destabilizing source of free energy.

\[
\frac{d}{dr} \left( \frac{\frac{n^2}{r^2} - \omega^2 \rho}{r \left( \frac{n^2}{r^2} + \frac{n^2}{r^2} - \omega^2 \rho \right)} \right) - \frac{1}{r} \left( \frac{n^2}{r^2} - \omega^2 \rho - D \rho \right) \left( r \vec{r}_r \right) = 0
\]

If \( D(r) \) is sufficiently large, even \( \omega^2 \rho \) will result in negative relative curvature. A value of \( \omega^2 \rho \) that has a solution satisfying the boundary conditions results in an unstable eigenmode of the system.

---

**Energy Principle**

Returning to the eigenmode problem for an arbitrary static equilibrium in arbitrary geometry,

\[
-\omega^2 \rho \hat{\omega} = \hat{\varphi}(\vec{r})
\]

with an appropriate boundary condition, such as \( \hat{\omega} \cdot n = 0 \) at the wall, we can use methods of variational calculus to provide an alternative approach. If we consider functions \( \hat{\varphi}(\vec{r}) \) from a suitable space (one with an appropriate level of continuity and restrictions for "essential" boundary conditions), we could recast the problem as finding \( (-\omega^2, \hat{\omega}) \) such that

\[
-\omega^2 \int d\vec{r} \rho \hat{\varphi} \cdot \hat{\omega} = \int d\vec{r} \hat{\varphi} \cdot \hat{\varphi}(\vec{r})
\]
For all $\hat{\psi}(\hat{z})$ in the suitable space. To turn this into a variational approach, consider the Rayleigh quotient,

$$ R(\hat{z}) = -\frac{\frac{1}{2} \delta \hat{z} \cdot \hat{\psi}(\hat{z})}{\frac{1}{2} \delta \hat{z} \cdot \hat{\phi} \cdot \hat{\psi}(\hat{z})} $$

for arbitrary $\hat{z}$ in the space. For $\hat{z} = \hat{z}_0$, we have $R(\hat{z}) = -\omega_0^2$. If we perturb $\hat{R}$ about an arbitrary $\hat{z}$,

$$ \delta R = -\frac{\delta \hat{z} \cdot \delta \hat{\psi} \cdot \hat{\psi}(\hat{z}) + \delta \hat{z} \cdot \delta \hat{\phi} \cdot \hat{\psi}(\hat{z})}{\delta \hat{z} \cdot \hat{\phi} \cdot \hat{\psi}(\hat{z})} $$

$$ + \frac{\delta \hat{z} \cdot \delta \hat{\psi} \cdot \hat{\phi}(\hat{z})}{(\delta \hat{z} \cdot \hat{\phi} \cdot \hat{\psi}(\hat{z}))^2} \left( \delta \hat{z} \cdot \hat{\phi} \cdot \delta \hat{\psi}(\hat{z}) \right) $$

($\delta \hat{\psi}(\hat{z}) = \hat{\psi}(\hat{z})$ uses the fact that $\hat{\psi}$ is a linear function of its argument.) The coefficient of the second term is equivalent to $-\frac{R(\hat{z})}{\delta \hat{z} \cdot \hat{\phi} \cdot \hat{\psi}(\hat{z})}$ from the definition of $R(\hat{z})$. Using the self-adjoint property of $\hat{\psi}$, $\delta \hat{z} \cdot \delta \hat{\psi} \cdot \hat{\phi}(\hat{z}) = \delta \hat{z} \cdot \delta \hat{\phi} \cdot \hat{\psi}(\hat{z})$, so

$$ \delta R = -\frac{\delta \hat{z} \cdot \delta \hat{\psi} \cdot [\hat{\psi}(\hat{z}) + \hat{\phi}(\hat{z}) \hat{\psi}(\hat{z})] + \delta \hat{z} \cdot \delta \hat{\phi} \cdot [\hat{\psi}(\hat{z}) + \hat{\phi}(\hat{z}) \hat{\psi}(\hat{z})]}{\delta \hat{z} \cdot \hat{\phi} \cdot \hat{\psi}(\hat{z})} $$

Although $\delta \hat{z} \cdot \delta \hat{\psi} \cdot [\hat{\psi}(\hat{z}) + \hat{\phi}(\hat{z}) \hat{\psi}(\hat{z})] = 0$,

$$ \hat{\psi}(\hat{z}) + \hat{\phi}(\hat{z}) \hat{\psi}(\hat{z}) \neq 0 \text{ in general, i.e.} $$

the local relation $\hat{\psi}(\hat{z}) + \hat{\phi}(\hat{z}) \hat{\psi}(\hat{z}) = 0$ only holds when $\hat{z}$ is an eigen mode. Since $\delta \hat{z} \cdot \hat{\psi}(\hat{z})$ is arbitrary, it can be very localized, so
\[
\int dx \, \left[ \dot{F}(\vec{\delta}) + R \rho_0 \delta_x^2 \right] + C.C.
\]
tests whether \( \ddot{\vec{\delta}} + R \rho_0 \delta^2 = 0 \) is valid locally, when we check all possible \( \delta_x^2 \).

To summarize, the variation (\( \delta K \)) of

\[
R(\vec{\delta}) = \frac{\delta W(\vec{\delta}^*, \vec{\delta})}{K(\vec{\delta}^*, \vec{\delta})}
\]

where

\[
\delta W(\vec{\delta}^*, \vec{\delta}) = -\frac{1}{2} \int dx \, \delta_{xx} \cdot F(\vec{\delta})
\]

\[
K(\vec{\delta}^*, \vec{\delta}) = \frac{1}{2} \int dx \, \rho_0 \delta_x^4 / \delta^2
\]

vanishes if and only if \( \vec{\delta} \) is an eigenmode of the system, in which case \( R(\vec{\delta}_0) = \omega^2 \vec{\delta}_0 \). \( (\omega^2, \vec{\delta}_0) \) are eigen pairs.

