}) - \mathbf{v} \cdot \nabla p + \mathbf{\Pi} : (\nabla \mathbf{v})^T$$

$$+ \left(\frac{1}{2}m_i v^2\right) \nabla \cdot (D_n \nabla n) + \mathbf{v} \cdot \mathbf{F} + \left(\frac{1}{2}m_i v^2\right) N \right)$$

(B.9)

The total kinetic energy then evolves as

$$\frac{\partial W_K}{\partial t} = -\int \hat{n} \cdot \left[ \left(\frac{1}{2}m_i n v^2\right)v + \mathbf{\Pi} \cdot \mathbf{v} \right] dA + \int \left[ \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) - \mathbf{v} \cdot \nabla p + \mathbf{\Pi} : (\nabla \mathbf{v})^T \right] d^3x$$

$$+ \int \left[ \left(\frac{1}{2}m_i v^2\right) \nabla \cdot (D_n \nabla n) + \mathbf{v} \cdot \mathbf{F} + \left(\frac{1}{2}m_i v^2\right) N \right] d^3x$$

(B.10)

### B.3 Internal Energy

The species internal energy is defined to be

$$W_{p,s} \equiv \int \frac{p_s}{\Gamma_s - 1} d^3x$$

(B.11)

The evolution of the internal energy density then

$$\frac{\partial w_{p,s}}{\partial t} = \frac{1}{\Gamma_s - 1} \frac{\partial p_s}{\partial t} = \frac{1}{\Gamma_s - 1} \frac{\partial}{\partial t} (n_s T_s) = \frac{n_s}{\Gamma_s - 1} \frac{\partial T_s}{\partial t} + \frac{T_s}{\Gamma_s - 1} \frac{\partial n_s}{\partial t}$$

(B.12)
Recalling the density evolution equation used in NIMROD it follows that

\[
\frac{T_s}{\Gamma_s - 1} \frac{\partial n_s}{\partial t} = \frac{T_s}{\Gamma_s - 1} \left[ - \nabla \cdot (n_s v_s) + \nabla \cdot (D_n \nabla n_s) + N_s \right]
\]

\[
= - \nabla \cdot \left( \frac{n_s T_s}{\Gamma_s - 1} v_s \right) + \frac{n_s}{\Gamma_s - 1} v_s \cdot \nabla T_s + \frac{T_s}{\Gamma_s - 1} \left[ \nabla \cdot (D_n \nabla n_s) + N_s \right]
\]  \hspace{1cm} (B.13)

The species temperature equation obeys

\[
\frac{n_s}{\Gamma_s - 1} \frac{\partial T_s}{\partial t} = - \frac{n_s}{\Gamma_s - 1} v_s \cdot \nabla T_s - n_s T_s \nabla \cdot v_s - \Pi_s : (\nabla v_s)^T - \nabla \cdot q_s + Q_s + Q_s \]  \hspace{1cm} (B.14)

The sum of these two is then

\[
\frac{n_s}{\Gamma_s - 1} \frac{\partial T_s}{\partial t} + \frac{T_s}{\Gamma_s - 1} \frac{\partial n_s}{\partial t} = - \nabla \cdot \left( \frac{n_s T_s}{\Gamma_s - 1} v_s \right) - n_s T_s \nabla \cdot v_s - \Pi_s : (\nabla v_s)^T - \nabla \cdot q_s
\]

\[
+ Q_s + Q_s + T_s \frac{T_s}{\Gamma_s - 1} \left[ \nabla \cdot (D_n \nabla n_s) + N_s \right]
\]

\[
= - \nabla \cdot \left( \frac{\Gamma_s}{\Gamma_s - 1} n_s T_s v_s \right) - v_s \cdot \nabla (n_s T_s) - \Pi_s : (\nabla v_s)^T - \nabla \cdot q_s
\]

\[
+ Q_s + Q_s + T_s \frac{T_s}{\Gamma_s - 1} \left[ \nabla \cdot (D_n \nabla n_s) + N_s \right]
\]  \hspace{1cm} (B.15)

The evolution of the species’ internal energy is then

\[
\frac{\partial W_{p,s}}{\partial t} = - \int \frac{\Gamma_s}{\Gamma_s - 1} n_s T_s v_s \cdot \hat{n} dA + \int \left[ v_s \cdot \nabla (n_s T_s) - \Pi_s : (\nabla v_s)^T - \nabla \cdot q_s + Q_s \right] d^3 x
\]

\[
+ \int \left[ Q_s + T_s \frac{T_s}{\Gamma_s - 1} \left[ \nabla \cdot (D_n \nabla n_s) + N_s \right] \right] d^3 x
\]  \hspace{1cm} (B.16)

**B.3.1 Two Temperature Modeling**

Recall that the the \( v_s \) in terms of \( v \) and \( J \) are:

\[
v_i = v + \left[ \frac{m_e}{m_i} \right] \frac{1}{ne} J \equiv v + \frac{\alpha_i}{ne} J \quad v_e = v - \left[ \frac{1}{1 + m_e/m_i} \right] \frac{1}{ne} J = v - \frac{\alpha_e}{ne} J
\]  \hspace{1cm} (B.17)
where the \( \alpha \) factors have been defined for convenience. Then neglecting the anomalous terms for simplicity:

\[
\frac{\partial W_{p,i}}{\partial t} = - \int \frac{\Gamma_i}{\Gamma_i - 1} nT_i \mathbf{v}_i \cdot \hat{n} dA + \int \left[ \mathbf{v}_i \cdot \nabla (nT_i) - \nabla \cdot (\nabla \mathbf{v}_i)^T - \nabla \cdot \mathbf{q}_i + Q_i \right] d^3 x
\]

\[
= - \int \frac{\Gamma_i}{\Gamma_i - 1} nT_i \left( \mathbf{v} + \frac{\alpha_i}{n_e} \mathbf{J} \right) \cdot \hat{n} dA
\]

\[
+ \int \left[ \left( \mathbf{v} + \frac{\alpha_i}{n_e} \mathbf{J} \right) \cdot \nabla (nT_i) - \nabla \cdot \left( \nabla \left( \mathbf{v} + \frac{\alpha_i}{n_e} \mathbf{J} \right) \right)^T - \nabla \cdot \mathbf{q}_i + Q_i \right] d^3 x
\]

(B.18)

and the electron internal energy becomes

\[
\frac{\partial W_{p,e}}{\partial t} = - \int \frac{\Gamma_e}{\Gamma_e - 1} nT_e \left( \mathbf{v} - \frac{\alpha_e}{n_e} \mathbf{J} \right) \cdot \hat{n} dA
\]

\[
+ \int \left[ \left( \mathbf{v} - \frac{\alpha_e}{n_e} \mathbf{J} \right) \cdot \nabla (nT_e) - \nabla \cdot \left( \nabla \left( \mathbf{v} - \frac{\alpha_e}{n_e} \mathbf{J} \right) \right)^T - \nabla \cdot \mathbf{q}_e + Q_e \right] d^3 x
\]

(B.19)

In the limit of small electron mass, \( \alpha_i \to 0 \) and \( \alpha_e \to 1 \), so that the sum of these two equations becomes

\[
\frac{\partial}{\partial t} (W_{p,i} + W_{p,e}) = - \int \left[ \left( \frac{\Gamma_i}{\Gamma_i - 1} p_i + \frac{\Gamma_e}{\Gamma_e - 1} p_e \right) \mathbf{v} - \frac{\Gamma_e}{\Gamma_e - 1} p_e \left( \frac{1}{n_e} \mathbf{J} \right) \right] \cdot \hat{n} dA
\]

\[
+ \int \left[ \mathbf{v} \cdot \nabla (p_i + p_e) - \frac{1}{n_e} \mathbf{J} \cdot \nabla p_e - \nabla \cdot \left( \nabla \mathbf{v} \right)^T - \nabla \cdot \left( \nabla \left( \mathbf{v} - \frac{1}{n_e} \mathbf{J} \right) \right)^T - \nabla \cdot \mathbf{q} + Q \right] d^3 x
\]

(B.20)

where \( \mathbf{q} = \mathbf{q}_i + \mathbf{q}_e \) and \( Q = Q_i + Q_e \). The \( \mathbf{J} \cdot \nabla p_e/n_e \) term exactly balances a similar term in the magnetic energy evolution. It is missed in our single-temperature evolution, as will be seen next. Also, there are \( \mathcal{O} \left( \frac{m_e}{m_i} \right) \) corrections to this equation that are not listed here, although they are present in NIMROD calculations with separate species’ temperatures.

### B.3.2 Single Temperature Modeling

Assume that collisions between species are sufficiently frequent to keep the species in thermal equilibrium with each other. Then the electron and ion species can be modeled as being at the same temperature. Each individual species pressure is then just half of the total pressure:

\[
p = \sum_s p_s = p_i + p_e = nT_i + nT_e = n(T_i + T_e) = 2nT
\]

(B.21)
for \( T \equiv T_i \). Only a single temperature equation is advanced:

\[
\frac{n}{\Gamma - 1} \left[ \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right] = -nT \mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{2} \Pi_s : (\nabla \mathbf{v}_s)^T - \nabla \cdot \mathbf{q} + \frac{1}{2} Q + \frac{1}{2} \mathcal{Q} \tag{B.22}
\]

The factors of \( \frac{1}{2} \) represent the ion fraction of the total energy and are included in NIMROD. Here the anomalous heat source term is given by

\[
\mathcal{Q} = -Q_{eq} + \frac{2n_{eq}}{\Gamma - 1} \mathbf{v}_{eq} \cdot \nabla T_{eq} + 2n_{eq} T_{eq} \mathbf{v}_{eq} + \Pi_{s,eq} : \nabla \mathbf{v}_{eq} + 2 \nabla \cdot \mathbf{q}_{eq} \tag{B.23}
\]

In this limit, the total internal energy is \( \frac{\partial W_p}{\partial t} = 2 \frac{\partial W_{p,s}}{\partial t} \) so that

\[
\frac{\partial W_p}{\partial t} = -\frac{\Gamma}{\Gamma - 1} \int (2nT) \mathbf{v} \cdot \hat{n} dA - \int 2\mathbf{q} \cdot \hat{n} dA + \int [\mathbf{v} \cdot \nabla (2nT) - \Pi_s : \nabla \mathbf{v} + \mathcal{Q}] d^3x \tag{B.24}
\]

\[
+ \int \left[ \mathcal{Q} + \frac{2T}{\Gamma - 1} [\nabla \cdot (D_n \nabla n) + \mathcal{N}] \right] d^3x \tag{B.25}
\]

### B.4 Total Energy of NIMROD Equations

Then we have the full system of equations being

\[
\frac{\partial W_B}{\partial t} = -\int \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \cdot \hat{n} dA + \int \left\{ -\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) + \Lambda_e \left[ \frac{1}{ne} \mathbf{J} \cdot \nabla p_e + \frac{m_e}{ne^2} \mathbf{J} \cdot \frac{\partial \mathbf{J}}{\partial t} \right] - \eta \mathbf{J}^2 
\right.
\]

\[
\left. + \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} - \frac{\kappa}{\mu_0} (\nabla \cdot \mathbf{B})^2 \right\} d^3x \tag{B.26}
\]

\[
\frac{\partial W_K}{\partial t} = -\int \hat{n} \cdot \left[ \left( \frac{1}{2} m_i n v^2 \right) \mathbf{v} + \Pi_i : \mathbf{v} \right] dA + \int [\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) - \mathbf{v} \cdot \nabla p + \Pi_i : \nabla \mathbf{v}] d^3x
\]

\[
+ \int \left[ \left( \frac{1}{2} m_i v^2 \right) \nabla \cdot (D_n \nabla n) + \mathbf{v} \cdot \mathcal{F} + \left( \frac{1}{2} m_i v^2 \right) \mathcal{N} \right] d^3x \tag{B.27}
\]

\[
\frac{\partial W_p}{\partial t} = -\frac{\Gamma}{\Gamma - 1} \int (2nT) \mathbf{v} \cdot \hat{n} dA - \int 2\mathbf{q} \cdot \hat{n} dA + \int [\mathbf{v} \cdot \nabla (2nT) - \Pi_s : \nabla \mathbf{v} + \mathcal{Q}] d^3x
\]

\[
+ \int \left[ \mathcal{Q} + \frac{2T}{\Gamma - 1} [\nabla \cdot (D_n \nabla n) + \mathcal{N}] \right] d^3x \tag{B.28}
\]

The terms in pink are the artificial diffusion terms and are not present in the physical system. The terms \( \mathcal{N}, \mathcal{F}, \mathcal{E}, \) and \( \mathcal{Q} \) result from the separation into steady-state and perturbed quantities, which may require anomalous sourcing for steady-state balance. The assumption
of equal species temperatures results in the $\mathbf{J} \cdot \nabla p_e$ contribution from the electron pressure term becoming an anomalous source or sink of energy. If we further neglect the viscous stress tensor $\Pi_i : \nabla \mathbf{v}$ in the temperature equation, along with the heat source $Q$, then the viscous stress term in the kinetic energy evolution and the resistive term in the magnetic energy evolution become loss mechanisms as well.
C EVOLUTION OF HELICITIES IN THE NIMROD SYSTEM

C.1 Magnetic Helicity

Recall the relative magnetic helicity is defined to be

\[ K_{\text{rel}} \equiv \int (A - A') \cdot (B + B') \, d^3x \equiv \int A_+ \cdot B_+ d^3x \]  

(C.1)

and the temporal evolution is given by

\[ \frac{\partial K_{\text{rel}}}{\partial t} = \int \left[ \frac{\partial A_-}{\partial t} \cdot B_+ + A_- \cdot \frac{\partial B_+}{\partial t} \right] d^3x. \]  

(C.2)

Note that the vector potential \( A \) is not a field that is solved for in NIMROD. It is constructed by solving the equations

\[ B = \nabla \times A \quad \text{and} \quad \nabla \cdot A = 0 \]  

(C.3)

at each point in time for a NIMROD simulation, using the full \( B \), with no separation into equilibrium and time-evolving fields, as described in Appendix ??

