IDEAL LINEAR ANALYSIS

The analysis of small deviations from an equilibrium of the ideal MHD system is among the most mathematically rigorous aspects of plasma theory. It begins with an assessment of properties of the linear force operator. The findings suggest a variational approach and, subsequently, a shortcut to analyzing the stability of the equilibrium.

The ideal linear system with nontrivial equilibrium current and pressure distributions but no equilibrium flow is

\[
\rho_0 \frac{\partial ^2 \mathbf{s}}{\partial t^2} = \mathbf{j} \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{b} - \nabla p
\]

\[
p = -\mathbf{s} \cdot \nabla \mathbf{B}_0 - \mathbf{V}_p \cdot \mathbf{V} \mathbf{s}
\]

\[
\mathbf{b} = \mathbf{V} \times (\mathbf{s} \times \mathbf{B}_0)
\]

where \( \mathbf{s} \) is the displacement vector, and the perturbed pressure and magnetic field equations have been integrated with respect to \( t \). [The adiabatic pressure equation is used for closure.] The perturbed pressure and magnetic field can be eliminated, resulting in

\[
\rho_0 \frac{\partial ^2 \mathbf{s}}{\partial t^2} = \frac{1}{\mu} \left[ \mathbf{j} \times \nabla \times (\mathbf{s} \times \mathbf{B}_0) \right] \times \mathbf{B}_0 + \mathbf{j}_0 \times \nabla \times (\mathbf{s} \times \mathbf{B}_0) + (\mathbf{g} \cdot \nabla) \mathbf{g} + (\mathbf{g}_p \cdot \mathbf{g})
\]

\[
= \mathbf{F} (\mathbf{s})
\]

where \( \mathbf{F} \) is the (equilibrium-dependent) linear force operator.
The equation for the displacement is a linear (but not simple) hyperbolic PDE in the independent variables \((\bar{x}, t)\). Solving this system starts with the usual separation of variables technique:

\[
\frac{\ddot{\bar{x}}(\bar{x}, t)}{\bar{x}(\bar{x}, t)} = \phi(\bar{x}) \psi(t)
\]

\[
\frac{\ddot{x}(t)}{x(t)} = \frac{\ddot{\bar{\phi}}(\bar{x})}{\ddot{\bar{\phi}}(\bar{x})} = -\omega^2
\]

\[
\ddot{x} + \omega^2 x = 0,
\]

\[
x = e^{-\omega t}
\]

For magnetic confinement, we are interested in bounded systems, which leads to boundary-value, hence eigenvalue problems. The acceptable values of \(\omega^2\) are then determined by the eigenvalue problems associated with:

\[
-\omega^2 \bar{\phi}(\bar{x}) = \ddot{x}(\bar{x})
\]

plus appropriate boundary conditions for the displacement \(\bar{\phi}\). Returning to convention, let \(\bar{\bar{x}}\) represent just the spatial part of the displacement function \((\bar{\phi} \Rightarrow \bar{\bar{x}})\).

One can prove that the operator \(\ddot{\bar{\Phi}}\) is self adjoint:

\[
\int d\bar{x} \bar{\bar{x}} \cdot \ddot{\bar{\Phi}}(\bar{\bar{x}}) = \int d\bar{x} \bar{\bar{x}} \cdot \bar{\phi}(\bar{\bar{x}})
\]

[See Appendix A of Froidberg.] From this alone, we know
1) The eigenvalues \( \tau^2 \) are real, and
2) The eigenfunctions are orthogonal

\[ \tilde{s}_i \cdot \tilde{s}_j = 0 \quad \text{for } i \neq j \]

Condition 1) means that \( \omega \) (\( \tau \) of eigenvalue) is either purely real or purely imaginary. Thus, the linear ideal MHD system for a static equilibrium supports purely oscillatory modes \( e^{-i \omega t} \) or growth (or decay) without oscillation \( e^{\pm i \omega t} \). There are no overstable modes, so

\[ \text{Re}(\omega) \neq 0 \]

as one approaches conditions of marginal stability. We shall see that this leads to a simplified computation for determining stability.