While the variational approach is used to compute spectra for arbitrary equilibria, we can recognize that \( K(\vec{\delta}^*, \vec{\delta}) > 0 \) for \( \vec{\delta} \neq \vec{0} \), so the existence of unstable modes \( (\omega^2 < 0) \) is determined by the sign of \( \delta W(\vec{\delta}^*, \vec{\delta}) \). The Energy Principle is the statement that an equilibrium is stable if and only if \( \delta W \geq 0 \) for all possible perturbations.

The physical interpretation is the same as that for the mechanical pendulum example described in the introductory part of the course. Since \( F \) is the force density, and it is a linear functional,
\[ \frac{1}{2} \int d\bar{\mathbf{x}} \bar{\mathbf{F}} \cdot \mathbf{F}(\bar{\mathbf{x}}) \] is the work done by the system in moving from the equilibrium to a nearby state that is a displacement \( \bar{\mathbf{s}} \) from the equilibrium. \[ \int d\bar{\mathbf{x}} \bar{\mathbf{F}} \cdot \mathbf{F}(\bar{\mathbf{x}}) \] is a real number because \( \mathbf{F}(\bar{\mathbf{x}}) \) is self-adjoint. As a conservative force field (in the sense that \( \int d\bar{\mathbf{x}} \bar{\mathbf{F}} \cdot \mathbf{F} \) is path independent), we identify
\[
\delta W(\bar{\mathbf{s}}, \bar{\mathbf{s}}) = -\frac{1}{2} \int d\bar{\mathbf{x}} \bar{\mathbf{F}} \cdot \mathbf{F}(\bar{\mathbf{x}})
\]
as the change in potential energy of the magneto-fluid system incurred over the displacement \( \bar{\mathbf{s}} \) from the equilibrium. Since total energy (kinetic + potential) is conserved, \( \delta W < 0 \) indicates an increase in kinetic energy when moving from the equilibrium, hence instability.

Verifying stability for a configuration requires a search for the minimum value of \( \delta W \) and checking that it is \( \geq 0 \). We cannot have \( \delta W < 0 \) without an eigenvalue \( \omega^2 < 0 \). The physics reasoning above supports this claim. Mathematically, the argument is complicated by the existence of continua. However, if we find any \( \omega \) such that \( \delta W < 0 \), we know the configuration is unstable without further investigation.

From here, we will find explicit forms for \( \delta W \) and use the Energy Principle to find some important and general results. Before proceeding, we should clarify our use of \( \bar{\mathbf{s}}(\mathbf{x}_0, t) \). In particular, we have
written expressions like $\nabla \hat{s}$, but $\nabla$ is an operator with respect to actual position, whereas $\hat{s}$ is defined to be a function of the undisturbed position, $\hat{s}_0 = \hat{s} - \hat{s}$. In most of the domain, the subtle difference between $\hat{s}$ and $\hat{s}_0$ does not matter. However, it is important at the interface between the plasma and vacuum regions.

In the original paper on the Energy Principle by Bernstein, Frieman, Krustal, and Kulsrud, the authors show that $\nabla$ transforms to $\nabla_0 - (V_0 \hat{s})$, $\nabla_0$ to first order in $\hat{s}$, where $\nabla_0 = \hat{i} \frac{\partial}{\partial \hat{x}_0} + \hat{j} \frac{\partial}{\partial \hat{y}_0} + \hat{k} \frac{\partial}{\partial \hat{z}_0}$. Thus, to lowest order, $\nabla \hat{s} = V_0 \hat{s}_1$, where $\hat{s}_1$ is a perturbed quantity. The transformation does provide a Lagrangian formulation, however, so the distortion of the plasma/vacuum interface is included in $\int \hat{s}_1$ integrals over the equilibrium plasma region ($R_{po}$).
[If the integrals were carried to finite displacement, a Jacobian, \( \frac{\partial (x, y, z)}{\partial (x_0, y_0, z_0)} \), would be required. Here, \( \delta \) is considered small, so the Jacobian is unity plus small corrections that are proportional to \( 13^\alpha, \alpha > 1 \).]

Starting from the Eulerian perspective, substituting \( \nabla_0 \cdot (\nabla_0 \delta) \cdot \nabla_0 = \nabla_0 \), and keeping only first order in \( \delta \),

\[
\delta W (\delta x, \delta y) = - \frac{1}{2} \int \delta \rho_0 \delta x \cdot \delta y \cdot \rho_0 \delta x \cdot \nabla_0 \times (\delta x \delta y) \times \delta x_0 + \nabla_0 \delta x_0 \times (\delta x \delta y)
\]

Defining \( \varphi = \nabla_0 \times (\delta x \delta y) \)

\[
\delta W = \frac{1}{2} \int \delta \rho_0 \left( \frac{1}{\rho_0} \varphi \cdot \rho_0 \varphi \cdot \nabla_0 \right) + \frac{1}{2} \delta \rho_0 \varphi \cdot \nabla_0 \left( \varphi \right) - \frac{1}{2} \delta \rho_0 \left[ \varphi \cdot \nabla_0 \varphi + \left( \varphi \cdot \nabla_0 \rho \right) \rho_0 \right]
\]

Our search for the minimum \( \delta W \) will be easier if we separate parallel and perpendicular (wrt \( \delta y \)) contributions to \( \delta_0 \times \delta x + \nabla_0 \delta \cdot \delta_0 \rho_0 \). [Freidberg Ch. 8 & Appendix A]

\[
\delta_0 \cdot \left[ (\delta_0 \times \delta y) + \nabla_0 \delta \cdot \delta_0 \rho_0 \right] = - \delta_0 \cdot \delta_0 \delta y + \nabla_0 \cdot \left[ \delta_0 \delta \cdot \delta_0 \rho_0 \right]
\]

\[
= - \delta_0 \cdot \delta_0 \rho_0 + \nabla_0 \cdot \left[ \delta_0 \delta \cdot \delta_0 \rho_0 \right]
\]

\[
= \nabla_0 \cdot \left[ \delta_0 \rho_0 \times (\delta x \delta y) + \delta_0 \delta \cdot \delta_0 \rho_0 \right]
\]