Using \( E = -\frac{\partial}{\partial t} A - \nabla \phi \), where \( E \) represents the total electric field, not just the equilibrium or time-evolving part, this becomes

\[ \frac{dK_{\text{rel}}}{dt} = \int \left[ (-E_- - \nabla \phi_-) \cdot B_+ - A_- \cdot \nabla \times E_+ + A_- \cdot \kappa \nabla (\nabla \cdot B_+) + A_- \cdot \nabla \times \mathcal{E} \right] d^3x \]

\[ = -\int E_- \cdot B_+ d^3x - \int [\nabla \cdot (\phi_- B_+) - \phi_- (\nabla \cdot B_+)] d^3x \]

\[ + \int [\nabla \cdot (A_- \times E_+) \cdot E_+ - \nabla \times A_-] d^3x \]

\[ + \int \left\{ \nabla \cdot [\kappa (\nabla \cdot B_+) A_-] - \kappa (\nabla \cdot B_+) (\nabla \cdot A_-) \right\} d^3x \]

(C.4)

[42] notes that the reference electric field is defined up to an arbitrary gauge freedom \( E' \rightarrow E' + \nabla \psi \). Consequently, we can choose the gauge for the reference field to be such that the \( \phi_- \) term vanishes everywhere and the tangential components of \( A_- \) drop out on the boundary. Using \( A' \cdot \hat{n} = 0, \nabla \cdot A' = 0, \) and \( \nabla \cdot B' = 0 \) identically, this becomes

\[ \frac{dK_{\text{rel}}}{dt} = -\int [E_- \cdot B_+ + E_+ \cdot B_-] d^3x - \int [\kappa A \cdot \nabla (\nabla \cdot B) + (A - A') \cdot \nabla \times \mathcal{E}] d^3x. \]  

(C.5)
The first set of terms can be rearranged as

$$E_\perp \cdot B_+ + E_+ \cdot B_\perp = (E - E') \cdot (B + B') + (E + E') \cdot (B - B') = 2 [E \cdot B - E' \cdot B'] .$$

(C.6)

Noting that there is no evolution of the reference magnetic field, $\nabla \times E' = 0$, the relative helicity evolution can be written as

$$\frac{dK_{rel}}{dt} = 2 \int [\nabla \cdot (A' \times E') - E \cdot B] d^3x$$
$$+ \int [\kappa A \cdot \nabla (\nabla \cdot B) + (A - A') \cdot \nabla \times E] d^3x .$$

(C.7)

### C.2 Cross Helicity

The cross-helicity,

$$\mathcal{X} = \int v \cdot B d^3x , $$

(C.8)

evolves as

$$\frac{\partial \mathcal{X}}{\partial t} = \int \left( \frac{\partial v}{\partial t} \cdot B + v \cdot \frac{\partial B}{\partial t} \right) d^3x .$$

(C.9)

In NIMROD, this is

$$\frac{\partial \mathcal{X}}{\partial t} = \int \left\{ -v \cdot \nabla v \cdot B + \frac{\mathcal{F}}{m_i n} \cdot B + v \cdot [-\nabla \times E + \nabla \times \mathcal{E} + \kappa \nabla (\nabla \cdot B)] \right\} d^3x .$$

(C.10)

The first term on the right can be rearranged as

$$(v \cdot \nabla v) \cdot B = \left[ \nabla \left( \frac{v^2}{2} \right) - v \times (\nabla \times v) \right] \cdot B$$
$$= \nabla \cdot \left( \frac{v^2}{2} B \right) - \left( \frac{v^2}{2} \right) \nabla \cdot B - [v \times (\nabla \times v)] \cdot B$$
$$= \nabla \cdot \left( \frac{v^2}{2} B \right) + (v \times B) \cdot \nabla \times v .$$

(C.11)
Then, using $\mathbf{v} \cdot \nabla \times \mathbf{E} = \nabla \cdot (\mathbf{E} \times \mathbf{v}) + \mathbf{E} \cdot \nabla \times \mathbf{v}$ and rearranging, we find

$$
\frac{\partial X}{\partial t} = - \int \nabla \cdot \left( \frac{\mathbf{v}^2}{2} \mathbf{B} + \mathbf{E} \times \mathbf{v} \right) d^3 x + \int \left[ - (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla \times \mathbf{v} + \frac{\mathbf{F}}{m_i} \cdot \mathbf{B} \right] d^3 x \\
+ \int \left\{ \frac{\mathbf{F}}{m_i} \cdot \mathbf{B} + \mathbf{v} \cdot [\nabla \times \mathbf{E} + \kappa \nabla \cdot (\nabla \cdot \mathbf{B})] \right\} d^3 x
$$

(C.12)

### C.3 Kinetic Helicity

The kinetic helicity,

$$
\mathcal{H} \equiv \int \mathbf{v} \cdot \nabla \times \mathbf{v} d^3 x,
$$

(C.13)

evolves as

$$
\frac{\partial \mathcal{H}}{\partial t} = \int \left[ \frac{\partial \mathbf{v}}{\partial t} \cdot \nabla \times \mathbf{v} + \mathbf{v} \cdot \frac{\partial}{\partial t} (\nabla \times \mathbf{v}) \right] d^3 x \\
= \int \left[ 2 \frac{\partial \mathbf{v}}{\partial t} \cdot \nabla \times \mathbf{v} - \nabla \cdot \left( \mathbf{v} \times \frac{\partial \mathbf{v}}{\partial t} \right) \right] d^3 x.
$$

(C.14)

In NIMROD, this becomes

$$
\frac{\partial \mathcal{H}}{\partial t} = - \int \nabla \cdot \left( \mathbf{v} \times \frac{\partial \mathbf{v}}{\partial t} \right) d^3 x + 2 \int \left[ - (\mathbf{v} \cdot \nabla) \cdot \nabla \times \mathbf{v} + \frac{\mathbf{F} + \mathcal{F}}{m_i} \cdot \nabla \times \mathbf{v} \right] d^3 x.
$$

(C.15)

Note

$$
[\mathbf{v} \cdot \nabla \mathbf{v}] \cdot \nabla \times \mathbf{v} = \left[ \nabla \left( \frac{\mathbf{v}^2}{2} \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) \right] \cdot \nabla \times \mathbf{v} = (\nabla \times \mathbf{v}) \cdot \nabla \left( \frac{\mathbf{v}^2}{2} \right) = \nabla \cdot \left( \frac{\mathbf{v}^2}{2} \nabla \times \mathbf{v} \right) - \frac{\mathbf{v}^2}{2} \nabla \cdot (\nabla \times \mathbf{v}) = \nabla \cdot \left( \frac{\mathbf{v}^2}{2} \nabla \times \mathbf{v} \right).
$$

(C.16)

Then

$$
\frac{\partial \mathcal{H}}{\partial t} = - \int \nabla \cdot \left( \mathbf{v} \times \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}^2 \nabla \times \mathbf{v} \right) d^3 x + 2 \int \frac{\mathbf{F} + \mathcal{F}}{m_i} \cdot \nabla \times \mathbf{v} d^3 x.
$$

(C.17)
We use mean-field analysis to construct the evolution of the magnetic and kinetic energies associated with individual spectral components. Quantities will be decomposed as

\[ A = \langle A \rangle + \tilde{A} \]
\[ \tilde{A} = \sum_{m,n} \tilde{A}_{mn}. \]  

(D.1)

In what follows, we will ignore the electron inertia and terms that are associated with the anomalous forcing in the NIMROD system. Consider the equations:

\[ m_i \frac{\partial v}{\partial t} = -m_i \mathbf{n} \cdot \nabla v + \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi}, \]  

(D.2)

\[ E = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} + \Lambda_e \mathbf{J} \times \mathbf{B} - \nabla p, \]  

(D.3)

The axisymmetric parts of these are given by

\[ \langle m_i \frac{\partial v}{\partial t} \rangle = -\langle m_i \mathbf{n} \cdot \nabla v \rangle + \langle \mathbf{J} \times \mathbf{B} \rangle - \langle \nabla p \rangle - \langle \nabla \cdot \mathbf{\Pi} \rangle, \]  

(D.4)

\[ \langle E \rangle = -\langle \mathbf{v} \times \mathbf{B} \rangle + \eta \langle \mathbf{J} \rangle + \Lambda_e \mathbf{J} \times \mathbf{B} - \Lambda_e \nabla p, \]  

(D.5)

and the fluctuating parts are given by

\[ \left( m_i \frac{\partial v}{\partial t} \right)_{mn} = -\left( m_i \mathbf{n} \cdot \nabla v \right)_{mn} + \left( \mathbf{J} \times \mathbf{B} \right)_{mn} - \left( \nabla p \right)_{mn} - \left( \nabla \cdot \mathbf{\Pi} \right)_{mn}, \]  

(D.6)

\[ \tilde{E}_{mn} = -\left( \mathbf{v} \times \mathbf{B} \right)_{mn} + \eta \mathbf{J}_{mn} + \Lambda_e \left( \frac{1}{ne} \left( \mathbf{J} \times \mathbf{B} \right) \right)_{mn} - \Lambda_e \left( \frac{1}{ne} \nabla p \right)_{mn}. \]  

(D.7)

The evolution of kinetic energy associated with the axisymmetric component of the flow can be found by dotting the axisymmetric momentum equation into \( \langle \mathbf{v} \rangle \):

\[ \left( m_i \frac{\partial \mathbf{v}}{\partial t} \right) \cdot \langle \mathbf{v} \rangle = -\left( m_i \mathbf{n} \cdot \nabla \mathbf{v} \right) \cdot \langle \mathbf{v} \rangle + \langle \mathbf{J} \times \mathbf{B} \rangle \cdot \langle \mathbf{v} \rangle - \langle \nabla p \rangle \cdot \langle \mathbf{v} \rangle - \langle \mathbf{v} \rangle \cdot \langle \nabla \cdot \mathbf{\Pi} \rangle. \]  

(D.8)
Note \( \langle n \frac{\partial \bar{v}}{\partial t} \rangle = \langle n \rangle \frac{\partial \langle \bar{v} \rangle}{\partial t} + \langle \bar{n} \frac{\partial \bar{v}}{\partial t} \rangle \) so

\[
\left\langle m_i n \frac{\partial \bar{v}}{\partial t} \right\rangle \cdot \langle \bar{v} \rangle = m_i \langle n \rangle \frac{\partial \langle \bar{v} \rangle}{\partial t} + m_i \left\langle \bar{n} \frac{\partial \bar{v}}{\partial t} \right\rangle \cdot \langle \bar{v} \rangle
\]

\[
= \frac{\partial}{\partial t} \left( \frac{1}{2} m_i \langle n \rangle \langle \bar{v} \rangle \cdot \langle \bar{v} \rangle \right) - \left( \frac{1}{2} m_i \langle \bar{v} \rangle \cdot \langle \bar{v} \rangle \right) \frac{\partial \langle n \rangle}{\partial t} + m_i \left\langle \bar{n} \frac{\partial \bar{v}}{\partial t} \right\rangle \cdot \langle \bar{v} \rangle.
\]

(D.9)

Then rearranging terms we have:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} m_i \langle n \rangle \langle \bar{v} \rangle \cdot \langle \bar{v} \rangle \right) = \left( \frac{1}{2} m_i \langle \bar{v} \rangle \cdot \langle \bar{v} \rangle \right) \frac{\partial \langle n \rangle}{\partial t} - m_i \langle \bar{n} \frac{\partial \bar{v}}{\partial t} \rangle \cdot \langle \bar{v} \rangle - \langle m_i n \bar{v} \cdot \nabla \bar{v} \rangle \cdot \langle \bar{v} \rangle + \langle J \times B \rangle \cdot \langle \bar{v} \rangle - \langle \bar{v} \rangle \cdot \langle \nabla p \rangle - \langle \bar{v} \rangle \cdot \langle \nabla \cdot \Pi_i \rangle.
\]

