Kinetic energy error in the NIMROD spheromak simulations

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10/25/00

Dmitri Ryutov recently provided a memo (attached below for convenience) on the kinetic energy error associated with solving compressible fluid equations while ignoring density evolution, as we are presently doing in NIMROD. He suggested that we compute this error from the simulation results and compare it with physical dissipation. This is a good check, so I have performed these computations for the *S*=5000, *H/R*=1.5, uniform electrode B_z flux-core result, that we have used in our manuscripts as typical of the strongly driven but steady class of results. This note reports on the findings. [The energy check is part of the standard diagnostics in the 0-β version of the DEBS code.]

The volume integrals of the three powers are:

$$\int dVol\, \sigma^{-1} J^2 = 3.46\times 10^7 \left(\text{Watts} -- \text{but this is arbitrary}\right)$$

$$\int dVol\, \eta (\nabla \mathbf{V})^T : \nabla \mathbf{V} = 5.04\times 10^6 , \quad \eta \equiv \nu\rho$$

$$\int dVol\, \frac{\rho V^2}{2} \nabla\cdot\mathbf{V} = -\,7.0\times 10^5$$

So the error is only about 2% of the total power being dissipated, and the sign indicates that it is associated with flow being compressed. The current extracted from the electrodes is 1.0×10^6 Amps, and the applied potential is 35 V. There is also some current extracted and returned along the outer wall (staying in the boundary layer where the viscous dissipation primarily occurs), which probably accounts for the discrepancy in total powers.

Locally, the kinetic error shows up as two "hot spots" along the electrodes, as shown in Fig. 1. Away from the electrodes, the error is quite small. The peak value has a magnitude of approximately 1.9×10^7, which is smaller than the peak resistive power (shown in Fig. 2), but somewhat larger than the sum of the resistive and viscous power densities at the location of the error hot spots. (Viscous power density is shown in Fig. 3, and the sum of the physical power densities is shown in Fig. 4.) The physical dissipation density is dominated by resistive dissipation at the peak of the current path (the "dough-hook" in SPHEX parlance).

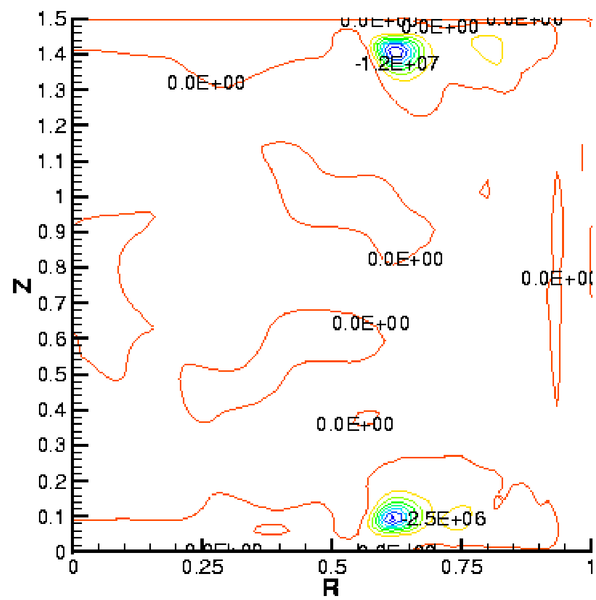

Fig. 1. Error power density, $\frac{\rho V^2}{2} \nabla \cdot \mathbf{V}$, plotted at a fixed toroidal angle (ϕ=π).

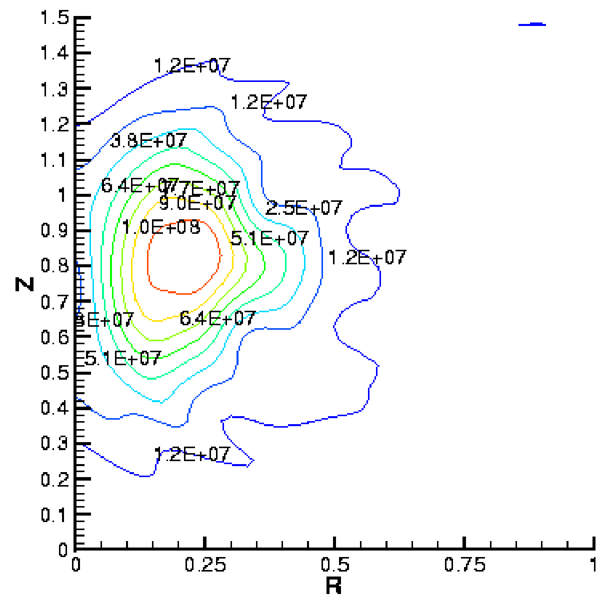

Fig. 2. Resistive power density, $\sigma^{-1} J^2$, plotted at a fixed toroidal angle (ϕ=π).