\[
= 0
\]
We can therefore write

\[ \hat{\mathbf{F}}^* \cdot \left[ \hat{\mathbf{S}}_e \times \hat{\mathbf{A}} + \nabla_\mathbf{r} \cdot \hat{\mathbf{S}}_e \cdot \nabla_\mathbf{r} \right] = \left[ \hat{\mathbf{F}}^* \cdot \left( \hat{\mathbf{S}}_e \times \hat{\mathbf{A}} + \nabla_\mathbf{r} \cdot \hat{\mathbf{S}}_e \cdot \nabla_\mathbf{r} \right) \right] \]

\[ = \hat{\mathbf{F}}^* \cdot \left( \hat{\mathbf{S}}_e \times \hat{\mathbf{A}} + \nabla_\mathbf{r} \cdot \hat{\mathbf{F}}^* \cdot \left( \hat{\mathbf{S}}_e \right) \right) - \left( \nabla_\mathbf{r} \cdot \hat{\mathbf{F}}^* \right) \cdot \hat{\mathbf{S}}_e \cdot \nabla_\mathbf{r} \]

Converting divergence terms to surface integrals, over the equilibrium plasma - vacuum interface in the \( \mathbf{r}_0 \) variables,

\[ \delta W = \frac{1}{2} \int_{\mathbf{r}_0} \left\{ \frac{1}{\mu_0} \hat{\mathbf{F}}^* \cdot \left( \hat{\mathbf{S}}_e \times \hat{\mathbf{A}} + \nabla_\mathbf{r} \cdot \hat{\mathbf{F}}^* \cdot \left( \hat{\mathbf{S}}_e \right) \right) \right\} d\mathbf{r}_0 \]

\[ + \frac{1}{2} \int_{\mathbf{S}_{\mathbf{r}_0}} \left\{ \hat{\mathbf{S}}_e \cdot \hat{\mathbf{F}}^* \times \left( \hat{\mathbf{S}}_e \times \hat{\mathbf{A}} \right) \right\} d\mathbf{S}_{\mathbf{r}_0} \]

Using \( \delta W_{\mathbf{r}_0} \) for the volume integral over \( \mathbf{r}_0 \), and \( \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{b}}_0 = 0 \),

\[ \delta W = \delta W_{\mathbf{r}_0} + \frac{1}{2} \int_{\mathbf{S}_{\mathbf{r}_0}} \left\{ \hat{\mathbf{S}}_e \cdot \left( \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{F}}^* \right) \right\} d\mathbf{S}_{\mathbf{r}_0} \]

The volume integral over the plasma region, \( \delta W_{\mathbf{r}_0} \), is in a standard form. The surface integral will be further manipulated into something more useful and intuitive by introducing interface relations. To apply these relations, we need to know how \( \mathbf{B} \) and \( \mathbf{P} \) change in the Lagrangian perspective. Adding \( \hat{\mathbf{S}}_e \cdot \nabla \mathbf{r}_0 \) and \( \hat{\mathbf{S}}_e \cdot \nabla \mathbf{r}_0 \) to the linear equations for \( \mathbf{p} \) and \( \mathbf{b} \), respectively, gives us

\[ \nabla \mathbf{r}_0 = -\nabla \mathbf{r}_0 \mathbf{r}_0 \hat{\mathbf{S}} \]

\[ \hat{\mathbf{b}}_l = -\hat{\mathbf{a}}_l \cdot \hat{\mathbf{S}} + \hat{\mathbf{a}}_l \cdot \hat{\mathbf{S}} = \hat{\mathbf{a}} + \hat{\mathbf{b}}_l \mathbf{r}_0 \]

182
\[ \delta W - \delta W_F = \frac{1}{2} \int dS_o (\hat{n}_o \cdot \vec{\mathbf{b}}_L) \left\{ -\left(\frac{\hat{b}_o \cdot \nabla \hat{b}_o}{\rho_o} - \frac{\hat{b}_o \cdot \vec{B}_o}{\rho_o} - \frac{\hat{b}_o \cdot \vec{b}_L}{\rho_o} \right) \right\} \]

and since \((\nabla \hat{b}_o) \cdot \vec{b}_o = \hat{b}_o \times (\nabla \times \hat{b}_o) \), \(\hat{b}_o \cdot \nabla \hat{b}_o = \frac{1}{2} \nabla \hat{b}_o^2 \),

\[ \delta W - \delta W_F = \frac{1}{2} \int dS_o (\hat{n}_o \cdot \vec{\mathbf{b}}_L) \left\{ -\frac{\hat{b}_o \cdot \nabla \hat{b}_o}{\rho_o} + \frac{\hat{b}_o \cdot \vec{b}_L}{\rho_o} \right\} \]

Note that \(\rho_L + \frac{\hat{b}_o \cdot \vec{b}_L}{\rho_o}\) is the 1st-order pressure at the interface. Since there is no flow through the interface, the interface relation gives us

\[ \frac{\hat{b}_o \cdot \vec{b}_L}{\rho_o} = \rho_L + \frac{\hat{b}_o \cdot \vec{b}_L}{\rho_o} \]

where \(\hat{b}_o\) and \(\vec{b}_L\) are the vacuum fields outside the interface.

We will also need the interface relation from Faraday's law,

\[ \hat{n}_L \times (\vec{E} + \nabla \times (\vec{B} - \vec{b}_L) - \vec{E}_p) = \vec{0} \]

(Eulerian)

where \(\vec{E}\) is the electric field in the vacuum in the lab frame (see p. 22). For ideal MHD, \(\vec{E}_p = -\nabla \times \vec{B}_p\), so

\[ \hat{n}_L \times \vec{E} = -\hat{n}_L \times (\nabla \times \vec{B}) \]

To return to the time-integrated variables, we introduce a vector potential \(\hat{A}\) such that \(\nabla \times \hat{A} = \vec{B}\) and choose
the $\phi = 0$ gauge so that $\hat{E} = -\frac{\partial \hat{A}}{\partial t}$. We then have

$$\hat{n}_0 \times \hat{A}_1 = \hat{n}_0 \times (\frac{\partial}{\partial t} \hat{B}_0)$$

for the first order (in $\frac{\partial}{\partial t}$) vector potential, $\hat{A}_1$. Since $\hat{A}$ is in an Eulerian frame, $(\frac{\partial \hat{B}}{\partial t} = \nabla \times \frac{\partial \hat{A}}{\partial t})$,

$$\hat{B}_0 = \nabla \times \hat{A}_1 + \frac{\partial}{\partial t} \nabla \times \hat{B}_0$$