(D.10)

The evolution of kinetic energy associated with the \( m, n \) fluctuating component can be found by dotting the fluctuating momentum equation into \( \tilde{v}_{mn} \):

\[
\left( m_i \frac{\partial \bar{v}}{\partial t} \right)_{mn} \cdot \tilde{v}_{mn} = - \left( m_i n \bar{v} \cdot \nabla \bar{v} \right)_{mn} \cdot \tilde{v}_{mn} + (J \times B)_{mn} \cdot \tilde{v}_{mn} - \tilde{v}_{mn} \cdot (\nabla p)_{mn} - \tilde{v}_{mn} \cdot (\nabla \cdot \Pi_i)_{mn}.
\]

(D.11)

Note that there are several components of the LHS,

\[
\left( n \frac{\partial \bar{v}}{\partial t} \right)_{mn} = \langle n \rangle \frac{\partial \tilde{v}_{mn}}{\partial t} + \tilde{n}_{mn} \frac{\partial \langle \bar{v} \rangle}{\partial t} + \left( \tilde{n} \frac{\partial \bar{v}}{\partial t} \right)_{mn},
\]

(D.12)

so after some rearranging we obtain

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} m_i \langle n \rangle \tilde{v}_{mn} \cdot \tilde{v}_{mn} \right) = \left( \frac{1}{2} m_i \tilde{v}_{mn} \cdot \tilde{v}_{mn} \right) \frac{\partial \langle n \rangle}{\partial t} - m_i \tilde{n}_{mn} \frac{\partial \langle \bar{v} \rangle}{\partial t} \cdot \tilde{v}_{mn} - m_i \left( \tilde{n} \frac{\partial \bar{v}}{\partial t} \right)_{mn} \cdot \tilde{v}_{mn} - \left( m_i n \bar{v} \cdot \nabla \bar{v} \right)_{mn} \cdot \tilde{v}_{mn} + (J \times B)_{mn} \cdot \tilde{v}_{mn} - \tilde{v}_{mn} \cdot (\nabla p)_{mn} - \tilde{v}_{mn} \cdot (\nabla \cdot \Pi_i)_{mn}.
\]

(D.13)

Dotting Faraday’s law with \( \langle B \rangle \) yields an equation for the evolution of magnetic energy associated with the axisymmetric fields:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2\mu_0} \langle B \rangle \cdot \langle B \rangle \right) = -\nabla \cdot \left( \frac{1}{\mu_0} \langle E \rangle \times \langle B \rangle \right) - \langle E \rangle \cdot \langle J \rangle.
\]

(D.14)
Using the equation for \( \langle E \rangle \) in the second term above we find:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2 \mu_0} \langle B \rangle \cdot \langle B \rangle \right) = - \nabla \cdot \left( \frac{1}{\mu_0} \langle E \rangle \times \langle B \rangle \right) + \langle v \times B \rangle \cdot \langle J \rangle - \eta \langle J \rangle \cdot \langle J \rangle - \frac{\Lambda_e}{n e} \langle J \rangle \cdot \langle J \rangle + \frac{1}{n e} \nabla p_e
\]

(D.15)

The evolution of magnetic energy associated with the \( m, n \) spectral component immediately follows:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2 \mu_0} \tilde{B}_{mn} \cdot \tilde{B}_{mn} \right) = - \nabla \cdot \left( \frac{1}{\mu_0} \tilde{E}_{mn} \times \tilde{B}_{mn} \right) + (\tilde{v} \times \tilde{B})_{mn} \cdot \tilde{J}_{mn} - \eta \tilde{J}_{mn} \cdot \tilde{J}_{mn} - \frac{\Lambda_e}{n e} \tilde{J}_{mn} \cdot \tilde{J}_{mn} + \frac{1}{n e} \nabla p_e
\]

(D.16)

Note that we can further decompose the quadratic terms in these equations based on their own axisymmetric and fluctuating representations. In the axisymmetric equations, we may expand terms as

\[
\langle v \times B \rangle \cdot \langle J \rangle = \left( \langle v \rangle + \tilde{v} \right) \times \left( \langle B \rangle + \tilde{B} \right) \cdot \langle J \rangle = \langle v \rangle \times \langle B \rangle \cdot \langle J \rangle + \langle \tilde{v} \times \tilde{B} \rangle \cdot \langle J \rangle.
\]

(D.17)

Similarly, in the equations for the fluctuations we may expand terms as

\[
(\tilde{v} \times \tilde{B})_{mn} \cdot \tilde{J}_{mn} = \left( \langle \tilde{v} \rangle \times \langle \tilde{B} \rangle \right)_{mn} \cdot \tilde{J}_{mn} + \tilde{v} \times \tilde{B} \cdot \langle \tilde{J} \rangle + \tilde{v} \times \tilde{B} \cdot \langle \tilde{J} \rangle
\]

(D.18)

Using this decomposition in our equations, we find that the kinetic energy associated with axisymmetric flows evolves as

\[
\frac{\partial}{\partial t} \left( \frac{m_i \langle n \rangle}{2} |\langle v \rangle|^2 \right) = \left( \frac{1}{2} m_i \langle v \rangle \cdot \langle v \rangle \right) \frac{\partial \langle n \rangle}{\partial t} - m_i \left( \hat{n} \cdot \frac{\partial \vec{v}}{\partial t} \right) \cdot \langle v \rangle + \langle J \rangle \times \langle B \rangle \cdot \langle v \rangle + \langle \vec{J} \times \vec{B} \rangle \cdot \langle v \rangle - m_i n \langle v \rangle \cdot \nabla p - \langle v \rangle \cdot \nabla \langle p \rangle - \langle v \rangle \cdot \nabla \langle \Pi_i \rangle.
\]

(D.19)
and the kinetic energy associated with the fluctuations evolves as

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{m_i}{2} \langle n \rangle |\tilde{v}_{mn}|^2 \right) &= \left( \frac{1}{2} m_i \tilde{v}_{mn} \cdot \tilde{v}_{mn} \right) \frac{\partial \langle n \rangle}{\partial t} - m_i \tilde{n}_{mn} \frac{\partial \langle v \rangle}{\partial t} \cdot \tilde{v}_{mn} - m_i \left( \tilde{n} \frac{\partial \tilde{v}}{\partial t} \right)_{mn} \cdot \tilde{v}_{mn} \\
&+ \tilde{J}_{mn} \times \langle B \rangle \cdot \tilde{v}_{mn} + \langle J \rangle \times \tilde{B}_{mn} \cdot \tilde{v}_{mn} + \left( \tilde{J} \times \tilde{B} \right)_{mn} \cdot \tilde{v}_{mn} \\
&- (m_i n v \cdot \nabla \tilde{v})_{mn} \cdot \tilde{v}_{mn} - \tilde{v}_{mn} \cdot (\nabla \tilde{p})_{mn} - \tilde{v}_{mn} \cdot (\nabla \cdot \Pi)_{mn}.
\end{align*}
\]  
(D.20)

Similarly, the magnetic energy associated with the axisymmetric field evolves as

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{1}{2 \mu_0} |\langle B \rangle|^2 \right) &= -\nabla \cdot \left( \frac{\langle E \rangle \times \langle B \rangle}{\mu_0} \right) - \eta |\langle J \rangle|^2 \\
&+ \langle v \rangle \times \langle B \rangle \cdot \langle J \rangle + \langle \tilde{v} \times \tilde{B} \rangle \cdot \langle J \rangle - \Lambda_e \langle \frac{J}{n} e \rangle \cdot \langle \tilde{J} \times \tilde{B} \rangle,
\end{align*}
\]  
(D.21)

and the energy associated with the fluctuating fields evolves as

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{1}{2 \mu_0} |\tilde{B}_{mn}|^2 \right) &= -\nabla \cdot \left( \frac{\tilde{E}_{mn} \times \tilde{B}_{mn}}{\mu_0} \right) - \eta |\tilde{J}_{mn}|^2 + \tilde{J}_{mn} \cdot \left( \frac{1}{ne} \nabla \tilde{p}_e \right)_{mn} \\
&+ \tilde{v}_{mn} \times \langle B \rangle \cdot \tilde{J}_{mn} + \langle v \rangle \times \tilde{B}_{mn} \cdot \tilde{J}_{mn} + \left( \tilde{v} \times \tilde{B} \right)_{mn} \cdot \tilde{J}_{mn} \\
&+ \Lambda_e \langle \frac{J}{n} e \rangle \cdot \tilde{J}_{mn} \times \tilde{B}_{mn} - \Lambda_e \tilde{J}_{mn} \langle \frac{J}{n} e \rangle \cdot \langle \tilde{J} \times \tilde{B} \rangle_{mn}.
\end{align*}
\]  
(D.22)

The color coding has been used to highlight the coupling.
E CONSTRUCTION OF THE VECTOR POTENTIAL

Here, we explain how the magnetic vector potential $\mathbf{A}$ is constructed within the NIMROD framework. Consider the variational problem

$$F[\mathbf{A}] = \int \left\{ \frac{\kappa A}{2} (\nabla \cdot \mathbf{A})^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - (\nabla \times \mathbf{A}) \cdot \mathbf{B} \right\} d^3 x$$

(E.1)

where we take $\mathbf{B}$ as a fixed source term. Note that the boundary conditions on $\mathbf{A}$ are fixed so that we must have all variations being zero on the boundary: i.e. $\delta \mathbf{A} = 0$ on the boundary. The variation is:

$$\delta F = \int \left\{ \kappa A (\nabla \cdot \mathbf{A}) \cdot (\nabla \cdot \delta \mathbf{A}) + (\nabla \times \delta \mathbf{A}) \cdot (\nabla \times \mathbf{A}) - (\nabla \times \delta \mathbf{A}) \cdot \mathbf{B} \right\} d^3 x$$

$$= \int \left\{ \kappa A [\nabla \cdot ((\nabla \cdot \mathbf{A}) \delta \mathbf{A}) - \delta \mathbf{A} \cdot \nabla (\nabla \cdot \mathbf{A})] + [\nabla \cdot (\delta \mathbf{A} \times (\nabla \times \mathbf{A} - \mathbf{B})) + \delta \mathbf{A} \cdot \nabla \times (\nabla \times \mathbf{A} - \mathbf{B})] \right\} d^3 x$$

(E.2)

$$= \int \delta \mathbf{A} \cdot [-\kappa A \nabla (\nabla \cdot \mathbf{A}) + \nabla \times (\nabla \times \mathbf{A} - \mathbf{B})] d^3 x.$$

The boundary terms vanish due to the requirement that $\delta \mathbf{A}$ vanish on the boundary. The minimum is found by setting $\delta F = 0$ and must be independent of $\delta \mathbf{A}$. Then we must have:

$$-\kappa A \nabla (\nabla \cdot \mathbf{A}) + \nabla \times (\nabla \times \mathbf{A} - \mathbf{B}) = 0.$$  

(E.3)

Clearly one solution of this equation is given by:

$$\nabla \times \mathbf{A} = \mathbf{B} \quad \quad \nabla \cdot \mathbf{A} = 0.$$  

(E.4)

We can find a vector potential that satisfies the above from the steady-state solution of an artificial time-dependent problem:

$$\frac{\partial \mathbf{A}}{\partial t} = \kappa A \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A} - \mathbf{B})$$

(E.5)

where again we have $\mathbf{B}$ a fixed source term. This equation is solved in NIMROD by noting
that it is very similar to the already implemented MHD advance with $v = 0$:

$$\frac{\partial \mathbf{B}}{\partial t} = \kappa \nabla \cdot \mathbf{B} \nabla (\nabla \cdot \mathbf{B}) - \nabla \times \left( -\mathbf{v} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B} + \mathbf{E}_{\text{applied}} \right)$$  \hspace{1cm} (E.6)

$$= \kappa \nabla \cdot \mathbf{B} \nabla (\nabla \cdot \mathbf{B}) - \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} + \mathbf{E}_{\text{applied}} \right).$$  \hspace{1cm} (E.7)

A modified version of NIMROD is written to advance only this time-dependent equation. The procedure is outlined below.