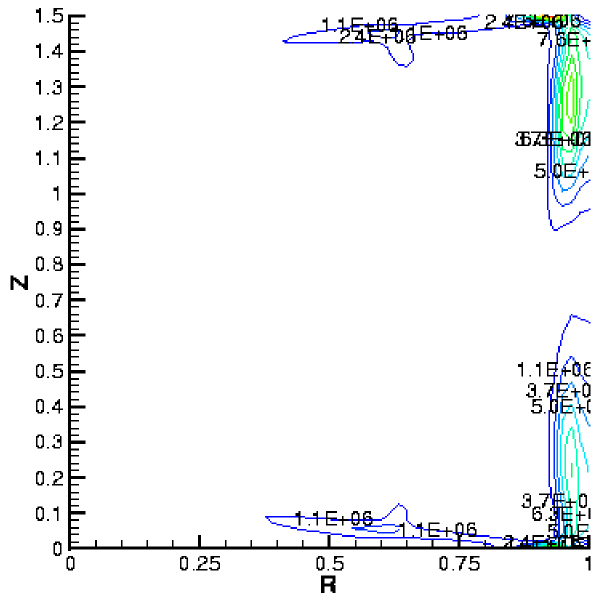

Fig. 3. Viscous power density, $\eta(\nabla\mathbf{V})^T : \nabla\mathbf{V}$, plotted at a fixed toroidal angle ($\phi=\pi$).

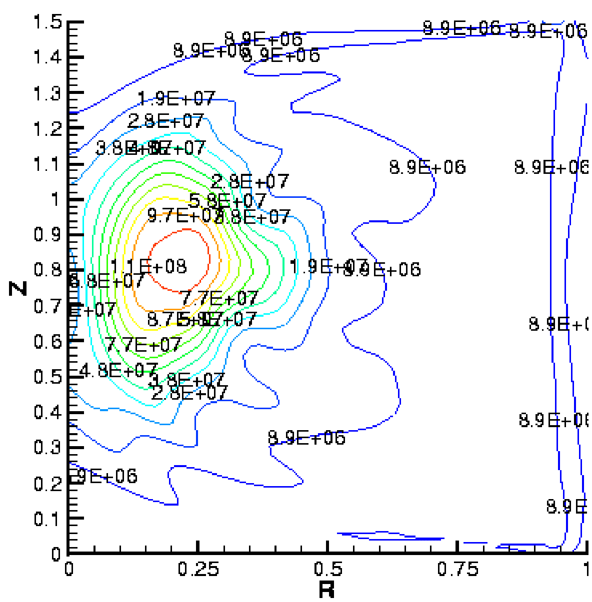

Fig. 4. Sum of resistive and viscous power densities plotted at a fixed toroidal angle ($\phi=\pi$).

The velocity vectors show the divergent behavior at the error hot spots. Figure 5 plots the n=0 Fourier component, which clearly contributes to the divergence. Flow from the outer wall appears to feed the toroidal flux supplied by the circuit into the regions where the toroidal **E**x**B** drifts are largest. A similar flow is present in the related axisymmetric pinch computation, but the velocity magnitude and energy error are smaller. I would therefore speculate that this poloidal flow is just part of how the system deals with the source of toroidal flux. [A related Gedanken experiment: Axial magnetic field in a cylinder is embedded in electrodes at the cylinder ends. Rotate the ends in opposite directions like solar footpoint motion to create azimuthal field. As the field is twisted, it collapses against the electrodes. If no-slip boundary conditions are imposed at the ends (as in the spheromak simulations), it may lead to compression in a boundary layer.] The maximum divergence always occurs near the electrodes, where non-fluid processes in an experiment may be largest, allowing divergent flow to be sustained.

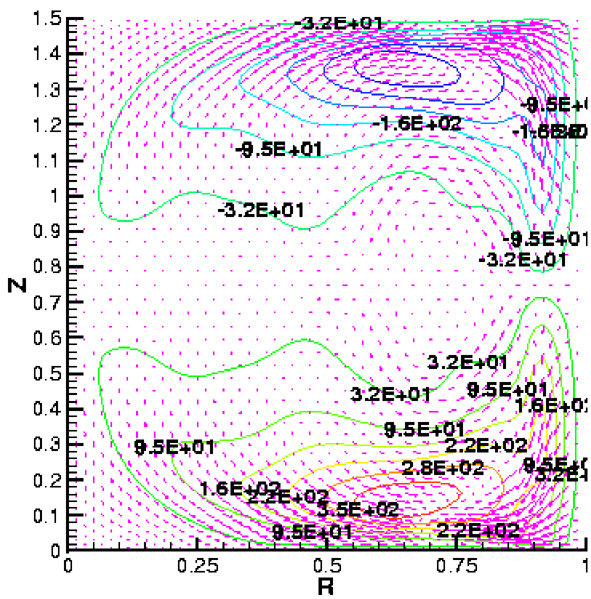

Fig 5. The n=0 part of **V**. Arrows show poloidal components, contours show the toroidal component, and the peak magnitudes of the toroidal and poloidal parts are similar.