Using the linear pressure balance, the surface term becomes

$$\delta W - \delta W_F = \frac{1}{2} \int dS_0 \left( \hat{u}_0 \cdot \hat{n}_0 \right) \left\{ \nabla \times \hat{B}_0 \left( \frac{\partial \hat{B}_0^2}{\partial \rho_0} - \frac{\partial \hat{B}_0^2}{\partial \rho_0} - \rho_0 \right) + \frac{\hat{B}_0 \cdot \hat{B}_0}{\rho_0} \right\}$$

$$\{| sp\}$$

$$= \frac{1}{2} \int dS_0 \left( \hat{u}_0 \cdot \hat{n}_0 \right) \left\{ \nabla \times \hat{B}_0 \left( \frac{\partial \hat{B}_0^2}{\partial \rho_0} - \frac{\partial \hat{B}_0^2}{\partial \rho_0} - \rho_0 \right) + \frac{\hat{B}_0 \cdot \hat{B}_0}{\rho_0} \right\}$$

$$\{| sp\}$$

The first term in this relation is left as a surface contribution to $\delta W$,

$$\delta W_s = \frac{1}{2} \int dS_0 \left( \hat{u}_0 \cdot \hat{n}_0 \right) \left[ \nabla \times \hat{B}_0 \left( \frac{\partial \hat{B}_0^2}{\partial \rho_0} + \rho_0 \right) \right]$$

$$\{| sp\}$$

where the tangential contribution to $\hat{E} \cdot \nabla [\hat{B}_0]$ vanishes by the zeroth-order pressure balance.

For the remaining term, we use the surface
relation for \( \mathbf{A}_1 \),

\[
\hat{\mathbf{n}}_0 \times \mathbf{A}_1 = \hat{\mathbf{n}}_0 \times \left( \frac{1}{2} \hat{\mathbf{B}}_0 \right) \Rightarrow -\mathbf{B}_0 \left( \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{r}} \right)
\]

\[
\frac{1}{2} \int_{S_{W_0}} \left( \hat{\mathbf{r}} \times \hat{\mathbf{n}}_0 \right) \cdot \mathbf{B}_0 \cdot \frac{\partial \mathbf{A}_1}{\partial \mathbf{r}} \frac{\partial \mathbf{A}_1}{\partial \mathbf{r}} = -\frac{1}{2} \int_{S_{W_0}} \hat{\mathbf{n}}_0 \times \mathbf{A}_1^* \cdot \frac{\partial \mathbf{A}_1^*}{\partial \mathbf{r}} \frac{\partial \mathbf{A}_1^*}{\partial \mathbf{r}}
\]

\[
= \frac{1}{2} \int_{S_{W_0}} \mathbf{D}_0 \cdot \left( \hat{\mathbf{r}} \times \frac{\partial \mathbf{A}_1^*}{\partial \mathbf{r}} \right)
\]

where the sign change results from going to a volume integral over the complementary vacuum volume.

Note that there are no contributions from conducting walls surrounding the vacuum, where

\[
\hat{\mathbf{n}} \times \mathbf{E} = -\hat{\mathbf{n}} \times \frac{\partial \mathbf{A}}{\partial t} = \mathbf{0}, \text{ hence}
\]

\[
\hat{\mathbf{n}} \times \mathbf{A} = \hat{\mathbf{n}} \times \mathbf{A}_0,
\]

\[
\hat{\mathbf{n}} \times \mathbf{A}_1 = \mathbf{0}.
\]

This vacuum contribution is then

\[
\delta W_0 = \frac{1}{2} \int_{S_{W_0}} \mathbf{D}_0 \cdot \frac{\partial \mathbf{A}_1^*}{\partial \mathbf{r}} \cdot \frac{\partial \mathbf{A}_1^*}{\partial \mathbf{r}}
\]

To summarize, the \( \delta W(\xi, \xi) \) energy integral has three parts: a fluid part \( \delta W_F \), a plasma/vacuum surface contribution \( \delta W_S \), and a vacuum contribution \( \delta W_0 \).
\[ \delta W_F = \frac{1}{2} \sum_{r_o} \left\{ \frac{1}{\rho_o} \delta_{1,1} \cdot \delta_{1,1} \cdot \partial \cdot \partial + \frac{\delta_{2,1}}{\rho_o} \right\} \]

\[
\uparrow \quad \text{perturbed magnetic energy in plasma}
\]

\[
\delta W_\phi = \frac{1}{2} \sum_{r_o} \left\{ \frac{1}{\rho_o} \partial \cdot \partial + \frac{\delta_{2,1}}{\rho_o} \right\} \]

\[
\uparrow \quad \text{work done by net equilibrium surface pressure arising from surface motion}
\]

\[
\delta W_v = \frac{1}{2} \sum_{r_o} \left\{ \frac{1}{\rho_o} \partial \cdot \partial + \frac{\delta_{2,1}}{\rho_o} \right\} \left( \hat{B}_1 \cdot \hat{B}_1 \right)
\]

\[
\uparrow \quad \text{perturbed magnetic energy in vacuum} \quad \hat{B}_1 = \nabla \times \hat{A}
\]