[Note that \( \tilde{\nu} \neq 0 \) or nonzero dissipation changes the character of the linear system; growing or damped oscillations are possible.]

For the sheared slab or cylindrical (1D) equilibria, we will be able to separate \( \tilde{s} \) into factors for the ignorable coordinate directions (Fourier basis functions, \( e^{in\theta} e^{ikz} \)) and a factor for the inhomogeneous direction. We are also able to solve for the \( \tilde{s} \) components in the homogeneous directions in terms of the remaining component. This leaves us with
with an ODE for each \((n/m)\) pair, that can be analyzed with all of the very well established mathematical tools for ODEs.

[Alan Glasser has developed a similar method for toroidally symmetric systems. Only the toroidal direction is separable, but a poloidal expansion turns the problem into a system of coupled ODEs. The method is described in the manuscript, "The Direct Criterion of Newcomb for the Stability of Axisymmetric Toroidal Plasmas," and is embodied in the OCON code.]

Before looking into the simplified computation of stability for general systems, let's consider a simple system in a general way. The intent is to preview how the mathematics informs us of a variety of different linear behavior.

The example is a cylinder filled with uniform axial field and a cold \((T=0)\) conducting fluid with a slight density gradient.

\[
\begin{array}{c}
\text{\(p(0)\)} \\
\text{\(\rho_0(x)\)} \\
\text{0} \\
\text{\(a\)}
\end{array}
\]

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The linear equations reduce to

\[ \vec{b} = \nabla \times (\vec{a} \times \vec{B}^2) \]

\[ \mu_0 \rho_0^2 \frac{\partial^2 \hat{\rho}}{\partial t^2} = (\nabla \times \vec{a}) \times \vec{B}^2 \]

\[ = (\nabla \times \nabla \times (\vec{a} \times \vec{B}^2)) \times \vec{B}^2 \]

\[ \times \rho = \rho_0(0) \]

If we let \( \rho_0(r) = \vec{r} \rho(r) \), where \( \rho(r) \) is a dimensionless function \( \approx 1 \), we can take dimensions out of the system

\[ \mu_0 \vec{B}^2 \frac{\partial^2 \vec{a}^2}{\partial t^2} = (\alpha \nabla \times \nabla \times (\vec{a} \times \vec{B}^2)) \times \vec{B}^2 \]

Taking \( t \rightarrow t^\text{new} = \frac{t}{\alpha} \) \( \rightarrow t \) and \( r \rightarrow \frac{r}{\alpha} \), \( \vec{z} \rightarrow \frac{\vec{z}}{\alpha} \) (so that

\[ \nabla \rightarrow \alpha \nabla \] , where \( \alpha \) is the wall radius,

\[ \rho \frac{\partial^2 \vec{a}^2}{\partial \vec{z}^2} = (\nabla \times \nabla \times (\vec{a} \times \vec{B}^2)) \times \vec{B}^2 \]

\[ \vec{a} \times \vec{B}^2 = \vec{a} \times \vec{B}^2 - \vec{a} \times \vec{B}^2 \]

\[ \nabla \times (\vec{a} \times \vec{B}^2) = \frac{\partial \vec{a}}{\partial \vec{z}} + \frac{\partial \vec{B}^2}{\partial \vec{z}} - \left[ \frac{1}{\rho \vec{a}} (\vec{a} \vec{B}^2) + \frac{1}{\rho \vec{B}^2} \vec{a} \right] \vec{B}^2 \]

\[ \nabla \times \nabla \times (\vec{a} \times \vec{B}^2) = \left[ - \frac{1}{\rho} \left[ \frac{1}{\rho \vec{a}} (\vec{a} \vec{B}^2) + \frac{1}{\rho \vec{B}^2} \vec{a} \right] - \frac{\partial^2 \vec{B}^2}{\partial \vec{z}^2} \right] \vec{B}^2 \]