We first store the perturbed magnetic field $\tilde{\mathbf{B}}$ in the perturbed velocity $\tilde{\mathbf{v}}$. Then $\tilde{\mathbf{B}}$ is set to zero, although in principal this is not necessary, as any initial condition should approach the same-steady state solution (assuming only one solution). The brhs.mhd routine in integrands.f is modified to solve a new advance equation with the explicit electric field constructed as

$$\mathbf{E} = \text{elecd} \left[ \nabla \times \tilde{\mathbf{B}} - \left( \mathbf{B}_{\text{eq}} + \tilde{\mathbf{v}} \right) \right].$$  \hspace{1cm} (E.8)

Then the full advance looks like

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \kappa \nabla \cdot \mathbf{B} \nabla (\nabla \cdot \tilde{\mathbf{B}}) - \nabla \times \left( \text{elecd} \left[ \nabla \times \tilde{\mathbf{B}} - \left( \mathbf{B}_{\text{eq}} + \tilde{\mathbf{v}} \right) \right] \right).$$  \hspace{1cm} (E.9)

Making note of the fact that we have transformed as

$$\tilde{\mathbf{B}} \rightarrow \mathbf{A} \hspace{1cm} \tilde{\mathbf{v}} \rightarrow \tilde{\mathbf{B}} \hspace{1cm} \mathbf{B}_{\text{eq}} \rightarrow \mathbf{B}_{\text{eq}}$$  \hspace{1cm} (E.10)

the equation we are solving is:

$$\frac{\partial \mathbf{A}}{\partial t} = \kappa \nabla \cdot \mathbf{B} \nabla (\nabla \cdot \mathbf{A}) - \nabla \times \left( \text{elecd} \left[ \nabla \times \mathbf{A} - \left( \mathbf{B}_{\text{eq}} + \tilde{\mathbf{B}} \right) \right] \right).$$  \hspace{1cm} (E.11)
Note that in the construction of $A$ used in NIMROD, a vector potential in the Coulomb gauge is obtained, that is $\nabla \cdot A = 0$. However, after obtaining this vector potential, any other allowable gauge transformation can be performed $A \rightarrow A + \nabla \chi$. Such transformations may change the magnetic helicity, which is undesirable if the helicity is to be a physically meaningful quantity. Here we discuss definitions of the magnetic helicity that eliminate the gauge variant terms and are thus physically meaningful.

### F.1 Change in Magnetic Helicity with Gauge Transform

In order for the magnetic helicity to be a physically meaningful construct, it must be invariant to the choice of gauge. Under a gauge transformation the helicity behaves as:

$$\int A \cdot B d^3x \rightarrow \int A \cdot B d^3x + \int \nabla \chi \cdot B d^3x \equiv K + \Delta K.$$  \hfill (F.1)

Note that

$$\Delta K = \int \nabla \cdot (\chi B) d^3x = \int \nabla \cdot (\chi \nabla \times A) d^3x = \int \nabla \cdot [\nabla \times (\chi A) + A \times \nabla \chi] d^3x$$
$$= \int \nabla \cdot (A \times \nabla \chi) d^3x = \int \langle A \times \nabla \chi \rangle \cdot \hat{n} dA$$ \hfill (F.2)

The last step follows from Stokes’ theorem, which requires a simply connected domain for validity. In the case of a torus or a periodic cylinder, a cut must be introduced in the domain at a fixed axial position or toroidal angle. Then

$$\Delta K = \int [A (a, \theta, \zeta) \times \nabla \chi (a, \theta, \zeta)] \cdot \nabla \rho J d\theta d\zeta$$
$$+ \int [A (\rho, \theta, 2\pi) \times \nabla \chi (\rho, \theta, 2\pi)] \cdot \nabla \zeta J d\rho d\theta - \int [A (\rho, \theta, 0) \times \nabla \chi (\rho, \theta, 0)] \cdot \nabla \zeta J d\rho d\theta$$ \hfill (F.3)

The endcap contributions vanish since $A$ and $\nabla \chi$ are both required to be periodic in $\zeta$, and

$$\Delta K = \int (A \times \nabla \chi) \cdot \nabla \rho J d\theta d\zeta.$$ \hfill (F.4)

A gauge-invariant definition of the magnetic helicity must eliminate this extraneous term.
F.2 Relative Helicity

According to Finn and Antonsen [42], a gauge-invariant definition of the magnetic helicity is:

\[ K_F = \int (A + A') \cdot (B - B') \, d^3x \]  \hspace{1cm} (F.5)

where

\[ B = \nabla \times A \quad B' = \nabla \times A' \quad \nabla \times B' = 0 \quad B \cdot \hat{n}|_{\text{boundary}} = B'|_{\text{boundary}} \]  \hspace{1cm} (F.6)

and the total toroidal and poloidal fluxes match.

Note that a gauge transformation \( A \rightarrow A + \nabla \chi \) and \( A' \rightarrow A' + \nabla \chi' \) leads to

\[ \Delta K_F = \int \left[ (A - A') \times \nabla (\chi + \chi') \right] \cdot \nabla \rho J \, d\theta d\zeta \]  \hspace{1cm} (F.7)

The normal components of magnetic field must be the same on the boundary

\[ \hat{n} \cdot \nabla \times (A - A') = \hat{n} \cdot (B - B') = 0 \quad \hat{n} \cdot \nabla \times [\nabla (\chi + \chi')] = 0 \]  \hspace{1cm} (F.8)

so that this integral can be transformed as (see Section F.3 for details)

\[ \Delta K_F = \int (A - A') \cdot d\ell^\theta \int \nabla (\chi + \chi') \cdot d\ell^\zeta - \int (A - A') \cdot d\ell^\zeta \int \nabla (\chi + \chi') \cdot d\ell^\theta \]  \hspace{1cm} (F.9)

The equality of the fluxes then removes the \( A - A' \) terms. Finn’s definition of relative helicity is gauge-invariant.

You [129] proposes a similar definition to Finn for the relative helicity in the interest of constructing a gauge-invariant quantity for the canonical species’ helicities. The relative helicity defined in that work is

\[ K_Y = \int (A - A') \cdot (B + B') \, d^3x \]  \hspace{1cm} (F.10)

where

\[ B = \nabla \times A \quad B' = \nabla \times A' \quad B \cdot \hat{n}|_{\text{boundary}} = B'|_{\text{boundary}} \]  \hspace{1cm} (F.11)

A gauge-transformation of this results in something similar to Finn’s \( \Delta K_F \) but not identical:

\[ \Delta K_Y = \int \left[ (A + A') \times \nabla (\chi - \chi') \right] \cdot \nabla \rho J \, d\theta d\zeta. \]  \hspace{1cm} (F.12)
This can not be converted into a double line integral as before since \( \hat{n} \cdot \nabla \times (A + A') \neq 0 \) in general. However, You requires that the tangential components of the actual field and the reference field match on the boundary: \( (A - A') \times \hat{n} = 0 \). This must hold even under gauge transformations so that \( \nabla (\chi - \chi') \times \hat{n} = 0 \). With this restriction, it clearly follows that \( \Delta K_Y = 0 \) and You’s definition is also gauge-invariant.

\section*{F.3 Transformation of the Surface Integral}

We wish to show that

\[
\int (A \times B) \cdot \hat{n} dA = \oint_P A \cdot d\ell \oint_T B \cdot d\ell - \oint_P B \cdot d\ell \oint_T A \cdot d\ell \tag{F.13}
\]

for arbitrary vector fields \( A \) and \( B \) that both satisfy \( \hat{n} \cdot \nabla \times A = 0 \) and \( \hat{n} \cdot \nabla \times B = 0 \). This means that both vector fields are surface gradients \( A_s = \nabla_s \chi \) and \( B_s = \nabla_s \phi \), where the subscript \( s \) refers only to the directions lying in the surface, not normal to it. (See Reiman [97] equation 15 or Hameiri [51] equation 7 for details.)

The most general form of \( \chi \) and \( \phi \) is:

\[
\chi = \alpha \theta + \beta \zeta + \Gamma (\theta, \zeta) \quad \quad \phi = \gamma \theta + \delta \zeta + \Delta (\theta, \zeta) \tag{F.14}
\]

with both \( \Gamma \) and \( \Delta \) periodic functions in \( \theta \) and \( \zeta \). Then the components of the fields that lie in the surface can be expressed as

\[
A_s = \left( \alpha + \frac{\partial \Gamma}{\partial \theta} \right) \nabla \theta + \left( \beta + \frac{\partial \Gamma}{\partial \zeta} \right) \nabla \zeta \quad \quad B_s = \left( \gamma + \frac{\partial \Delta}{\partial \theta} \right) \nabla \theta + \left( \delta + \frac{\partial \Delta}{\partial \zeta} \right) \nabla \zeta \tag{F.15}
\]

We have used equation A8 from Boozer [11] for the general definition of \( \nabla f \) in curvilinear coordinates. In general \( A = A_\rho \nabla \rho + A_s \) and \( B = B_\rho \nabla \rho + B_s \).

Then using the definition of the area element \( dA^1 = \frac{\partial x}{\partial x^2} \times \frac{\partial x}{\partial x^3} d\theta d\zeta = \nabla x^1 J dx^2 dx^3 \) we have

\[
\int (A \times B) \cdot \hat{n} dA = \int (A \times B) \cdot \nabla \rho J d\theta d\zeta
\]

\[
= \int (A_\rho \nabla \rho \times B_s) \cdot \nabla \rho J d\theta d\zeta + \int (B_\rho A_s \times \nabla \rho) \cdot \nabla \rho J d\theta d\zeta + \int (A_s \times B_s) \cdot \nabla \rho J d\theta d\zeta
\]

\[
= \int (A_s \times B_s) \cdot \nabla \rho J d\theta d\zeta.
\]
Using Eq. F.15, this becomes

\[ \int (A \times B) \cdot \hat{n} dA = \int \left[ \left( \alpha + \frac{\partial \Gamma}{\partial \theta} \right) \left( \delta + \frac{\partial \Delta}{\partial \zeta} \right) \right] \nabla \theta \times \nabla \zeta \cdot \nabla \rho J d\theta d\zeta \\
+ \int \left[ \left( \beta + \frac{\partial \Gamma}{\partial \zeta} \right) \left( \gamma + \frac{\partial \Delta}{\partial \theta} \right) \right] \nabla \zeta \times \nabla \theta \cdot \nabla \rho J d\theta d\zeta. \]

(F.17)

Recalling that \( J^{-1} = \nabla x^1 \cdot \nabla x^2 \times \nabla x^3 \) we find that

\[ \int (A \times B) \cdot \hat{n} dA = \int \left[ \left( \alpha + \frac{\partial \Gamma}{\partial \theta} \right) \left( \delta + \frac{\partial \Delta}{\partial \zeta} \right) - \left( \beta + \frac{\partial \Gamma}{\partial \zeta} \right) \left( \gamma + \frac{\partial \Delta}{\partial \theta} \right) \right] d\theta d\zeta \\
= \int \left[ \alpha \delta - \beta \gamma \right] + \left[ \frac{\partial \Delta}{\partial \zeta} + \frac{\partial \Gamma}{\partial \theta} \frac{\partial \Delta}{\partial \zeta} - \frac{\partial \Delta}{\partial \theta} \frac{\partial \Gamma}{\partial \zeta} \right] \right] d\theta d\zeta \\
= \int \left[ \alpha \delta - \beta \gamma \right] d\theta d\zeta + \int \left[ \frac{\partial \Gamma}{\partial \theta} \frac{\partial \Delta}{\partial \zeta} - \frac{\partial \Delta}{\partial \theta} \frac{\partial \Gamma}{\partial \zeta} \right] d\theta d\zeta. \]

(F.18)

The cross terms have vanished due to the periodicity of \( \Gamma \) and \( \Delta \). Noting that \( \alpha, \beta, \gamma, \) and \( \delta \) are independent of \( \theta \) and \( \zeta \) the first term above can be rewritten as

\[ \int [\alpha \delta - \beta \gamma] d\theta d\zeta = \int \alpha d\theta \int \delta d\zeta - \int \gamma d\theta \int \beta d\zeta. \]

(F.19)

We can add to the integrand any function which vanishes when integrated over the limits of the integral without changing the value of the integral: \( \int_0^{2\pi} ad\theta = \int_0^{2\pi} [a + \cos (m\theta)] d\theta. \) Then

\[ \int [\alpha \delta - \beta \gamma] d\theta d\zeta = \int \left[ \alpha + \frac{\partial \Gamma}{\partial \theta} \right] d\theta \int \left[ \delta + \frac{\partial \Delta}{\partial \zeta} \right] d\zeta - \int \left[ \gamma + \frac{\partial \Delta}{\partial \theta} \right] d\theta \int \left[ \beta + \frac{\partial \Gamma}{\partial \zeta} \right] d\zeta \\
= \int A \cdot d\ell_\theta \int B \cdot d\ell_\zeta - \int B \cdot d\ell_\theta \int A \cdot d\ell_\zeta \]

(F.20)

where we have made use of the fact that \( \int B_i dx^i = \int B \cdot dx^i \) (Boozer [11]).