The perturbed magnetic energy terms and the fluid compression term are positive-definite, hence they are always stabilizing. A few more manipulations on \( \delta W_F \) produces what is called its "intuitive" form,

\[
\delta W_F = \frac{1}{2} \sum_{r_o} \left\{ \frac{1}{\rho_o} \delta_{1,1}^2 + \frac{\delta_{2,1}^2}{\rho_o} \left[ \nabla \cdot \left( \frac{\hat{B}_1}{\rho_o} \right) + 2 \delta_{1,1} \cdot \hat{B}_1 \right] \right\} \]

\[
- \frac{1}{2} \left( \delta_{1,1} \cdot \nabla \rho_o \right) \left( \nabla \cdot \frac{\hat{B}_1}{\rho_o} \right) - \nabla \cdot \left( \frac{\hat{B}_1}{\rho_o} \right) \cdot \hat{A}_1^2
\]

where \( \hat{B}_1 = \frac{\hat{B}_o}{|B_o|} \cdot \nabla \left( \frac{\hat{B}_o}{|B_o|} \right) \) is the curvature of \( \hat{B}_o \).
In the intuitive form, it is clear that the potentially destabilizing terms in $SW$ are

$$- \frac{1}{2} \int_{\Omega} d^2 x \left\{ 2 \left( \hat{\nabla}_i \cdot D_0 A^*_i \right) (\hat{\nabla}_i \cdot \hat{A}_i) - J_{10} \left( \hat{\nabla}_i \times \frac{\hat{A}_i}{|\hat{A}_i|} \right) \cdot \hat{A}_i \right\}$$

since only these terms have indeterminate sign.

**Comparison Theorems**

We are now ready to investigate some of the consequences of the Energy Principle and its mathematical form just derived. We can start with some very general results that help build intuition regarding MHD stability. The "comparison theorems" in the Bornstein paper provide information on the effect of the vacuum region.

To prove the comparison theorems, we first need to show that the perturbed vacuum energy is the smallest possible magnetic energy in $\Omega_{V_0}$, subject to the boundary conditions on the vacuum vector potential:

$$SW_V = \frac{1}{2\mu_0} \int_{\Omega_{V_0}} d^2 x \left| D_0 \times \hat{A}_i \right|^2$$

has a minimum where

$$\delta (SW_V) = 0$$

$$\delta (SW_V) = \frac{1}{2\mu_0} \int_{\Omega_{V_0}} d^2 x \left( D_0 \times \hat{A}_i \right) \left( D_0 \times \hat{A}_i \right)$$

$$= \frac{1}{2\mu_0} \int_{\Omega_{V_0}} d^2 x \left[ D_0 \cdot \left( \hat{A}_i \times D_0 \times \hat{A}_i \right) + \hat{A}_i \times D_0 \times \hat{A}_i \right]$$

$$+ \frac{1}{2\mu_0} \int_{\Omega_{V_0}} d^2 x \left[ \hat{A}_i \times D_0 \times \hat{A}_i \right]$$

$$= \frac{1}{2\mu_0} \int_{\Omega_{V_0}} d^2 x \left[ \hat{A}_i \times D_0 \times \hat{A}_i \right]$$

$$+ \frac{1}{2\mu_0} \int_{S_{V_0}} dS \left( \hat{A}_i \times D_0 \times \hat{A}_i \right)$$
\[
S(SW) = \frac{1}{\mu_0} \int_{S_v} \left( \mathbf{S} \mathbf{A} \cdot \nabla \mathbf{D} + \mathbf{D} \mathbf{A} \cdot \nabla \mathbf{S} \right) \, dS
\]

The boundary conditions specify \( \hat{n} \times \mathbf{A} \) at the conducting wall and at the plasma-vacuum interface, so we only consider \( \mathbf{A} \) such that \( \mathbf{A} \times \mathbf{S} = 0 \) along \( S_v \). We then have

\[
S(SW) = \frac{1}{\mu_0} \text{Re} \left\{ \int_{S_v} \mathbf{D} \cdot \nabla \mathbf{A} \, dS \right\}
\]

The minimum (where \( S(SW) = 0 \)) then satisfies the Euler equation

\[
\mathbf{D} \times \mathbf{D} \times \mathbf{A} = 0
\]

Thus, zero current density is not only consistent with the fact that there are no charge carriers inside the vacuum region, it is also the state of minimum magnetic energy (regardless of the medium) for this volume.

This leads to the first comparison theorem:

I. Replacing the vacuum region with a pressureless conducting fluid may increase \( SW \), but it cannot reduce \( SW \).

- The pressureless (and static) fluid has magnetic energy density only, like a vacuum. If the
Vector potential in this pressureless fluid, 
\( \mathbf{A}_1 = \frac{1}{2} \times \mathbf{B}_0 \), satisfies the same boundary conditions as \( \mathbf{A}_1 \), the magnetic energy is equal to or greater (i.e., \( \nabla \times \nabla \times \mathbf{A}_1 \), \( \mathbf{A}_1 \)) than that of the vacuum.

- This comparison theorem is useful if it is easier to calculate \( \delta W \) with a pressureless fluid in the vacuum region. Then if \( \delta W < 0 \) with the substitution, the original configuration is also unstable.

Analogous two configurations that are the same except for the location of the conducting wall surrounding the vacuum region, the one with the more distant wall will be less stable.

- We can find a distribution of \( \mathbf{A}_1 \) for the larger-vacuum-region configuration such that \( \mathbf{A}_1 \) is the same as \( \mathbf{A}_1 \) in the other case for the corresponding volume and such that \( \nabla \times \mathbf{A}_1 = \mathbf{0} \) for the remaining volume.

1D illustration (sheared slab)
This distribution of \( \vec{A} \) for the larger-vacuum configuration has internal currents corresponding to eddy currents at the wall of the smaller-vacuum configuration, so

a) it is not the physical vacuum distribution for the larger-vacuum configuration.

b) the true vacuum distribution has less magnetic energy.

Thus, \( SW_{\text{larger}} < \frac{1}{2\mu} \int_{\Sigma_0} 10x\vec{A} \cdot 1 = SW_{\text{smaller}} \), so

withdrawing the wall reduces the stabilizing influence of \( SW \).

Schmidt adds the following to the list of comparisons:

for motions with nonzero \( \nabla \cdot \vec{V} \), increasing \( \gamma \) increases \( SW \), which makes these motions more stable. While this is correct, we can obtain more specific information (Bernstein and Freidberg, Ch. 8).

