Implementing a Stress Tensor Force in NIMROD

C. Sovinec, 1/17/03

This quick note describes an appropriate weak form of a stress tensor with the divergence integrated by parts. This formulation is presently used for the simple viscous damping term, but it hasn't been used for scalar pressure, the Lorentz force, or the inertia. It could be used for the scalar pressure. Spatially varying density would complicate its implementation for inertia, but it may be possible. However, we probably don't want to use it for the Lorentz force, since the residual div(**b**) errors would introduce an unphysical parallel force—see Brackbill and Barnes JCP **35** note, 426 (1980).

Let $\alpha_{\mathrm{p,n}}(\mathbf{x})$ denote the NIMROD scalar basis functions including the factors for a 2D finite element basis function (p=1,2, …, # of f.e. bases in a mesh) and a Fourier basis function (n=0, 2, …, N); see the JCP manuscript dated 12/02 (UW-CPTC-02-5) or the developers tutorial, UW-CPTC-01-3. Our vector basis functions with right-handed (r,z,ϕ) cylindrical coordinates are

$$\mathbf{A}_{r,\mathrm{p,n}} = \alpha_{\mathrm{p,n}}(\mathbf{x})\hat{\mathbf{r}}(\phi)$$
$$\mathbf{A}_{z,\mathrm{p,n}} = \alpha_{\mathrm{p,n}}(\mathbf{x})\hat{\mathbf{z}}$$
$$\mathbf{A}_{\phi,\mathrm{p,n}} = \alpha_{\mathrm{p,n}}(\mathbf{x})\hat{\boldsymbol{\phi}}(\phi)$$

Although each is parallel to the respective unit direction vector, keep in mind that all have three components (two of which are zero). For slab geometry, (r,z,ϕ) becomes (x,y,z), and the unit vectors are independent of position.

A weak formulation of a stress tensor force will appear as

$$-\int d\mathbf{x}\, \mathbf{A}^*_{\nu,\mathrm{p,n}} \cdot \nabla \cdot \mathbf{T} = \int d\mathbf{x} \left(\nabla \mathbf{A}^*_{\nu,\mathrm{p,n}}\right)^T : \mathbf{T} \quad - \quad \oint d\mathbf{S} \cdot \mathbf{T} \cdot \mathbf{A}^*_{\nu,\mathrm{p,n}} \quad ,$$

where $\nu=r,z,\phi$. Note that the Cartesian-tensor-notation form of the total derivative is useful for identifying which components are contracted:

$$-A_{\nu pn_i}\delta_{ik}\frac{\partial}{\partial x_{\mathrm{j}}}T_{jk} = T_{jk}\delta_{ik}\frac{\partial}{\partial x_j}A_{\nu pn_i} \quad - \quad \frac{\partial}{\partial x_j}\left(A_{\nu pn_i}T_{jk}\delta_{ik}\right),$$

where i is the vector component index of the basis vectors. When the code carries out the integrals element by element, surface integrals between elements cancel. Whether a nonzero surface term appears at the boundary depends on the boundary conditions. A no-slip boundary condition makes the surface term identically zero for all possible tensors. For cylindrical geometry, the $\nabla \mathbf{A}^*_{\nu,\mathrm{p,n}}$ tensor is

$$\nabla \mathbf{A}^*_{\nu,\mathrm{p,n}} = \begin{pmatrix} \dfrac{\partial A^*_{\nu pn_r}}{\partial r} & \dfrac{\partial A^*_{\nu pn_z}}{\partial r} & \dfrac{\partial A^*_{\nu pn_\phi}}{\partial r} \\ \dfrac{\partial A^*_{\nu pn_r}}{\partial z} & \dfrac{\partial A^*_{\nu pn_z}}{\partial z} & \dfrac{\partial A^*_{\nu pn_\phi}}{\partial z} \\ -\dfrac{1}{r}\left(inA^*_{\nu pn_r} + A^*_{\nu pn_\phi}\right) & -\dfrac{1}{r}\left(inA^*_{\nu pn_z}\right) & -\dfrac{1}{r}\left(inA^*_{\nu pn\phi} - A^*_{\nu pn_r}\right) \end{pmatrix},$$

using the Fourier basis dependence to evaluate the ϕ-derivatives, including the conjugate operation. The required double-dot product can then be evaluated by finding the matrix product of $\left(\nabla \mathbf{A}^*_{\nu,\mathrm{p,n}}\right)^T$ and $\mathbf{T}$, and then finding its trace. When implementing this in a NIMROD rhs integrand routine, the first index of the "int" array corresponds to the ν index, which distinguishes the three basis vectors. The "iv" (formerly for 'vertex' but 'node' is more descriptive with high-order elements) index corresponds to the p index here.

The integrated-by-parts form of the force has a couple of advantages. First, it makes the weak form conservative in that fluxes across element boundaries are cancelled identically. For viscous damping, for example, the flux term represents transfer of momentum due to collisions. The remaining term is symmetric and negative definite, and it can only relax gradients (assuming a stable time-advance). This form should be convenient for complicated stress models, like neoclassical or simulation-particle fluxes, in that one does not need to compute gradients of $\mathbf{T}$. This may also improve accuracy for these models, since one does not have to project the distribution onto the bases used for the solution space.