Consider now the terms involving products of derivatives of \( \Gamma \) and \( \Delta \):

\[ \frac{\partial \Gamma}{\partial \theta} \frac{\partial \Delta}{\partial \zeta} - \frac{\partial \Gamma}{\partial \zeta} \frac{\partial \Delta}{\partial \theta} = \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial \Delta}{\partial \zeta} \right) \right] - \left[ \frac{\partial}{\partial \zeta} \left( \frac{\partial \Delta}{\partial \theta} \right) \right] - \left[ \frac{\partial}{\partial \zeta} \left( \frac{\partial \Delta}{\partial \theta} \right) \right] + \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial \Delta}{\partial \zeta} \right) \right] \]

(F.21)
so

\[
\int \left[ \frac{\partial \Gamma}{\partial \theta} \frac{\partial \Delta}{\partial \zeta} - \frac{\partial \Gamma}{\partial \zeta} \frac{\partial \Delta}{\partial \theta} \right] d\theta d\zeta = \int \left[ \frac{\partial}{\partial \theta} \left( \Gamma \frac{\partial \Delta}{\partial \zeta} \right) - \frac{\partial}{\partial \zeta} \left( \Gamma \frac{\partial \Delta}{\partial \theta} \right) \right] d\theta d\zeta
\]

\[
= \int \left[ \left( \Gamma \frac{\partial \Delta}{\partial \zeta} \right) \big|_{\theta=0} \right] d\zeta - \int \left[ \left( \Gamma \frac{\partial \Delta}{\partial \theta} \right) \big|_{\zeta=0} \right] d\theta = 0 \tag{F.22}
\]

Then it clearly follows that

\[
\int (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n} dA = \oint_P \mathbf{A} \cdot d\ell \oint_T \mathbf{B} \cdot d\ell - \oint_P \mathbf{B} \cdot d\ell \oint_T \mathbf{A} \cdot d\ell \tag{F.23}
\]

for arbitrary vector fields \( \mathbf{A} \) and \( \mathbf{B} \) that both satisfy \( \mathbf{n} \cdot \nabla \times \mathbf{A} = 0 \) and \( \mathbf{n} \cdot \nabla \times \mathbf{B} = 0 \).
Constructing straight-field coordinates requires knowledge of generalized coordinate systems, and an excellent reference for this is D’haeseleer et al. [34]. We begin with the equation of a magnetic field line: \( d\mathbf{R} = c\mathbf{B} \), for an arbitrary constant \( c \). Expanding \( d\mathbf{R} \) according to its contravariant components, \( d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u^i} du^i \), and using \( \nabla u^i \cdot \frac{\partial \mathbf{R}}{\partial u^j} = \delta^j_i \), it follows that \( \nabla u^i \cdot d\mathbf{R} = du^i = c\mathbf{B} \cdot \nabla u^i \). Rearranging this yields the field-line equation:

\[
C = \frac{du^i}{\mathbf{B} \cdot \nabla u^i}.
\] (G.1)

Equating any two \((i \text{ and } j)\) components yields

\[
\frac{du^i}{du^j} = \frac{\mathbf{B} \cdot \nabla u^i}{\mathbf{B} \cdot \nabla u^j}.
\] (G.2)

From this, it follows that \( \dot{\varphi} = d\zeta/d\theta \).

For axisymmetric devices, the elementary toroidal angle \( \zeta \) is already ignorable and it is logical to choose \( \zeta_f \equiv \zeta \). We can construct the \( \theta_f \) coordinate by following the magnetic field line. Consider

\[
d\zeta = \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} d\theta \quad \rightarrow \quad \Delta \zeta = \int_{\theta_0}^{\theta} \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} d\theta
\] (G.3)

and

\[
d\zeta_f = \frac{\mathbf{B} \cdot \nabla \zeta_f}{\mathbf{B} \cdot \nabla \theta_f} d\theta_f \quad \rightarrow \quad \Delta \zeta_f = \int_{\theta_{f,0}}^{\theta_f} \frac{\mathbf{B} \cdot \nabla \zeta_f}{\mathbf{B} \cdot \nabla \theta_f} d\theta_f = q(\rho)(\theta_f - \theta_{f,0}).
\] (G.4)

Equating the two yields an integral equation for \( \theta_f \) in terms of \( \theta \)

\[
\theta_f(\theta) = \theta_{f,0} + \frac{1}{q(\rho)} \int_{\theta_0}^{\theta} \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} d\theta'.
\] (G.5)

Eq. (G.5) may be used to find \( \theta_f \) as a function of any other measure of poloidal angle, \( \theta \).

We will now use Eq. (G.5) to find a set of \( N \) evenly spaced points in \( \theta_f \), which we will label by \( \theta^j_f = 2\pi j/N \) for \( j = 0, ..., N-1 \). Clearly, each of these has a corresponding discrete point in any other measure of the poloidal angle, which we will denote by \( \theta^j \), that is, from Eq. (G.5), \( \theta_f(\theta^j) = \theta^j_f \). The discrete mapping from \( \theta \) to \( \theta_f \) is specified by the \( N \) values \( \theta^j \), and the continuous mapping is found in the limit that \( N \to \infty \).
Subtracting $\theta_f^j$ from $\theta_f^{j+1}$, we find

$$\frac{2\pi}{N} = \theta_f^{j+1} - \theta_f^j = \frac{1}{q(\rho)} \int_{\theta_f}^{\theta_f^{j+1}} \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} d\theta',$$  \hspace{1cm} (G.6)

so

$$\int_{\theta_f}^{\theta_f^{j+1}} \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} d\theta' = \frac{2\pi}{N} q(\rho).$$  \hspace{1cm} (G.7)

Given a starting coordinate, $\theta^0 = \theta_0^j$, we may find the $\theta^j$ coordinates by integrating the LHS of Eq. (G.7) by an amount $2\pi q(\rho)/N$.

NIMROD utilizes an $(R, Z, \phi)$ system where $\nabla R = \hat{R}$, $\nabla Z = \hat{Z}$, and $\nabla \phi = \frac{1}{R} \hat{\phi}$. We first map from NIMROD’s coordinate system to an equivalent $(\rho, \theta, \phi)$ system, and then we map $\theta$ to $\theta_f$. The specification of the $N$ points $(R^j, Z^j)$ corresponding to the $N$ discrete $\theta_f^j$ completes the mapping.

Let $\rho$ be the distance from the magnetic axis, $\rho^2 \equiv (R - R_{\text{magaxis}})^2 + (Z - Z_{\text{magaxis}})^2$, and $\theta$ be the geometric angle measured about that axis. Then $\hat{\rho} = \cos(\theta) \hat{R} + \sin(\theta) \hat{Z}$, $\hat{\theta} = -\sin(\theta) \hat{R} + \cos(\theta) \hat{Z}$, and $\nabla \theta = \frac{1}{\rho} \hat{\theta}$ analogous to a cylindrical system. Then using this in Eq. (G.2), we find

$$\frac{dR}{d\theta} = \frac{\mathbf{B} \cdot \nabla R}{\mathbf{B} \cdot \nabla \theta} = \rho \left( \mathbf{B} \cdot \hat{R} \right) / \left( \mathbf{B} \cdot \hat{\theta} \right),$$  \hspace{1cm} (G.8)

$$\frac{dZ}{d\theta} = \frac{\mathbf{B} \cdot \nabla Z}{\mathbf{B} \cdot \nabla \theta} = \rho \left( \mathbf{B} \cdot \hat{Z} \right) / \left( \mathbf{B} \cdot \hat{\theta} \right),$$  \hspace{1cm} (G.9)

$$\frac{d\phi}{d\theta} = \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} = \rho \left( \mathbf{B} \cdot \hat{\phi} \right) / \left[ R \left( \mathbf{B} \cdot \hat{\theta} \right) \right].$$  \hspace{1cm} (G.10)

NIMFL is used to integrate these equations over a flux surface (in poloidal angle $\theta$ from 0 to $2\pi$). When the integral of Eq. (G.10) increases by an amount $2\pi q/N$, we make note of the coordinates $R$ and $Z$ to obtain the the coordinates $(R^j, Z^j)$ corresponding to straight-field poloidal angle $\theta_f^j$. 
H CYLINDRICAL PHASE RELATIONS

We show that the condition \( \mathbf{n} \cdot \mathbf{J} \big|_{r=a} = 0 \) in cylindrical geometry restricts the amplitude and phase of the perturbations such that they satisfy

\[
\sin (\delta_{\theta} - \delta_{\zeta}) = 0, \quad \frac{\tilde{B}_{\zeta}}{\tilde{B}_{\theta}} = \frac{nr}{mR_0} \cos (\delta_{\theta} - \delta_{\zeta}). \tag{H.1}
\]

In cylindrical geometry

\[
\mu_0 J_r = \frac{1}{r} \frac{\partial}{\partial \theta} B_z - \frac{\partial}{\partial z} B_{\theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \tilde{B}_{\zeta} - \frac{1}{R_0} \frac{\partial}{\partial \zeta} \tilde{B}_{\theta}, \tag{H.2}
\]

where we have related the axial coordinate \( z \) to the angle coordinate \( \zeta \) via \( \zeta = \frac{2\pi z}{L} = \frac{z}{R_0} \).

The perturbed fields may be described by

\[
\tilde{B}_{\theta} = \tilde{B}_{\theta} (r) \cos (m\theta + n\zeta + \delta_{\theta}), \quad \tilde{B}_{\zeta} = \tilde{B}_{\zeta} (r) \cos (m\theta + n\zeta + \delta_{\zeta}) \tag{H.3}
\]

so that

\[
\mu_0 \tilde{J}_r = \frac{-m}{r} \tilde{B}_{\zeta} (r) \sin (m\theta + n\zeta + \delta_{\zeta}) + \frac{n}{R_0} \tilde{B}_{\theta} (r) \sin (m\theta + n\zeta + \delta_{\theta}) \tag{H.4}
\]

Using the trigonometric identity \( \sin (a + b) = \sin (a) \cos (b) + \cos (a) \sin (b) \) with \( \chi \equiv m\theta + n\zeta \), this becomes

\[
\mu_0 \tilde{J}_r = -\sin (\chi) \left[ \frac{m}{r} \tilde{B}_{\zeta} \cos (\delta_{\zeta}) - \frac{n}{R_0} \tilde{B}_{\theta} \cos (\delta_{\theta}) \right] - \cos (\chi) \left[ \frac{m}{r} \tilde{B}_{\zeta} \sin (\delta_{\zeta}) - \frac{n}{R_0} \tilde{B}_{\theta} \sin (\delta_{\theta}) \right] \tag{H.5}
\]

If \( \tilde{J}_r = 0 \) at the boundary, then this must vanish for any value of \( \chi \). Then we require that

\[
\frac{m}{r} \tilde{B}_{\zeta} (a) \cos (\delta_{\zeta}) = \frac{n}{R_0} \tilde{B}_{\theta} (a) \cos (\delta_{\theta}) \tag{H.6}
\]
\[
\frac{m}{r} \tilde{B}_{\zeta} (a) \sin (\delta_{\zeta}) = \frac{n}{R_0} \tilde{B}_{\theta} (a) \sin (\delta_{\theta}). \tag{H.7}
\]

Both of these may be solved for the ratio of the amplitudes to give

\[
\frac{\tilde{B}_{\zeta}}{\tilde{B}_{\theta}} = \frac{nr \cos (\delta_{\theta})}{mR_0 \cos (\delta_{\zeta})} = \frac{nr \sin (\delta_{\theta})}{mR_0 \sin (\delta_{\zeta})}. \tag{H.8}
\]
With similar trigonometric identities, one may show

\[
\frac{\cos (\delta_\theta)}{\cos (\delta_\zeta)} = \cos (\delta_\theta - \delta_\zeta) - \sin (\delta_\theta - \delta_\zeta) \frac{\sin (\delta_\zeta)}{\cos (\delta_\zeta)}, \quad (H.9)
\]

\[
\frac{\sin (\delta_\theta)}{\sin (\delta_\zeta)} = \sin (\delta_\theta - \delta_\zeta) \frac{\cos (\delta_\zeta)}{\sin (\delta_\zeta)} + \cos (\delta_\theta - \delta_\zeta). \quad (H.10)
\]

Using this in Eq. (H.8) we find

\[
0 = \sin (\delta_\theta - \delta_\zeta) \left[ \frac{\cos (\delta_\zeta)}{\sin (\delta_\zeta)} + \frac{\sin (\delta_\zeta)}{\cos (\delta_\zeta)} \right] = \frac{\sin (\delta_\theta - \delta_\zeta)}{\sin (\delta_\zeta) \cos (\delta_\zeta)} \left[ \sin^2 (\delta_\zeta) + \cos^2 (\delta_\zeta) \right] \quad (H.11)
\]

with the solution

\[
\sin (\delta_\theta - \delta_\zeta) = 0, \quad \frac{\tilde{B}_\zeta}{\tilde{B}_\theta} = \frac{nr}{mR_0} \cos (\delta_\theta - \delta_\zeta). \quad (H.12)
\]

This completes the proof.
I EXTENDED MHD WAVES

Neglecting the ion gyroviscosity, fluid viscosity, and the electron inertia, and taking the isothermal limit, the extended MHD system of equations takes the form (Eqs. 3.1-3.6),

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0
\]  
(I.1)

\[
m_i n \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla p
\]  
(I.2)

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left[ -\mathbf{v} \times \mathbf{B} + \Lambda_e \frac{1}{me} (\mathbf{J} \times \mathbf{B} - \nabla p_e) + \eta \mathbf{J} \right].
\]  
(I.3)

The system is closed with the low-frequency Ampère’s law, \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \), appropriate for the slow MHD dynamics of interest. This approximation formally removes light waves from the system of equations. The marker \( \Lambda_e \) is used to indicate explicitly where two-fluid effects enter into the system.