Examining \( SW_F, SW_S, \) and \( SW_V \), we find that \( \tilde{\Sigma} \) appears in the \( D_0 \|D_0\|^2 \) term only. In cases where we can choose

\[ D_0 \cdot \nabla_0 \vec{B}_0 = \vec{B}_0 \cdot D_0 \left( \frac{\tilde{\Sigma}}{|B_x|} \right) = -D_0 \cdot \tilde{\Sigma}_2 \]

so that \( D_0 \cdot \vec{3} = 0 \), we achieve the minimum \( SW \) with
respect to $S_{ii}$. In some cases, certain classes of $\tilde{s}$ have $\bar{B}_0 \cdot D_0 \left( \frac{\delta u}{100} \right) = 0$, so we cannot make $D_0 \cdot \tilde{s} = 0$. In toroidal geometry, we also have to recognize that

$$\bar{B}_0 \cdot D_0 \left( \frac{\delta u}{100} \right) = F$$

is a magnetic differential equation for $\delta u/100$ that requires $\langle F \rangle$ (flux surface average) to be zero—see D'haeseleer. We then have to separate

$$D_0 \cdot \tilde{s}_2 = D_0 \cdot \tilde{s}_2 + \langle D_0 \cdot \tilde{s}_2 \rangle,$$

where $\tilde{s}_2$ is the periodic part, and solve

$$D_0 \cdot D_0 \left( \frac{\delta u}{100} \right) = -D_0 \cdot \tilde{s}_2$$

The minimum $\int d^2 x_0 \delta \rho_0 \left| \bar{B}_0 \cdot \tilde{s}_2 \right|^2$ is then $\int d^2 x_0 \delta \rho_0 \left( \bar{B}_0 \cdot \tilde{s}_2 \right)^2$.

**Heuristic Interchange Analysis**

We can also learn about the class of local modes known as interchange modes through a heuristic application of the energy principle. (after Schmidt, Seeit.5-4). Consider two adjacent flux tubes (volumes with $\delta \times \delta^2 = 0$ on their surfaces) arranged along an inhomogeneous direction. [It is easy to use a z-pinch as a specific example.] An interchange displacement slips the flux tubes—along with their fluid and magnetic field lines—past each other, so that they exchange positions. In this
heuristic approach, we compare the potential energy before and after the interchange to find the sign of $\delta W$.

If the cross sectional area of each flux tube is small, we can approximate the magnetic energy in each tube with a piecewise constant $B$ profile.

$$W_m = \frac{1}{2\mu_0} \int d\ell \, B^2 = \frac{1}{2\mu_0} \int d\ell \, \bar{B}^2 A$$

$$= \frac{1}{2\mu_0} \bar{B}^2 \int d\ell \frac{1}{A}$$

where $\bar{B}$ is the average magnetic induction in the tube, $A$ is the cross sectional area, and the integral $\int d\ell$ is along the flux tube. The magnetic flux through the tube, $\bar{B}$, is conserved by the ideal motion, so changes in $W_m$ for a tube result from changes in the geometric quantity $\frac{\int d\ell}{A}$ during the displacement. If the interchange is complete,
the geometry of tube 1 becomes that of tube 2 before the interchange, and vice versa. We then have

$$SW_{m_1} = \frac{\pi^3}{2\pi_0} \Delta \left( \frac{dA}{A} \right)$$

$$SW_{m_2} = -\frac{\pi^2}{2\pi_0} \Delta \left( \frac{dA}{A} \right)$$

where $\Delta (\cdot)$ signifies the difference between the initial tube 2 and initial tube 1,

$$SW_m = SW_{m_1} + SW_{m_2} = \frac{1}{2\pi_0} \left( \frac{F_2^2 - F_1^2}{2} \right) \Delta \left( \frac{dA}{A} \right)$$

If the displacement is incompressible, the flux tubes have equal volume, or $A_1 dl_1 = A_2 dl_2$, and

$$\Delta \left( \frac{dA}{A} \right)_{\text{incompress}} = \int \frac{dl_2}{A_2} - \int \frac{dl_1}{A_1};$$

$$\approx \int \left[ \frac{dl_2}{(A_2 dl_2)} dl_2 - \frac{dl_1}{A_1} \right]$$

$$\approx \int \left[ \frac{(dl_2)^2}{(A_2)} - 1 \right] \frac{dl_1}{A_1}$$

$$SW_{m_{\text{incompress}}} = \frac{1}{2\pi_0} \left( \frac{F_2^2 - F_1^2}{2} \right) \int \left[ 1 - \left( \frac{dl_2}{dl_1} \right)^2 \right] \frac{dl_1}{A_1}$$

In configurations where $F$ and $dl$ increase simultaneously, $SW_{m_{\text{incompress}}} < 0$. Furthermore, with no volume change, there is no change in internal energy, so $SW = SW_{m_{\text{incompress}}}$, and these configurations are unstable to interchange modes. Configurations where
\( \hat{\mathbf{R}} \) points towards \( \nabla \mathbf{P} \) have \( B \) and \( d\ell \) increasing simultaneously; the normal \( z \)-pinch is an example.

Configure where \( \hat{\mathbf{R}} \) points in the opposite direction of \( \nabla \mathbf{P} \) have \( B \) increasing where \( d\ell \) decreases. These configurations are stable to interchange. The importance of the sign of \( \hat{\mathbf{R}} \cdot \nabla \mathbf{P} \) leads to the notion of \'good curvature\' vs. \'bad curvature\'.

We could also estimate \( SWP \) (see Schnell), but we should also keep in mind that though useful for developing intuition, these estimates lack the precision needed to analyze more complicated configurations. Tokamaks, for example, have bad poloidal and toroidal curvature on their outboard side and good toroidal but bad poloidal curvature on their inboard side. Magnetic shear is also very important, as anticipated by the Kruskal–Schwarzschild instability.
Cylindrical Configurations

From this point, our analysis of ideal MHD stability will focus on 1D equilibria in cylindrical geometry. This approach is a reasonable compromise in that we will be able to use straightforward analytic manipulations and still arrive at physically meaningful results in several cases. We begin by minimizing $\mathbf{S}$ with respect to all components of $\mathbf{S}$, except for $S_r$. The Euler-Lagrange equation is then a second-order ODE for $S_r$ only. We use this equation to analyze interchange and kink modes, and we consider the comprehensive Newcomb approach to provide necessary and sufficient conditions for stability.