Energy equation in NIMROD

We discuss here a steady state solution found in NIMROD simulations. The following set of equations (in CGS units) describes this steady-state solution:

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{c}\mathbf{j}\times\mathbf{B} + \eta\nabla^2\mathbf{v}$$
$$\mathbf{j} = \sigma(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}) \qquad (1)$$
$$\mathbf{E} = -\nabla\varphi$$

The quantities ρ, σ, and η are assumed to be constant. No mass continuity equation is used. Taking a scalar product of the first equation with **v**, using the identity **v·(j×B)**=-**j·(v×B)**, and substituting **v×B** from the second equation, one finds:

$$\rho\nabla\cdot(\mathbf{v}\frac{\mathrm{v}^2}{2}) - \frac{\rho\mathrm{v}^2}{2}\nabla\cdot\mathbf{v} = -\frac{j^2}{\sigma} - \nabla\cdot(\mathbf{j}\varphi) + \eta\mathbf{v}\cdot\nabla^2\mathbf{v} \qquad (2)$$

We have used equation $\nabla\cdot\mathbf{j}=0$, but not the mass continuity equation (because the latter is not a part of the set considered in NIMROD).

The viscous dissipation term can be rewritten as:

$$\mathbf{v}\cdot\nabla^2\mathbf{v} = \mathrm{v}_\alpha\frac{\partial^2\mathrm{v}_\alpha}{\partial x_\beta^2} = \frac{\partial}{\partial x_\beta}\left(\mathrm{v}_\alpha\frac{\partial\mathrm{v}_\alpha}{\partial x_\beta}\right) - \left(\frac{\partial\mathrm{v}_\alpha}{\partial x_\beta}\right)^2 \qquad (3)$$

Integrating this equation over the whole volume, and, where appropriate, transforming volume integrals to surface integrals, one finds:

$$\int_V\left[-\frac{\rho\mathrm{v}^2}{2}\nabla\cdot\mathbf{v} + \frac{j^2}{\sigma} + \eta\left(\frac{\partial\mathrm{v}_\alpha}{\partial x_\beta}\right)^2\right]dV =$$
$$-\int_S\varphi\mathbf{j}\cdot d\mathbf{S} - \int_S\frac{\rho\mathrm{v}^2}{2}\mathbf{v}\cdot d\mathbf{S} + \int_S\frac{\partial\mathrm{v}_\alpha}{\partial x_\beta}\mathrm{v}_\alpha dS_\beta \qquad (4)$$

In the first surface integral φ is constant over the surface of each electrode, so that the integration over these surfaces yields the term *IU*, where *U* is the applied voltage, and *I* is the total current between the electrodes; the contribution of the insulating gap is zero, because the normal component of the current is zero there. The second and the third surface integrals are zero, because boundary condition of zero velocity on the material surface is used. We end up with the following energy equation:

$$IU = \int_V \left[\frac{j^2}{\sigma} + \eta \left(\frac{\partial v_\alpha}{\partial x_\beta} \right)^2 \right] dV - \int_V \frac{\rho v^2}{2} \nabla \cdot \mathbf{v} dV \qquad (5)$$

The l.h.s. represents the work produced by an external source, and the first integral represents Ohmic and viscous losses. In a normal incompressible (ρ=const) hydrodynamics only this integral would be present in the r.h.s.. The second integral appears because $\nabla \cdot \mathbf{v}$ is not set to zero in NIMROD equations. It would be interesting to evaluate this term based on NIMROD results. If it is, say, 10% of the *IU* value, then it is probably harmless. On the other hand, if it is larger, then it may be a sign of a significant inconsistency in the basic equations. To be sure that this term is really not very important, it is better to apply a local constraint,

$$\left| \frac{\rho v^2}{2} \nabla \cdot \mathbf{v} \right| << \left[\frac{j^2}{\sigma} + \eta \left(\frac{\partial v_\alpha}{\partial x_\beta} \right)^2 \right] \qquad (6)$$

One can easily generalize Eq.(5) to a non-steady-state case. The electric field will not be curl-free any more and one has to drop the last equation of the set (1). One will need to use Maxwell equations. The result reads:

$$-\int_{S_{ins}} c \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \cdot d\mathbf{S} = \frac{d}{dt} \int_V \left(\frac{\rho v^2}{2} + \frac{B^2}{8\pi} \right) dV + \int_V \left[\frac{j^2}{\sigma} + \eta \left(\frac{\partial v_\alpha}{\partial x_\beta} \right)^2 \right] dV - \int_V \frac{\rho v^2}{2} \nabla \cdot \mathbf{v} dV \qquad (7)$$

The l.h.s. represents energy influx through the insulating gap (the sign is determined by that we orient the surface element along the outer normal). The last (unphysical) term is again present.

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