I.1 Linearization

We will consider small perturbations about an equilibrium by linearizing all quantities, \( f = f_0 + \tilde{f} \), where \( f_0 \) is the steady-state part, \( \frac{df_0}{dt} = 0 \), and \( \tilde{f} \) is the time varying piece with \( |\tilde{f}|/|f_0| \ll 1 \). Here we will restrict our analysis to simple slab geometry (a Cartesian \((x, y, z)\) system). Periodicity will be imposed in \( y \) and \( z \), but the boundary conditions in \( x \) will depend on the problem under consideration.

Equilibrium quantities may only vary in \( x \), but perturbations will take the form \( \tilde{f} (x, y, z, t) = \tilde{f} (x) e^{ik_y y - i\omega t} \), that is, there is no variation in the \( \hat{z} \) direction. A consequence of the cold ion and massless electron assumption is that we may transform to a frame of reference in which there is no center of mass background flow, \( \mathbf{v}_0 = 0 \), even with a background pressure gradient.

I.1.1 Density Equation

The linearized density equation can immediately be rearranged to yield

\[
\frac{i\omega \tilde{n}}{n_0} = \nabla \cdot \tilde{\mathbf{v}} + \tilde{v}_x L_n^{-1}
\]  
(I.4)

where \( L_n^{-1} = \frac{1}{n_0} \frac{dn_0}{dx} \).
I.1.2 Induction Equation

The linearized electric field is

\[ \tilde{E} = -\tilde{v} \times B_0 + \frac{\Lambda_e}{n_0 e} (\tilde{J} \times B_0 + J_0 \times \tilde{B} - \nabla \tilde{p}_e) + \eta \tilde{J} \]  

(I.5)

and its curl is given by

\[ -\nabla \times \tilde{E} = -B_0 (\nabla \cdot \tilde{v}) + i k_y B_{0y} \tilde{v} + \frac{\eta}{\mu_0} \nabla^2 \tilde{B} \]

\[ + \frac{\Lambda_e}{n_0 e} \left[-i k_y B_{0y} \tilde{J} + \tilde{J}_x \frac{d B_0}{d x} - \tilde{B}_x \frac{d J_0}{d x} + i k_y J_{0y} \tilde{B} \right] \]

\[ + \frac{\Lambda_e}{n_0 e} L^{-1}_n \left[-\tilde{J}_x B_0 + \tilde{B}_x J_0 - i k_y \tilde{p}_e \hat{z} \right]. \]

(I.6)

Note that Ampère’s law, with \( \nabla \cdot \tilde{J} = 0 \) and \( \nabla \cdot \tilde{B} = 0 \) gives

\[ \mu_0 \tilde{J}_x = i k_y \tilde{B}_z \quad \mu_0 \tilde{J}_y = -\frac{\partial}{\partial x} \tilde{B}_z \quad \mu_0 \tilde{J}_z = -\frac{1}{i k_y} \nabla^2 \tilde{B}_x. \]

(I.7)

The \( \hat{x} \) component of the induction equation is

\[ -i \omega \tilde{B}_x = i k_y B_{0y} \tilde{v}_x + \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_x - \Lambda_e \frac{i k_y}{n_0 e \mu_0} \left[i k_y B_{0y} \tilde{B}_z + \frac{d B_{0z}}{d x} \tilde{B}_x \right]. \]

(I.8)

We can see that the presence of the Hall term \( \Lambda_e = 1 \) couples this into the parallel magnetic field perturbations, \( \tilde{B}_z \). The equation for \( \tilde{B}_z \) comes from the \( \hat{z} \) component of Eq. (I.6):

\[ -i \omega \tilde{B}_z = -B_{0z} (\nabla \cdot \tilde{v}) + i k_y B_{0y} \tilde{v}_z - \tilde{v}_x \frac{d B_{0z}}{d x} + \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_z \]

\[ + \Lambda_e \frac{1}{\mu_0 n_0 e} \left[B_{0y} \left(\nabla^2 - \frac{1}{B_{0y}} \frac{d^2 B_{0y}}{d x^2}\right) \tilde{B}_x \right] \]

\[ + \Lambda_e \frac{L^{-1}_n}{n_0 e \mu_0} \left[-i k_y B_{0z} \tilde{B}_z + \frac{d B_{0y}}{d x} \tilde{B}_x - i k_y \mu_0 n_0 T_e \tilde{n} \right]. \]

(I.9)

There are several important features worth mentioning here. The first is the presence of the \( \nabla \cdot \tilde{v} \) term with the factor \( B_{0z} \), which is often taken to be quite large (strong guide field limit). Unless the motions are very nearly incompressible, \( \nabla \cdot \tilde{v} \approx 0 \), this term provides a very large contribution. We are interested in slow polarizations, where the divergence remains small. Consequently, we will assume that the flow is very nearly incompressible, although the exact incompressible limit will remove some interesting effects.
I.1.3 Momentum Equation

Consider now the linearized momentum equation:

\[-i\omega_0 m_i n_0 \tilde{v} = \tilde{B} \cdot \nabla B_0 + B_0 \cdot \nabla \tilde{B} - \nabla \left( \tilde{B} \cdot B_0 + \mu_0 \tilde{p} \right)\]

\[= \tilde{B}_x \frac{dB_0}{dx} + ik_y B_{0y} \tilde{B} - \nabla \left[ \tilde{B} \cdot B_0 + \mu_0 n_0 T_{e0} \frac{n}{n_0} \right]. \tag{I.10}\]

When we operate with \(z \cdot \nabla \times\) on this equation, we find two contributions. The first comes from the usual variation of \(\tilde{v}\) and the second comes from background density variations. The left hand side transforms as

\[\tilde{z} \cdot \nabla \times (-i\omega_0 m_i n_0 \tilde{v}) = -i\omega_0 m_i n_0 \left[ \frac{1}{ik_y} \left( \frac{\partial}{\partial x} (\nabla \cdot \tilde{v}) - \frac{\partial^2 \tilde{v}_x}{\partial x^2} \right) - ik_y \tilde{v}_x \right] - i\omega_0 m_i n_0 L^{-1} \frac{1}{ik_y} \left( (\nabla \cdot \tilde{v}) - \frac{\partial \tilde{v}_x}{\partial x} \right) \tag{I.11}\]

where we have used \(\tilde{v}_y = (\nabla \cdot \tilde{v}) - \frac{\partial}{\partial x} \tilde{v}_x / ik_y\) to eliminate \(\tilde{v}_y\) in favor of \(\nabla \cdot \tilde{v}\). Taking the \(z \cdot \nabla \times\) operator on the right hand side of Eq. (I.10) and rearranging results in

\[-i\omega_0 m_i n_0 \nabla^2 \tilde{v}_x = -ik_y B_{0y} \left. \left[ \nabla^2 - \frac{1}{B_{0y}} \frac{d^2 B_{0y}}{dx^2} \right] \tilde{B}_x \right. - i\omega_0 m_i n_0 \left. \left( L^{-1} + \frac{\partial}{\partial x} \right) (\nabla \cdot \tilde{v}) - L^{-1} \frac{\partial \tilde{v}_x}{\partial x} \right]. \tag{I.12}\]

As we will see later, in the usual tearing mode analysis, only the terms on the first line of Eq. (I.12) are kept. The basic tearing mode does not require background density gradients, which can provide either a stabilizing or destabilizing response [27], and the most unstable modes of the single-fluid MHD system have \(\nabla \cdot \tilde{v} = 0\) [7].

The \(y\) component of the momentum equation is

\[-i\omega_0 m_i n_0 \tilde{v}_y = \tilde{B}_z \frac{dB_{0y}}{dx} + ik_y B_{0y} \tilde{B}_y - ik_y \left[ \tilde{B}_y B_{0y} + \tilde{B}_z B_{0z} + \mu_0 n_0 T_{e0} \frac{n}{n_0} \right]. \tag{I.13}\]
Replacing \( \vec{v}_y \) by \( \nabla \cdot \vec{v} \) and \( \vec{v}_x \) and using \( \nabla \cdot \vec{B} = 0 \) to express \( \tilde{B}_y \) in terms of \( \tilde{B}_x \), this becomes

\[
-i\omega \mu_0 m_i n_0 \left[ \nabla \cdot \vec{v} - \frac{\partial}{\partial x} \vec{v}_x \right] = ik_y \tilde{B}_x \frac{dB_{0y}}{dx} - ik_y B_{0y} \left( \frac{\partial \tilde{B}_x}{\partial x} \right) + k_y^2 \left( \frac{-1}{ik_y} \frac{\partial \tilde{B}_x}{\partial x} \right) B_{0y} + \tilde{B}_z B_{0z} + \mu_0 n_0 T_0 \frac{\tilde{n}}{n_0}.
\]

As we will see presently, the last terms in the brackets plays a role in the dispersion relation for the fast magnetoacoustic waves. These waves can be ordered out of the system by assuming that balance is achieved between the term in brackets. We will not make this assumption immediately. Lastly, the \( \hat{z} \) component of the momentum equation is

\[
-i\omega \mu_0 m_i n_0 \tilde{v}_z = \tilde{B}_x \frac{dB_{0z}}{dx} + ik_y B_{0y} \tilde{B}_z.
\]

### I.1.4 Full Equations

Our two-fluid linearized system then consists of

\[
-i\omega \tilde{B}_x = ik_y B_{0y} \tilde{v}_x + \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_x - \Lambda_e \frac{ik_y}{n_0 e \mu_0} \left[ ik_y B_{0y} \tilde{B}_x + \frac{dB_{0z}}{dx} \tilde{B}_x \right]
\]

\[
-i\omega \tilde{B}_z = - B_{0z} (\nabla \cdot \tilde{v}) + ik_y B_{0y} \tilde{v}_z - \tilde{v}_x \frac{dB_{0z}}{dx} + \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_z + \frac{1}{\mu_0 n_0 e} \left[ \frac{L_{-1}}{n_0 e \mu_0} \left( -ik_y B_{0z} \tilde{B}_z + \frac{dB_{0y}}{dx} \tilde{B}_z - ik_y \mu_0 n_0 T_0 \frac{\tilde{n}}{n_0} \right) \right]
\]

\[
-i\omega \mu_0 m_i n_0 \nabla^2 \tilde{v}_x = ik_y B_{0y} \left[ \nabla^2 - \frac{1}{B_{0y}} \frac{d^2 B_{0y}}{dx^2} \right] \tilde{B}_x
\]

\[
-i\omega \mu_0 m_i n_0 \left[ \left( L_{-1}^{-1} + \frac{\partial}{\partial x} \right) (\nabla \cdot \tilde{v}) - L_{-1}^{-1} \frac{\partial \tilde{v}_x}{\partial x} \right]
\]

\[
-i\omega \mu_0 m_i n_0 (\nabla \cdot \tilde{v}) = - i\omega \mu_0 m_i n_0 \left( \frac{\partial \tilde{v}_x}{\partial x} \right) + ik_y \tilde{B}_z \frac{dB_{0y}}{dx} - ik_y B_{0y} \left( \frac{\partial \tilde{B}_x}{\partial x} \right)
\]

\[
+i \omega \frac{\tilde{n}}{n_0} = \tilde{v}_x L_{-1}^{-1} + \nabla \cdot \tilde{v}.
\]
There are six equations here and six unknowns, so this constitutes a completely closed system. We will first examine this system in the ideal limit.

### I.2 Ideal System, Uniform Background

Consider now Eqs. (I.16)-(I.21) in the ideal limit \((\eta = 0)\) and without background gradients. We can set \(k = k_y \hat{y}\) here without loss of generality (i.e. we set \(\frac{\partial}{\partial x} = 0\)) as we allow \(B_0 = B_0 \cos (\theta_b) \hat{z} + B_0 \sin (\theta_b) \hat{y}\) to orient the magnetic field and the wave-vector \(k\). Then Eqs. (I.16)-(I.21) become

\[
-\omega \tilde{B}_x = ik_y B_{0y} \tilde{v}_x + \Lambda_e \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \tilde{B}_z \tag{I.22}
\]

\[
-\omega \tilde{B}_z = -B_{0z} (\nabla \cdot \tilde{v}) + ik_y B_{0y} \tilde{v}_z + \Lambda_e \frac{B_{0y}}{\mu_0 n_0 e} \nabla^2 \tilde{B}_x \tag{I.23}
\]

\[
-\omega \mu_0 m_i n_0 \tilde{v}_x = ik_y B_{0y} \tilde{B}_x \tag{I.24}
\]

\[
-\omega \mu_0 m_i n_0 \nabla \cdot \tilde{v} = k_y^2 \left[ B_z B_{0z} + \mu_0 n_0 T_e \frac{\tilde{n}}{n_0} \right] \tag{I.25}
\]

\[
-\omega \mu_0 m_i n_0 \tilde{v}_z = ik_y B_{0y} \tilde{B}_z \tag{I.26}
\]

\[
i \omega \frac{\tilde{n}}{n_0} = \nabla \cdot \tilde{v} \tag{I.27}
\]

Combining Eqs. (I.22) and (I.24) results in

\[
\left( \omega^2 - \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right) \tilde{B}_x = \Lambda_e (i\omega) \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \tilde{B}_z. \tag{I.28}
\]

In the single-fluid system, the term on the left hand side yields the shear Alfvén wave dispersion relation, \(\omega = \pm k_i V_A\). Two-fluid effects couple the radial magnetic perturbation to the parallel field perturbation, and the resulting dispersion relation is more complicated.