Since we have already shown that the difference between $D$ and $D_0$ leads to higher-order terms in $15$, with the possible exception of the plasma-vacuum interface, we will drop the $D_0$ subscript and use the Lagrangian frame only where required. Furthermore, the cylindrical symmetry of the geometry and of the equilibria permit us to use Fourier series in $\theta$ and Fourier series or Fourier transform in $z$ (depending on whether the $z$-direction is periodic or infinite, respectively).
Note that for 
\[ \mathbf{\hat{s}}(r, \theta, z) \rightarrow \sum \sum \mathbf{\hat{s}}_{m,n}(r) e^{i\theta + ikz} \]
for the periodic cases, for example,
\[ \mathbf{\hat{s}}_{-m,n}(r) = \mathbf{\hat{s}}_{m,n}(r)^* \]
so that \( \mathbf{\hat{s}} \) is a real displacement. In addition, the symmetry implies
\[ F(\mathbf{\hat{s}}_{m,n}(r)e^{i\theta + ikz}) \propto e^{i\theta + ikz} \]
so that \( \mathbf{\hat{s}} \mathbf{\hat{w}}(\mathbf{\hat{m}}, \mathbf{\hat{s}}_{m,n}(r)e^{i\theta + ikz}) \neq 0 \) for \( \mathbf{\hat{m}} = \mathbf{\hat{s}}_{m,n}(r)e^{-i\theta - ikz} \) only. We can then check stability for each \((m,k)\) pair separately.

For the general case where \( B_{0,r}(r) \neq 0, B_{0,z}(r) \neq 0 \), we can use the intuitive form of \( \mathbf{\hat{s}} \mathbf{\hat{w}} \) and immediately minimize \( \mathbf{\hat{s}} \mathbf{\hat{w}} \) wrt to \( \mathbf{\hat{s}}_{il} \) for \((m,k) \neq (0,0)\). This results from the fact that
\[ \mathbf{\hat{B}}_{0} \cdot \mathbf{\nabla} (\mathbf{\hat{s}}_{il}/10) \rightarrow i(\frac{B_{0,r,m}}{r} + B_{0,z}(r)) (\mathbf{\hat{s}}_{il}/10) \]
for \( \mathbf{\hat{s}}_{il} \rightarrow \mathbf{\hat{s}}_{il}(r)e^{i\theta + ikz} \)
so \( \mathbf{\hat{B}}_{0} \cdot \mathbf{\nabla} (\mathbf{\hat{s}}_{il}/10) \rightarrow 0 \) only at isolated locations where
\[ \frac{B_{0,r}(r)m}{r} + B_{0,z}(r) k = 0 \]
For periodic cylinders $k_n = \frac{2\pi n}{L} = \frac{n}{R_o}$, so the isolated locations occur where

$$\frac{B_{\phi}(r)}{n} + \frac{B_z(r)}{R_o} = 0$$

$$\frac{m}{n} = -\frac{\int B_z(r)}{R_o B_\phi(r)} = -q(r)$$

We will soon see that these locations are singular points in the ODE, and we will relate them to resonances in the limit of $\omega \rightarrow 0$.

The fact that $\hat{B}_\phi \cdot V(\frac{S_n}{16\pi}) = 0$ at isolated locations means that we can choose $S_n$ such that $V \cdot \hat{S} = 0$ everywhere except at these isolated points. The finite nonzero $V \cdot \hat{S}$ at isolated locations then contributes only negligibly to the SW integral.

Following Friedberg, Appendix B, eliminating $S_n$ gives us

$$SW_F = \frac{1}{2} \int d\xi \left\{ \frac{1}{2} \frac{\hat{S}_{\xi}^2}{\rho_s} + \frac{B_z^2}{R_o^2} \right\} d\xi + \frac{\hat{S}_n^z}{R_o} \cdot \hat{Q}_n$$

$$\hat{S}_n = S_n \hat{e}_n + \eta \hat{e}_m$$

$$\hat{e}_m = \hat{b} \times \hat{e}_n = \frac{B_{z_0} \hat{e}_n - B_{\phi_0} \hat{e}_z}{B_0}$$

$$\eta = e_m \cdot \hat{S}_n = \frac{1}{B_0} \left( B_{z_0} S_n - B_{\phi_0} S_z \right)$$
\[ \frac{\partial}{\partial t} \mathbf{\Phi} = \mathbf{b} \cdot \nabla \mathbf{\Phi} = -\frac{\mathbf{B}_{\text{eo}}^2}{c \mathbf{\beta}_{\text{eo}}^2} \mathbf{e}_r \]

\[ \mathbf{Q}_\perp = \left[ \nabla \times \left( \mathbf{\beta}_r \times \mathbf{\Phi}_0 \right) \right]_\perp \]

\[ = \left[ -\mathbf{\Phi}_0 \left( \nabla \mathbf{\beta}_r \right) + \mathbf{\Phi}_0 \cdot \nabla \mathbf{\beta}_r - \mathbf{\beta}_r \cdot \nabla \mathbf{\Phi}_0 \right]_\perp \]

\[ = \left[ \mathbf{i} \mathbf{F} \left( \mathbf{5}_r \mathbf{e}_r + m \mathbf{e}_m \right) - \mathbf{\Phi}_0 \mathbf{\beta}_{\text{eo}} \mathbf{e}_r + \frac{\mathbf{\Phi}_0 \mathbf{\beta}_{\text{eo}} \mathbf{e}_m}{r} \right] \]

\[ - \mathbf{\beta}_r \left( \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} \mathbf{e}_r + \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} \mathbf{e}_r \right) + \mathbf{\Phi}_0 \mathbf{\beta}_{\text{eo}} \mathbf{e}_m \]

\[ \mathbf{F} = \frac{\mathbf{m} \mathbf{B}_{\text{eo}}}{r} + \mathbf{k} \mathbf{B}_{\text{zo}} = \frac{\mathbf{\Omega}_0 \mathbf{B}_0}{r} \]