\[ + \left[ \frac{\partial^2 \vec{a}}{\partial \vec{z}^2} + \frac{\partial}{\partial \vec{z}} \left[ \frac{1}{\rho \vec{a}} (\vec{a} \vec{B}^2) + \frac{1}{\rho \vec{B}^2} \vec{a} \right] \right] \vec{B}^2 \]

\[ + \vec{a} \cdot \vec{B}^2 \]

\[ (\nabla \times \nabla \times (\vec{a} \times \vec{B}^2)) \times \vec{B}^2 = \left[ \frac{\partial^2 \vec{a}}{\partial \vec{z}^2} + \frac{\partial}{\partial \vec{z}} \left[ \frac{1}{\rho \vec{a}} (\vec{a} \vec{B}^2) + \frac{1}{\rho \vec{B}^2} \vec{a} \right] \right] \vec{B}^2 \]

\[ + \left[ \frac{1}{\rho} \left[ \frac{\partial^2 \vec{a}}{\partial \vec{z}^2} \vec{B}^2 + \frac{\partial^2 \vec{B}^2}{\partial \vec{z}^2} \vec{a} \right] \right] \vec{B}^2 \]

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For an infinite cylinder, we would use \( e^{j\omega t + ikr} \) with \( k \) continuous. Here we consider a periodic cylinder using \( e^{j\omega t + jnraz} \), where \( R = L/L_{2m} \) and \( L \) is the periodic axial dimension. Also using the \( \omega^2 \) constant from separating \( T(t) \), we have

\[
-w^2 \rho \hat{s}_r = \frac{d}{dr}\left( \frac{1}{r} \frac{d}{dr}\left( r \hat{s}_r \right) \right) - \frac{m^2}{R^2} \hat{s}_r + \frac{d}{dr}\left( \frac{in}{r} \hat{s}_\theta \right)
\]

\[
-w^2 \rho \hat{s}_\theta = - \left( \frac{m^2}{R^2} + \frac{m^2}{R^2} \right) \hat{s}_\theta + \frac{im}{r} \frac{d}{dr}(r \hat{s}_r)
\]

Eliminating \( \hat{s}_\theta \):

\[
\left( \frac{m^2}{R^2} + \frac{n^2}{R^2} - w^2 \right) \hat{s}_\theta = \frac{im}{r} \frac{d}{dr}(r \hat{s}_r)
\]

\[
\left( \frac{n^2}{R^2} - w^2 \right) \hat{s}_r = \frac{d}{dr}\left( \frac{1}{r} \frac{d}{dr}\left( r \hat{s}_r \right) \right) - \frac{d}{dr}\left( \frac{m^2}{R^2} \frac{d}{dr}(r \hat{s}_r) \right)
\]

\[
\frac{d}{dr}\left( \frac{1}{r} \left( 1 - \frac{m^2}{R^2} \frac{n^2}{R^2} \right) \frac{d}{dr}(r \hat{s}_r) \right) + \frac{1}{r} (w^2 - \frac{n^2}{R^2}) (r \hat{s}_r) = 0
\]

\[
\frac{d}{dr}\left( \frac{1}{r \left( \frac{w^2 - m^2}{R^2} \frac{n^2}{R^2} \right)} \left( w^2 - \frac{n^2}{R^2} \right) \frac{d}{dr}(r \hat{s}_r) \right) + \frac{1}{r} (w^2 - \frac{n^2}{R^2}) (r \hat{s}_r) = 0
\]

This ODE for \( r \hat{s}_r \) has the form

\[
\frac{d}{dr}\left( u \frac{d}{dr}(r \hat{s}_r) \right) - \nu \hat{s}_r + \omega^2 \hat{s}_r = 0
\]

where \( u, \nu, \) and \( \omega \) are continuous functions of the independent variable \( r \). Thus, our problem is cast as a Sturm-Liouville problem. This simple example is not exceptional in
this regard. More complicated IO equilibria will also produce Sturm-Liouville problems, too. With a wall at $r=a$, the outer boundary condition is $\xi_0(a) = 0$. At $r=0$, we have a (not uncommon) singular point, where we demand regular behavior from $\xi_1$.

