Numerical Computation for Two-Fluid Reconnection and Dynamo

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Objectives

The objective of this study is to understand the influence of two-fluid effects in nonlinear magnetic relaxation events where there is a guide field, such as RFP dynamo and spheromak flux amplification.

Outline

• Introduction--motivation, equations, solution method
• Linear benchmarking--problem set-up, results
• Nonlinear evolution--temporal behavior, two-fluid signatures, dynamo
• Conclusions
Introduction--motivation

- Fast reconnection without a guide field is well established in space and other natural settings and has been studied extensively by many groups.

- Two-fluid reconnection with a guide field is expected for the tokamak sawtooth and has been modeled by the CMRS group.

- The MST group has measured correlation of magnetic field and current density fluctuations leading to Hall dynamo effects, i.e. $\langle \tilde{j} \times \tilde{b} \rangle / ne$, at RFP relaxation events.

- Quasilinear calculations by Mirnov, Hegna, and Prager predict strong but very localized Hall dynamo from two-fluid reconnection in the presence of a large guide field.

- Numerical computation is needed to investigate nonlinear conditions in single and multi-helicity states.
In this numerical study, we model reconnection and relaxation with a non-ideal Hall-MHD system of equations.

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left( \eta \mathbf{J} - \mathbf{V} \times \mathbf{B} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla p_e \right)
\]

Faraday’s / Ohm’s law

\[
\mu_0 \mathbf{J} = \nabla \times \mathbf{B}
\]

low-\(\omega\) Ampere’s law

\[
\nabla \cdot \mathbf{B} = 0
\]

divergence constraint

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \mathbf{J} \times \mathbf{B} - \nabla p + \nabla \cdot \nu \rho \mathbf{W}
\]

flow evolution

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = \nabla \cdot D \nabla n
\]

particle continuity

\[
\frac{n}{\gamma - 1} \left( \frac{\partial T_\alpha}{\partial t} + \mathbf{V}_\alpha \cdot \nabla T_\alpha \right) = -p_\alpha \nabla \cdot \mathbf{V}_\alpha + \nabla \cdot n \chi \nabla T_\alpha
\]

temperature evolution

\[
W \equiv \nabla \mathbf{V} + \nabla \mathbf{V}^T - \frac{2}{3} \mathbf{I} \nabla \cdot \mathbf{V}
\]

\[
\bullet \text{ We will set } D, \chi, \text{ and } \nu << \eta/\mu_0.
\]
Introduction—solution method

• We apply the NIMROD code (www.nimrodteam.org) to solve the HMHD equations with a high-order finite-element spatial representation.

• The advance temporally staggers velocity from number density, magnetic field, and temperature, and implicit stepping provides stability at large $\Delta t$.

\[
m_i n^{j+1/2} \left( \frac{\Delta V}{\Delta t} + \frac{1}{2} V^j \cdot \nabla \Delta V + \frac{1}{2} \Delta V \cdot \nabla V^j \right) - \Delta t L^{j+1/2}(\Delta V) + \nabla \cdot \Pi_i(\Delta V) = J^{j+1/2} \times B^{j+1/2}
\]

\[
- m_i n^{j+1/2} V^j \cdot \nabla V^j - \nabla p^{j+1/2} - \nabla \cdot \Pi_i(V^j)
\]

\[
\frac{\Delta n}{\Delta t} + \frac{1}{2} \nabla \cdot \left( V^{j+1} \cdot \Delta n - D \nabla \Delta n \right) = -\nabla \cdot \left( V^{j+1} \cdot n^{j+1/2} - D \nabla n^{j+1/2} \right)
\]

\[
\frac{3n}{2} \left( \frac{\Delta T_\alpha}{\Delta t} + \frac{1}{2} V_{\alpha}^{j+1} \cdot \nabla \Delta T_\alpha \right) + \frac{n}{2} \Delta T_\alpha \nabla \cdot V_{\alpha}^{j+1} + \frac{1}{2} \nabla \cdot q_\alpha(\Delta T_\alpha)
\]

\[
= -\frac{3n}{2} V_{\alpha}^{j+1} \cdot \nabla T_\alpha^{j+1/2} - n T_\alpha^{j+1/2} \nabla \cdot V^{j+1} - \nabla \cdot q_\alpha(T_\alpha^{j+1/2}) + Q_\alpha^{j+1/2}
\]

\[
\frac{\Delta B}{\Delta t} - \frac{1}{2} \nabla \times \left( V^{j+1} \times \Delta B \right) + \frac{1}{2} \nabla \times \frac{1}{ne} \left( J^{j+1/2} \times \Delta B + \Delta J \times B^{j+1/2} \right) + \frac{1}{2} \nabla \times \eta \Delta J
\]

\[
= -\nabla \times \left[ \frac{1}{ne} \left( J^{j+1/2} \times B^{j+1/2} - \nabla p_e \right) - V^{j+1} \times B^{j+1/2} + \eta J^{j+1/2} \right]
\]
Linear Benchmarking