The fast and slow waves are found from the remaining four equations. Replacing \(\tilde{v}_z\) and \(\nabla \cdot \tilde{v}\) in Eq. (I.23) yields

\[
-\omega \tilde{B}_z = -\omega B_{0z} \frac{\tilde{n}}{n_0} - \frac{ik_y^2 B_{0y}^2}{\omega \mu_0 m_i n_0} \tilde{B}_z + \Lambda_e \frac{B_{0y}}{\mu_0 n_0 e} \nabla^2 \tilde{B}_x. \tag{I.29}
\]

and replacing \(\nabla \cdot \tilde{v}\) in Eq. (I.25) gives

\[
\omega^2 \frac{\tilde{n}}{n_0} = \frac{k_y^2}{\mu_0 m_i n_0} \left[ B_z B_{0z} + \mu_0 n_0 T_e \frac{\tilde{n}}{n_0} \right]. \tag{I.30}
\]
The terms in the bracket on the right hand side of Eq. (I.30) represent the total perturbed pressure. If we assume that the total perturbed pressure is balanced, then the resulting dispersion relation simplifies considerably. For now, we will keep it in place. Using this in a rearranged Eq. (I.29) we find

\[
\{ \omega^4 - \omega^2 \left[ \frac{k_y^2 B_0^2}{\mu_0 m_i n_0} + \frac{k_y^2 T_{e0}}{m_i} \right] + \frac{k_y^2 T_{e0} B_{0y}}{\mu_0 m_i n_0^2} \} \dot{B}_z = -\Lambda_e (i\omega) \left( \omega^2 - \frac{k_y^2 T_{e0}}{m_i} \right) \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \dot{B}_x
\]  

(I.31)

where we have used \( \nabla^2 \to -k_y^2 \).

The dispersion relation results from combining Eqs. (I.28) and (I.31). After some algebra, one finds

\[
\left( \omega^2 - k^2 V_A^2 \right) \left\{ \omega^4 - \omega^2 \left[ k^2 V_A^2 (1 + \beta) \right] + k^2 k^2_{\parallel} \beta V_A^2 \right\} = \Lambda_e^2 \omega^2 \left( \omega^2 - \beta k^2 V_A^2 \right) k^2 V_A^2 d_i^2
\]

(I.32)

where we have used

\[
\frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \equiv k^2_{\parallel} V_A^2  \quad \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \equiv k^2 V_A^2  \quad \beta \equiv \frac{\mu_0 n_0 T_{e0}}{B_0^2} \quad \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \equiv k k_{\parallel} V_A d_i.
\]

It is clear that two-fluid effects (\( \Lambda_e = 1 \)) alter the character of the dispersion relation considerably. The waves that the full two-fluid system admits are not immediately apparent, but it will be helpful to consider some simple limits.

I.2.1 Single-Fluid Limit

In the single-fluid limit, Eq. (I.32) becomes

\[
\left( \omega^2 - k_{\parallel}^2 V_A^2 \right) \left\{ \omega^4 - \omega^2 \left[ k^2 V_A^2 (1 + \beta) \right] + k^2 k_{\parallel}^2 \beta V_A^2 \right\} = 0
\]

(I.33)

which is separable for the shear Alfvén waves, and the fast and slow waves. The term in the curly braces is a quadratic equation in \( \omega^2 \) with two solutions, the fast and slow magnetoacoustic waves, given exactly by

\[
\omega^2 = \frac{k^2 V_A^2 (1 + \beta)}{2} \left[ 1 \pm \sqrt{1 - 4 \frac{k_{\parallel}^2 \beta}{k^2 (1 + \beta)^2}} \right].
\]

(I.34)

In the low-\( \beta \) limit, Eq. (I.34) becomes

\[
\omega^2 = \frac{k^2 V_A^2 (1 + \beta)}{2} \left[ 1 \pm \left( 1 - 2 \frac{k_{\parallel}^2 \beta}{k^2 (1 + \beta)^2} + O(\beta^2) \right) \right]
\]

(I.35)
which has the solutions

\[ \omega^2 = k^2 V_A^2 + \mathcal{O}(\beta) \quad \omega^2 = \frac{\beta k^2 V_A^2}{1 + \beta} + \mathcal{O}(\beta^2). \tag{I.36} \]

### I.2.2 Pressure Balance

Consider now the total perturbed pressure in Eq. (I.30). If this is in balance, then

\[ \frac{\tilde{n}}{n_0} = -\frac{B_0}{\mu_0 n_0 T_e_0} \tilde{B}_z. \tag{I.37} \]

This can be used directly in Eq. (I.29) to give

\[ \left( \omega^2 - k^2 V_A^2 \right) \left( \omega^2 (1 + \beta) - \beta k^2 V_A^2 \right) = -\Lambda_e (i\omega) \beta k^2 V_A d_i \tilde{B}_z. \tag{I.38} \]

Combining Eq. (I.38) with Eq. (I.28) then yields a two-fluid dispersion relation

\[ \left( \omega^2 - k^2 V_A^2 \right) \left[ \omega^2 (1 + \beta) - \beta k^2 V_A^2 \right] = \Lambda_e^2 \omega^2 \beta k^2 V_A^2 d_i^2. \tag{I.39} \]

In the single-fluid limit, we recognize that the term in brackets on the left hand side corresponds to the slow wave from Eq. I.34, in the low-\( \beta \) limit, with \( \omega^2 = \beta k^2 V_A^2 / (1 + \beta) \). The fast wave behavior has been removed from the system.

Expanding this and grouping factors, we find

\[ \omega^4 (1 + \beta) - \omega^2 \left\{ k^2 V_A^2 \left[ (1 + \beta) + \beta k^2 d_i^2 \Lambda_e^2 \right] \right\} + \beta k^2 V_A^4 = 0. \tag{I.40} \]

This can be solved as a quadratic in \( \omega^2 \) to give

\[ \omega^2 = \frac{k^2 V_A^2 \left[ (1 + \beta) + \beta k^2 d_i^2 \Lambda_e^2 \right]}{2 (1 + \beta)} \left[ 1 \pm \sqrt{\frac{1 - 4 \beta}{(1 + \beta) + \beta k^2 d_i^2 \Lambda_e^2} \right]. \tag{I.41} \]

Expanding this for small \( \beta \), we find two roots. The first is a modification of the shear Alfvén wave with

\[ \omega^2 = \frac{k^2 V_A^2 (1 + \beta k^2 d_i^2)}{1 + \beta} + \mathcal{O}(\beta^2). \tag{I.42} \]

This is commonly referred to as the kinetic Alfvén wave (KAW). The second is the familiar slow wave where two-fluid effects only modify this at higher order in \( \beta \). Note that this expansion requires \( k^2 d_i^2 \ll 1 \).
Note that for large $k^2d_i^2$ or small $\beta$, we can ignore the second term in the square root of Eq. (I.41) to find

$$\omega^2 \approx k_{\parallel}^2 V_A^2 \left(1 + \beta k^2 d_i^2\right)$$  \hspace{1cm} (I.43)

or $\omega \sim k_{\parallel} k$, that is the frequency goes with the square of the wavenumber. At small spatial scales (high wavenumber) the frequencies of the KAW can become quite large, and this is posited to be a mechanism for the observed faster rates of reconnection than those predicted by single-fluid MHD.
where \( \beta \), we will take \( \epsilon \) resistive drift dispersion relation

\[
L = L - 1 = L - 1 - 1 = 1 - B \equiv L - B - B \frac{L}{L} - B \frac{L}{L} - B \frac{L}{L}.
\]

Here we examine a dispersion relation that results from the complete linearized system, Eqs. (I.16)-(I.21). Some orderings will be useful. First, we will order the component of magnetic field along \( k \) much smaller than the component perpendicular to it, i.e.

\[
B_0 = B_0 \hat{z} + B_0 y \hat{y} = B_0 \sqrt{1 - \epsilon^2 \hat{z}} + \epsilon B_0 y \hat{y}
\]

for \( \epsilon \ll 1 \). Shear is introduced into the system by having \( \epsilon = \epsilon(x) \); for simplicity, however, we will take \( \epsilon \) to be a constant. The pressure gradient is supported by variation in \( B_0 \): \( B^{-1} = \frac{1}{B_0} \frac{dB_0}{dx} = L - B^{-1} \equiv L - B^{-1} \). From equilibrium force balance, one can show \( L_c^{-1} = -\beta L^{-1} \), where \( \beta \equiv \mu_0 p_0 / B_0 \) differs from the usual definition by a factor of 2. It follows then, that \( |L^{-1}_B| \ll |L^{-1}_B| \) for low-\( \beta \) conditions, and we may justifiably neglect terms with \( L^{-1}_B \) compared to terms that contain \( L^{-1}_n \).

We will eliminate \( \tilde{v}_z \) and \( \nabla \cdot \hat{v} \) from Eq. (I.17).

\[
- i\omega \tilde{B}_z = -i\omega B_0 \frac{n}{n_0} + B_0 z - L^{-1}_n \tilde{v}_x - i \left( \frac{k_B^2 B_0^2}{\omega \mu_0 m_1 n_0} \right) \tilde{B}_z - \frac{k_y B_0 y B_0' y}{\omega \mu_0 m_1 n_0} \tilde{B}_x - \tilde{v}_x B'_0 y + \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_z
\]

\[
+ \Lambda_c \frac{B_0 y}{\mu_0 n_0} \left( \nabla^2 - \frac{1}{B_0 y} \frac{d^2 B_0}{dx^2} \right) \tilde{B}_x + \Lambda_c \frac{L^{-1}_n}{n_0 e \mu_0} \left[ -i k_y B_0 y B'_0 y + B'_0 y \tilde{B}_x - i k_y \mu_0 n_0 T \tilde{v}_x \tilde{B}_z \right].
\]

(Multiply by \( i\omega \) and group some terms)

\[
\omega^2 \tilde{B}_z = \omega^2 B_0 \frac{n}{n_0} + i \omega B_0 z \left( L^{-1}_n - L^{-1}_B \right) \tilde{v}_x + \left( \frac{k_B^2 B_0^2}{\omega \mu_0 m_1 n_0} \right) \tilde{B}_z - \left[ \frac{i k_y B_0 y B'_0 y}{\omega \mu_0 m_1 n_0} \right] \tilde{B}_x + i \omega \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_z
\]

\[
+ \Lambda_c \frac{i \omega B_0 y}{\mu_0 n_0 e} \left( \nabla^2 - \frac{1}{B_0 y} \frac{d^2 B_0}{dx^2} \right) \tilde{B}_x + \Lambda_c \frac{i \omega L^{-1}_n}{n_0 e \mu_0} \left[ -i k_y B_0 y B'_0 y + B'_0 y \tilde{B}_x - i k_y \mu_0 n_0 T \tilde{v}_x \tilde{B}_z \right].
\]

Now group all terms

\[
\left[ \omega^2 - \left( \frac{k_B^2 B_0^2}{\omega \mu_0 m_1 n_0} \right) - \omega \Lambda_c \frac{k_y L^{-1}_n B_0 y}{\mu_0 n_0 e} - i \omega \frac{\eta}{\mu_0} \nabla^2 \right] \tilde{B}_z = B_0 z \left[ \omega^2 + \omega \Lambda_c \frac{k_y L^{-1}_n B_0 y}{\mu_0 n_0 e} \frac{\beta B_0}{B_0 z} \right] \tilde{B}_z
\]

\[
+ i \omega B_0 z \left( L^{-1}_n - L^{-1}_B \right) \tilde{v}_x + \left[ -i k_y B_0 y B'_0 y + \Lambda_c \frac{i \omega B_0 y}{\mu_0 n_0 e} \left( \nabla^2 - \frac{B_0'' y}{B_0 y} \right) + \Lambda_c \frac{i \omega L^{-1}_n B_0 y}{\mu_0 n_0 e} \tilde{B}_z. \right)
\]