Besides the properties arising from $F(\xi)$ being self-adjoint, the SL character ensures a finite minimum eigenvalue, $(\omega^2)_{\text{min}}$. This makes sense on physical grounds in that an unstable mode $(\omega^2>0)$ cannot grow at an infinite rate; inertia is nonzero, and the free energy is finite.

What makes our basic problem interesting is the form of $u(r)$. For a given $p(r)$ profile, this function,

$$\frac{1}{r(\omega^2 p(r) - \frac{\omega^2}{R^2}) (\omega^2 p(r) - \frac{\omega^2}{R^2})}$$

Can be nonzero across the domain $(a \leq r \leq 1)$, or it may go through 0 or encounter a singular point and change sign, depending on $(m,n)$ and $\omega^2$-values.

We could enlist a computer to find the entire spectrum of $\omega^2$ for each $(m,n)$ pair. However,
we can gain a lot of insight quickly without solving the problem by examining different regimes:

1) At sufficiently large $\omega^2$, $\omega^3 \rho > \frac{a^3}{R^2} + \frac{b^3}{R^2}$, and $u(r) \to \frac{1}{r}$. The ODE reduces to

$$\frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r \delta_r) \right) + \omega^3 \rho \delta_r = 0$$

or

$$\frac{d}{dr} \left( \frac{1}{r^2} \frac{d}{dr} \delta_r \right) + \omega^3 \rho \delta_r = 0$$

$$r^2 \frac{d^2 \delta_r}{dr^2} + r \frac{d \delta_r}{dr} + (\omega^2 r^2 - 1) \delta_r = 0$$

For $\rho$ with a weak $r$-dependence, $\rho(r) \approx 1$, this is approximately Bessel's equation with order 1 Bessel function solutions. The boundary conditions quantize the (positive) eigenvalues, and the large values imply rapid oscillation of $\delta_r$ in $r$. This is normal, i.e., discrete spectrum, behavior of Sturm-Liouville solutions.

2) For $m = 0$, we also have $u(r) \to \frac{1}{r}$. Here we find

$$\frac{\delta''_r}{r^2} + \frac{\delta'_r}{r} - \frac{\delta_r}{r^2} + (\omega^2 \rho - \frac{a^2}{R^2}) \delta_r = 0$$

using $\frac{d}{dr} \to \frac{1}{r}$.
\[ r^2 s'' + r s' + \left[ \frac{(\omega^2 - \frac{n^2}{R^2})r^2 - 1}{\lambda} \right] s = 0 \]

If we again consider \( p(r) = 1 \), we have Bessel's equation, except that nonzero \( n^2 \) leads to an upward shift in the frequency \( (\omega) \) as compared with the eigenvalue of the ODE, \( \lambda = \omega^2 - \frac{n^2}{R^2} \).

3) Checking the \( s_\theta \) equation for \( m = 0 \) shows that we can also expect torsional waves that are not coupled to any radial motion.

\[
(\omega^2 p(r) - \frac{n^2}{R^2}) s_\theta = 0
\]

Since \( p(r) \) varies from \( \frac{p_0(r)}{p_0(0)} \) to 1, there is a continuous range of frequencies that satisfy the dispersion relation, \( \omega^2 p(r) - \frac{n^2}{R^2} = 0 \):

\[
\frac{n}{R} \leq \omega \leq \frac{n}{R} \sqrt{\frac{p_0(0)}{p_0(r)}}
\]

4) For \( m \neq 0 \), the range \( \frac{n}{R} \leq \omega \leq \frac{n}{R} \sqrt{\frac{p_0(0)}{p_0(r)}} \) also produces curious behavior. In this range, \( u(r) \) vanishes at some value of \( r \) in the domain, producing a singular point in the ODE. At this point, changes in \( \frac{d}{dr}(u(r)) \) can get extremely large, while
the \( \frac{d}{dr} (u \frac{d}{dr}(r \cdot r_n)) \) term remains finite. In a physical system, 'other' physical mechanisms (such as dissipation, electron fluid effects, or kinetic effects) need to be considered in a layer of small enough spatial scale that MHD is inadequate. These 'other' effects remove the singularity without changing the global character.