We benchmark our linear computations over a wide range of parameters against asymptotic results for two-fluid tearing in sheared slabs. [Mirnov, Hegna, and Prager, Phys. Plasmas 11, 4481 (2004)]

- The sheared equilibrium component is $B_y(x) = B_y\infty \tanh(x/L)$
- $B_y\infty << B_z$, but $\nabla \cdot \nabla \neq 0$
- The equilibrium has finite pressure, but it is uniform, so drift effects are not considered.
- Computations have walls at $|x| >> L$, whereas analytically, $-\infty < x < +\infty$
- Analytically, the perturbed magnetic flux in the outer ideal region has solutions

$$\psi \sim e^{\mp kx} \left[ 1 + \frac{1}{kL} \tanh\left( \frac{\pm x}{L} \right) \right]$$

$$\Delta' L = L \frac{\psi' + - \psi'}{\psi} \bigg|_{x=0} = \frac{2}{kL} - 2kL$$

$k = 2\pi / L_y$
A schematic of the parameter space shows where different effects become important. (Based on Mirnov, et al.)

\[ \beta \equiv \left( \frac{c_s}{v_A} \right)^2, \quad \delta \equiv \sqrt{d_e^2 + D_\eta / \gamma} \rightarrow \sqrt{D_\eta / \gamma}, \quad d_{i,e} \equiv c / \omega_{i,e}, \quad \rho_s \equiv c_s / \Omega_i \]

- The regime most relevant for MST has small $\Delta'$ and large $\beta$. 
Our numerically computed growth rates are in good agreement with the analytical dispersion relation, provided that the tearing layer is sufficiently small with respect to the equilibrium scale.

<table>
<thead>
<tr>
<th>CASE</th>
<th>$\Delta L$</th>
<th>$kL$</th>
<th>$\beta$</th>
<th>$\rho_s/L$</th>
<th>$S \equiv \tau / \tau_0$</th>
<th>$\delta/L$</th>
<th>$\Gamma \equiv \gamma \tau_0 / k \rho_s$</th>
<th>$\Gamma_{ana}$*</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.28</td>
<td>0.93</td>
<td>0.002</td>
<td>0.56</td>
<td>$6.7 \times 10^3$</td>
<td>0.184</td>
<td>$8.51 \times 10^{-3}$</td>
<td>9.04 x $10^{-3}$</td>
</tr>
<tr>
<td>B</td>
<td>0.28</td>
<td>0.93</td>
<td>0.083</td>
<td>3.6</td>
<td>$6.7 \times 10^3$</td>
<td>0.172</td>
<td>$1.50 \times 10^{-3}$</td>
<td>1.61 x $10^{-3}$</td>
</tr>
<tr>
<td>B2</td>
<td>0.28</td>
<td>0.93</td>
<td>0.083</td>
<td>3.6</td>
<td>$2.7 \times 10^4$</td>
<td>0.119</td>
<td>$7.77 \times 10^{-4}$</td>
<td>7.84 x $10^{-4}$</td>
</tr>
<tr>
<td>C</td>
<td>5.3</td>
<td>0.33</td>
<td>0.083</td>
<td>3.6</td>
<td>670</td>
<td>0.11</td>
<td>0.106</td>
<td>0.111</td>
</tr>
<tr>
<td>D</td>
<td>5.3</td>
<td>0.33</td>
<td>0.083</td>
<td>3.6</td>
<td>67</td>
<td>0.25</td>
<td>0.203</td>
<td>0.221</td>
</tr>
<tr>
<td>E</td>
<td>5.3</td>
<td>0.33</td>
<td>0.083</td>
<td>3.6</td>
<td>6.7</td>
<td>0.65</td>
<td>0.297</td>
<td>0.371</td>
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<tr>
<td>F</td>
<td>24</td>
<td>0.083</td>
<td>0.083</td>
<td>3.6</td>
<td>6.7</td>
<td>1.11</td>
<td>0.401</td>
<td>0.524</td>
</tr>
<tr>
<td>G</td>
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<td>0.33</td>
<td>0.83</td>
<td>11.4</td>
<td>6.7</td>
<td>0.48</td>
<td>0.169</td>
<td>0.174</td>
</tr>
</tbody>
</table>

*Computed from the dispersion relation from Mirnov, et al.