Now use the decoupling condition (Eq. (I.37), \( B_0 \tilde{n} = 0 \)) which requires \( \epsilon \ll 1 \) to be
valid here. Then we find

$$\left[ \omega^2 - \left( \frac{k_y B_{0y}^2}{\mu_0 m_i n_0} \right) - \omega \frac{k_y L_n^{-1} B_{0z}}{\mu_0 n_0 e} - \frac{i \omega \eta}{\mu_0} \nabla^2 \right] \tilde{B}_z = - \left( 1 - \frac{\epsilon^2}{\beta} \right) \left[ \omega^2 + \omega \frac{k_y L_n^{-1} B_0}{\mu_0 n_0 e} \frac{\beta}{\sqrt{1 - \epsilon^2}} \right] \tilde{B}_z$$

$$+ i \omega B_{0z} \left( L_n^{-1} - L_B^{-1} \right) \tilde{v}_x + \left\{ - \frac{i k_y B_{0y} B_{0z}'}{\mu_0 m_i n_0} + \Lambda_e \frac{i \omega B_{0y}}{\mu_0 n_0 e} \left( \nabla^2 - \frac{B_{0y}''}{B_{0y}} \right) + \Lambda_e \frac{i \omega L_n^{-1} B_0'}{n_0 e \mu_0} \right\} \tilde{B}_x.$$  

(J.5)

There is an exact cancellation between the two $\Lambda_e L_n^{-1} \tilde{B}_z$ terms. After this cancellation, and rearranging and multiplying through by $k_y B_{0y}$, we find

$$\left\{ \omega^2 \left( 1 + \frac{\beta}{1 - \epsilon^2} \right) - \frac{\beta}{1 - \epsilon^2} \left[ \frac{k_y B_{0y}^2}{\mu_0 m_i n_0} + \frac{i \omega \eta}{\mu_0} \nabla^2 \right] \right\} k_y B_{0y} \tilde{B}_z = \frac{i \omega \beta B_{0z} L_n^{-1}}{1 - \epsilon^2} (1 + \beta) \left( k_y B_{0y} \tilde{v}_x \right)$$

$$+ \frac{\beta}{1 - \epsilon^2} \left\{ - \frac{i k_y B_{0y} B_{0z}'}{\mu_0 m_i n_0} + \Lambda_e \frac{i \omega k_y B_{0y}'}{\mu_0 n_0 e} \left( \nabla^2 - \frac{B_{0y}''}{B_{0y}} \right) + \Lambda_e \frac{i \omega k_y B_{0y} L_n^{-1} B_0'}{n_0 e \mu_0} \right\} \tilde{B}_x.$$  

(J.6)

Some manipulation is needed. Note that the first of the $\tilde{B}_x$ term can be recast as

$$\left\{ \omega^2 \left( 1 + \frac{\beta}{1 - \epsilon^2} \right) - \frac{\beta}{1 - \epsilon^2} \left[ \frac{k_y B_{0y}^2}{\mu_0 m_i n_0} + \frac{i \omega \eta}{\mu_0} \nabla^2 \right] \right\} k_y B_{0y} \tilde{B}_z$$

$$= \frac{i \omega \beta B_{0z} L_n^{-1}}{1 - \epsilon^2} \left( 1 + \beta \right) \left( \omega \tilde{B}_x + k_y B_{0y} \tilde{v}_x \right) - \frac{i \beta B_{0z} L_n^{-1}}{1 - \epsilon^2} \left[ \omega^2 (1 + \beta) - \frac{\beta k_y B_{0y}^2}{\mu_0 m_i n_0} \right] \tilde{B}_x$$

$$+ \frac{\beta}{1 - \epsilon^2} \left\{ \Lambda_e \frac{i \omega k_y B_{0y}^2}{\mu_0 n_0 e} \left( \nabla^2 - \frac{B_{0y}''}{B_{0y}} \right) + \Lambda_e \frac{i \omega k_y B_{0y} L_n^{-1} L_B^{-1}}{\mu_0 n_0 e} \right\} \tilde{B}_x.$$  

(J.7)
Now multiply by $\Lambda_e i \omega k_y / \mu_0 n_0 e$:

$$
\left\{ \omega^2 \left( 1 + \beta \frac{1}{1 - \epsilon^2} \right) - \frac{\beta}{1 - \epsilon^2} \left[ \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} + i \omega \frac{\eta}{\mu_0} \nabla^2 \right] \right\} \left( \Lambda_e i \omega \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \right) \tilde{B}_z \\
= \left( \frac{\beta k_y B_{0z} L_{n}^{-1}}{\mu_0 n_0 e} \right) \frac{\omega \Lambda_e}{1 - \epsilon^2} \left\{ (1 + \beta) \left( \omega^2 \tilde{B}_z + \omega k_y B_{0y} \tilde{v}_x \right) - \left[ \omega^2 \left( 1 + \beta \right) - \frac{\beta k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right] \tilde{B}_z \right\} \quad (J.9)
$$

Now we will drop all terms that are $O(\epsilon^2)$, and ignore the resistive diffusion of $\tilde{B}_z$:

$$
\left\{ \omega^2 \left( 1 + \beta \right) - \left[ \beta \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right] \left( \Lambda_e i \omega \frac{k_y B_{0y}}{\mu_0 n_0 e} \right) \tilde{B}_z - \left[ \omega^2 \left( 1 + \beta \right) - \frac{\beta k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right] \left( \Lambda_e i \omega \frac{k_y B_{0z} L_{n}^{-1}}{\mu_0 n_0 e} \right) \tilde{B}_z \\
= - \left( \Lambda_e i \omega \frac{\beta k_y B_{0z} L_{n}^{-1}}{\mu_0 n_0 e} \right) (1 + \beta) \left( \omega^2 \tilde{B}_z + \omega k_y B_{0y} \tilde{v}_x \right) \\
- (\omega^2 \beta) \left( \Lambda_e \frac{k_y B_{0y}}{\mu_0 n_0 e} \right)^2 \left[ \nabla^2 - \frac{B_{0y}''}{B_{0y}} + L_{n}^{-1} L_{B}^{-1} \right] \tilde{B}_z.
$$

(J.10)

This must be combined with the equation for $\tilde{B}_x$ (Eq. (I.16)), which, after some rearrangement is

$$
(\omega^2 \tilde{B}_x + \omega k_y B_{0y} \tilde{v}_x) = i \omega \frac{\eta}{\mu} \nabla^2 \tilde{B}_x + \Lambda_e \left( i \omega \frac{k_y^2 B_{0y}}{\mu_0 n_0 e} \right) \tilde{B}_z - \Lambda_e \left( \omega \frac{\beta k_y B_{0z} L_{n}^{-1}}{\mu_0 n_0 e} \right) \tilde{B}_x. \quad (J.11)
$$

Multiplying this by $\omega^2 (1 + \beta) - \beta \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0}$, we see that the last two terms are equal to the first terms in the $\tilde{B}_z$ equation. We can simply combine them to get

$$
\left[ \omega^2 (1 + \beta) - \left( \beta \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right) \right] (\omega^2 \tilde{B}_x + \omega k_y B_{0y} \tilde{v}_x) = \left[ \omega^2 (1 + \beta) - \left( \beta \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right) \right] \left( i \omega \frac{\eta}{\mu_0} \nabla^2 \right) \tilde{B}_x \\
- \left( \Lambda_e i \omega \frac{\beta k_y B_{0z} L_{n}^{-1}}{\mu_0 n_0 e} \right) (1 + \beta) \left( \omega^2 \tilde{B}_z + \omega k_y B_{0y} \tilde{v}_x \right) \\
- (\omega^2 \beta) \left( \Lambda_e \frac{k_y B_{0y}}{\mu_0 n_0 e} \right)^2 \left[ \nabla^2 - \frac{B_{0y}''}{B_{0y}} + L_{n}^{-1} L_{B}^{-1} \right] \tilde{B}_x.
$$

(J.12)
The combination terms of $\tilde{B}_x$ and $\tilde{v}_x$ can be combined
\[
\left[ \omega^2 (1 + \beta) - \left( \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right) \right] + \Lambda_e \omega (1 + \beta) \frac{\beta k_y B_{0z} L_{-1}^{-1}}{\mu_0 n_0 e} \left( \omega^2 \tilde{B}_x + \omega k_y B_{0y} \tilde{v}_x \right) = \\
\left[ \omega^2 (1 + \beta) - \left( \beta \frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \right) \right] \left( i \omega \frac{\eta}{\mu_0} \nabla^2 \right) \tilde{B}_x - \left( \omega^2 \beta \right) \left( \Lambda_e \frac{k_y B_{0y}}{\mu_0 n_0 e} \right)^2 \left[ \nabla^2 - \frac{B''_{0y}}{B_{0y}} + L_{-1}^{-1} L_{-1}^{-1} \right] \tilde{B}_x.
\]
\[(J.13)\]

Note that
\[
\frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} = k_y^2 V_A^2, \quad -\beta k_y B_{0z} L_{-1}^{-1} \frac{1}{\mu_0 n_0 e} \equiv \omega_{\varepsilon}^{(n)}(\beta), \quad k_y B_{0y} = k_y V_A d_i.
\]

Using this in the above, we find
\[
\left[ \omega^2 (1 + \beta) - \beta k_y^2 V_A^2 - \Lambda_e \omega \omega_{\varepsilon}^{(n)} (1 + \beta) \right] \left( \omega^2 \tilde{B}_x + \omega k_y B_{0y} \tilde{v}_x \right) = \\
\left( \omega^2 (1 + \beta) - \beta k_y^2 V_A^2 \right) \left( i \omega \frac{\eta}{\mu_0} \nabla^2 \right) \tilde{B}_x - \omega^2 \left( k_y^2 V_A^2 \beta d_i^2 \Lambda_e^2 \right) \left[ \nabla^2 - \frac{B''_{0y}}{B_{0y}} + L_{-1}^{-1} L_{-1}^{-1} \right] \tilde{B}_x.
\]
\[(J.14)\]

All that remains is to use the equation for $\tilde{v}_x$ (Eq. (I.18)) in here. However, that equation couples to $\nabla \cdot \tilde{v}$, which will bring in the density fluctuations, and hence additional $\tilde{B}_x$ terms which will complicate the algebra quite a bit. Instead, we will simplify and assume that the largest terms in the $\tilde{v}_x$ equation are the $\nabla^2$ pieces, which must be in balance, so that
\[
-i \omega \mu_0 m_i n_0 \tilde{v}_x = i k_y B_{0y} \tilde{B}_x \quad \Rightarrow \quad \omega k_y B_{0y} \tilde{v}_x = -\frac{k_y^2 B_{0y}^2}{\mu_0 m_i n_0} \tilde{B}_x = -k_y^2 V_A^2 \tilde{B}_x.
\]
\[(J.15)\]

Using this in Eq. (J.14), we find
\[
\left( \omega^2 - k_y^2 V_A^2 \right) \left[ \omega^2 (1 + \beta) - \Lambda_e \omega \omega_{\varepsilon}^{(n)} (1 + \beta) - \beta k_y^2 V_A^2 \right] \tilde{B}_x = \\
i \omega \left( \omega^2 (1 + \beta) - \beta k_y^2 V_A^2 \right) \frac{\eta}{\mu_0} \nabla^2 \tilde{B}_x - \omega^2 \left( k_y^2 V_A^2 \beta d_i^2 \Lambda_e^2 \right) \left[ \nabla^2 - \frac{B''_{0y}}{B_{0y}} + L_{-1}^{-1} L_{-1}^{-1} \right] \tilde{B}_x.
\]
\[(J.16)\]
Figure K.1: Location of rational surfaces and the parallel current profile, $a\lambda$, before and after each event. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 0$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)
Figure K.2: Location of rational surfaces and the parallel current profile, $a\lambda$, before and after each event. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_c = 1$, $\Lambda_i = 0$, $S = 20,000$, $P_m = 1.0$)

Figure K.3: Location of rational surfaces and the parallel current profile, $a\lambda$, before and after each event. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_c = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 1.0$)
Figure K.4: Location of rational surfaces and the parallel current profile, $a\lambda$, before and after each event. (Parameters: $a\lambda_0 = 3.88$, $A_e = 1$, $A_i = 1$, $S = 20,000$, $P_m = 0.1$)

Figure K.5: Magnetic energy spectra averaged over the relaxation events. (Parameters: $a\lambda_0 = -3.88$, $A_e = 0$, $A_i = 0$, $S = 20,000$, $P_m = 1.0$)
Figure K.6: Magnetic energy spectra averaged over the relaxation events. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1, \Lambda_i = 0, S = 20,000, P_m = 1.0$)

Figure K.7: Magnetic energy spectra averaged over the relaxation events. (Parameters: $a\lambda_0 = -3.88$, $\Lambda_e = 1, \Lambda_i = 1, S = 20,000, P_m = 1.0$)
Figure K.8: Magnetic energy spectra averaged over the relaxation events. (Parameters: $a\lambda_0 = 3.88$, $\Lambda_e = 1$, $\Lambda_i = 1$, $S = 20,000$, $P_m = 0.1$)
REFERENCES