5) Where the denominator of \( u \cdot r_n \)

\[ \omega^2 \rho(r) - \frac{m^2}{r^2} - \frac{n^3}{R^2} \quad (m \neq 0) \]

goes through 0, \( |u| \rightarrow \infty \) and \( u(r) \) changes sign. Here, \( \frac{d}{dr}(r \cdot r_n) \) must be 0 to keep \( \frac{d}{dr}(u \frac{d}{dr}(r \cdot r_n)) \) finite. Since the second term in the equation, \( \frac{1}{r} (\omega^2 \rho - \frac{n^3}{R^2}) (r \cdot r_n) \), has a coefficient that does not change sign at this point, the character of the ODE changes,

\[ + \frac{d}{dr} \left( |u| \frac{d}{dr} y \right) + y = 0 \quad \text{locally negative, } \frac{d}{dr} y \]

\[ \text{local curvature, } \frac{d}{dr} y \]
\[- \frac{\partial}{\partial r} \left( |u| \frac{\partial}{\partial r} \gamma \right) + \gamma = 0 \rightarrow \text{locally positive relative curvature}\]

Note that for sufficiently small \( \omega^2 \), the character of the equation can be determined from

\[ \frac{d}{dr} \left( \frac{1}{r(1 + \frac{m^2 \gamma^2}{\omega^2})} \right) - \frac{1}{r} \left( \frac{m^2 \gamma^2}{\omega^2} - \omega^2 \right) (r^2 \gamma) = 0 \]

> 0

and the relative curvature of \( r \gamma \) is positive throughout. This behavior is inconsistent with a boundary value problem, so there is no solution for sufficiently small \( \omega^2 \).

For general 1D equilibria, the ODEs are much more complicated, but the form of the equation is similar. Therefore, we can expect a similar mix of discrete eigenmodes and continuous bands with singular eigenfunctions.

We also have instabilities arising in the general case. Consider, again, the previous example with an added term, \( O(r) 3r \), representing a
destabilizing source of free energy.

\[
\frac{d}{dr} \left( \frac{\frac{n^2}{r^2} - \omega^2 \rho}{r \left( \frac{n^2}{r^2} + \frac{n^2}{\rho^2} - \omega^2 \rho \right)} \right) - \frac{1}{r} \left( \frac{n^2}{r^2} - \omega^2 \rho - D \rho \right)(r^2) = 0
\]

If \( D(r) \) is sufficiently large, even \( \omega^2 < 0 \) will result in negative relative curvature. A value of \( \omega^2 < 0 \) that has a solution satisfying the boundary conditions results in an unstable eigenmode of the system.

**Energy Principle**

Returning to the eigenmode problem for an arbitrary static equilibrium in arbitrary geometry,

\[-\omega^2 \rho \frac{1}{\rho^2} = \bar{F}(\frac{1}{\rho})\]

with an appropriate boundary condition, such as \( \vec{3} \cdot \vec{n} = 0 \) at the wall, we can use methods of variational calculus to provide an alternative approach. If we consider functions \( \vec{n}(\vec{x}) \) from a suitable space (one with an appropriate level of continuity and restrictions for "essential" boundary conditions), we could recast the problem as finding \(-\omega_j^2, \vec{3}_j\) such that

\[-\omega_j^2 \int \text{d}x \rho \vec{n} \cdot \vec{3}_j = \int \text{d}x \vec{n} \cdot \bar{F}(\frac{1}{\rho})\]