- The definition of $S$ used here is $S \equiv L B_{\gamma_{\infty}} / D \eta \sqrt{\mu_0 \rho}$.
- Resistivity is the only dissipation mechanism in Cases A, B, and B2.
- Computations such as Case B are spatially resolved with a $48 \times 6$ mesh of bicubic elements, packed such that $\Delta x_0 \sim 0.1 \delta \sim 10^{-3} x_{\max}$.
- Temporal resolution is achieved at compressional wave-CFL # $\sim 10^6$. 
Viscous dissipation was anticipated for the nonlinear computations, and a parameter scan shows very little impact on linear growth rates with some broadening of the $V$ profile of the eigenfunction.

Eigenfunction profiles for $P_m=1$ show moderate broadening of $V_x$ relative to a case with $P_m=0$ over the equilibrium scale $L=0.33$. \hspace{1cm} (Pm $\equiv \nu \mu_0 / \eta$)

Both computations show the quadrupole structure of out-of-plane magnetic field within the reconnection scale $\delta = 5.7 \times 10^{-3}$. 
Nonlinear Evolution

Nonlinear computations in the small-Δ' large-β regime have the parameters of linear Case B with moderate additional dissipation for nonlinear numerical stability.

- Viscosity: Pm=0.1; particle: $\frac{D}{D_\eta}=0.01$; thermal: $(\gamma-1)\frac{\chi}{D_\eta}=0.01$
- There is an effective nonuniform emf that attempts to maintain the equilibrium current profile against resistive dissipation.
- The best-resolved computation so far has a packed 100×40 mesh of biquartic finite elements.

Plot of element borders over 1/2 of the domain indicates the multi-scale resolution requirement.
The nonlinear computation progresses from a linear eigenfunction through weakly nonlinear behavior to saturation over approximately 36,000 Alfvén times.

The temporal evolution of kinetic energy shows non-steady behavior that is presently the subject of further numerical checking.

In this case, the resulting magnetic island has a width that is small with respect to $L$. 
Nonlinear changes to out-of-plane $\mathbf{B}$ on the equilibrium scale exceed the quadrupole profile, but current density contours indicate its persistence along the island separatrix.

Out-of-plane magnetic field on the equilibrium scale at $t=18,000 \ \tau_a$. [$B_{eq}(0)=1.$]

Contours of constant $J_y$ at $t=18,000 \ \tau_a$ near the $X$-point show fine structure in the late nonlinear stage.
Correlation of current density and magnetic field fluctuations and flow and magnetic field fluctuations lead to Hall and MHD dynamo effects, respectively.

Quasilinear dynamo computations, based on the linear eigenfunction, show very strong and localized Hall dynamo, consistent with the analytical prediction. [Average over the y-direction is plotted.]

In nonlinear saturation, the Hall dynamo expands to the equilibrium scale $L$, and the two dynamos become comparable in magnitude. [For reference, the driving emf is of order $10^5$.]
Examining the dynamo contributions locally in $y$ shows that the two act in phase with respect to $y$ in the quasilinear result; though, the $x$-scales and magnitudes are very different.

Contours of local fluctuation-induced Hall electric field, parallel to the equilibrium magnetic field.

Contours of local fluctuation-induced MHD electric field, parallel to the equilibrium magnetic field.
In the nonlinear state, the local Hall electric field acts primarily near the separatrix and remains large in magnitude relative to the MHD contribution.

Contours of local fluctuation-induced Hall electric field. Comparison with the $y$-average indicates that most of it leads to shaping but not net redistribution of current density with respect to $x$.

Contours of local fluctuation-induced MHD electric field. The broad profile is expected for current density redistribution.
Conclusions

• The implicit leapfrog algorithm accurately reproduces two-fluid tearing in slab geometry with a large guide field, at large time-step.
• Nonlinear computations require greater spatial resolution to reproduce structure at the island separatrix.
• When the mode is at small amplitude, the computations reproduce the quasilinear prediction for Hall and MHD dynamo effects.
• With nonlinear saturation, the net Hall dynamo effect broadens and decreases in magnitude, becoming comparable to the net MHD dynamo effect; though, locally the amplitude of the Hall electric field remains far greater.
• Nonlinear single mode computations should also be performed with $L > \rho_s$ and in the other parameter regimes.
• Multi-helicity two-fluid computations are needed to understand relaxation through nonlinear interactions among fluctuations.