\[ \sim \text{meaning} \frac{\mathbf{m} \mathbf{e}_m}{r} + \mathbf{k} \mathbf{e}_z \]

\[ = \left[ \mathbf{i} \mathbf{F} \left( \mathbf{5}_r \mathbf{e}_r + m \mathbf{e}_m \right) - \mathbf{\beta}_r \left( \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} \mathbf{e}_r + \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} \mathbf{e}_r \right) \right] \mathbf{[I]} \]

\[ = \mathbf{i} \mathbf{F} \left( \mathbf{5}_r \mathbf{e}_r + m \mathbf{e}_m \right) - \mathbf{\beta}_r \left[ \mathbf{r} \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} \mathbf{e}_r + \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} \mathbf{e}_r \right] \]

\[ \mathbf{e}_r - \frac{\mathbf{\Phi}_0 \mathbf{\beta}_{\text{eo}}}{\mathbf{\beta}_{\text{eo}}^2} = \frac{\left( \mathbf{\beta}_{\text{eo}}^2 + \mathbf{\beta}_{\text{eo}}^2 \right) \mathbf{e}_r - \mathbf{\Phi}_0 \mathbf{e}_m - \mathbf{\Phi}_0 \mathbf{e}_m}{\mathbf{\beta}_{\text{eo}}^2} \]

\[ \mathbf{e}_m = \frac{\mathbf{\beta}_{\text{eo}} \mathbf{e}_m}{\mathbf{\beta}_{\text{eo}}} \]

\[ \mathbf{e}_z - \frac{\mathbf{\Phi}_0 \mathbf{\beta}_{\text{eo}}}{\mathbf{\beta}_{\text{eo}}^2} = \frac{\left( \mathbf{\beta}_{\text{eo}}^2 + \mathbf{\beta}_{\text{eo}}^2 \right) \mathbf{e}_z - \mathbf{\Phi}_0 \mathbf{e}_m - \mathbf{\Phi}_0 \mathbf{e}_m - \mathbf{\Phi}_0 \mathbf{e}_m}{\mathbf{\beta}_{\text{eo}}^2} \]

\[ \mathbf{\Phi}_0 \mathbf{\beta}_{\text{eo}} \mathbf{e}_m \]

\[ \mathbf{Q}_\perp = \mathbf{i} \mathbf{F} \left( \mathbf{5}_r \mathbf{e}_r + m \mathbf{e}_m \right) + \frac{\mathbf{5}_r}{\mathbf{\beta}_{\text{eo}}} \left[ \mathbf{\beta}_{\text{eo}} \frac{\partial \mathbf{B}_{\text{eo}}}{\partial r} - \mathbf{\beta}_{\text{eo}} \mathbf{B}_{\text{eo}} \right] \]

\[ \nabla \cdot \mathbf{\Phi}_0 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \mathbf{\Phi}_0 \right) + \frac{m}{\mathbf{\beta}_{\text{eo}}} \left( \mathbf{\beta}_{\text{eo}} \mathbf{e}_m - \mathbf{\Phi}_0 \mathbf{e}_m \right) \]

\[ \mathbf{G}_\perp \]
Using these expressions and integrating \( \delta W_F \) over \( \Theta \) and \( \varphi \),

\[
\delta W_F = \frac{2\pi^3 R_0}{\rho_0} \int_0^a r dr \left\{ F^2 |s_r|^2 + \left| iF \left\{ \frac{G}{bo} \left[ B_0 B_{z_o} - B_{z_o} \frac{(bo)^2}{r} \right] \right\} \right|^2 \\
+ \frac{\rho_0}{G} \left( \frac{iG}{bo} + \frac{iG}{bo} \frac{b_0^2}{bo^2} \frac{s_r}{r} \right) \frac{1}{r} \left( r s_r \right) - \frac{b_0^2}{bo^2} \frac{s_r}{r} \right| \right\} \\
+ \mu_0 J_0 \left\{ 2F \text{Im} ( s_r m^*) + \frac{|s_r|^2}{bo} \left( B_0 B_{z_o} - B_{z_o} \frac{(bo)^2}{r} \right) \right\}
\]

To minimize this expression with respect to \( m \), we need

\[
\frac{\delta}{\delta m} \text{ of the integrand}
\]

\[
\delta \left\{ \right\} = \left( -iF s_m \right) \left( iF m + \frac{G}{bo} \left[ B_0 B_{z_o} - B_{z_o} \frac{(bo)^2}{r} \right] \right) \\
+ \left( -iG \frac{b_0}{bo} s_m \right) \left( iG \frac{b_0}{bo} + \frac{iG}{bo^2} \frac{b_0^2}{bo^2} \frac{s_r}{r} \right) \\
- \mu_0 J_0 F s_r s_m^* + \text{c.c.} \quad (\text{c.c. = complex conjugate of everything after } \bar{s}^n)
\]

The minimum occurs when \( \int_0^a r dr \delta \left\{ \right\} \) vanishes for all possible \( s_m \), i.e. \( s_m^* \). Setting the coefficient of \( \delta m^* \) to zero achieves this:

\[
\left(F^2 + G^2\right) m - \left( iF \frac{s_p}{bo} \left[ B_0 B_{z_o} - B_{z_o} \frac{(bo)^2}{r} \right] \right) - \left( iG \frac{b_0}{bo} \left[ \frac{1}{r} \left( r s_r \right) - \frac{b_0^2}{bo^2} \frac{s_r}{r} \right] \right) \\
- \left( iF \frac{b_0}{bo} \left[ \left. - b_0 B_{z_o} \right| + \frac{b_0^2}{bo^2} \left( r B_0 \right) \right| \right) \right\} = 0
\]

199
\[ \begin{align*}
\hbar^2 \frac{\partial^2 m}{\partial r^2} &= i \frac{cG_s}{\nu_0} \left[ 2 \frac{\partial \phi_{\nu_0}}{\partial r} \right] + i \frac{cG_s}{\nu_0} (r \delta_r)' - 2i \frac{cG_s}{\nu_0} \frac{\partial^3 \phi_{\nu_0}}{\partial r^3} \delta_r \\
&= i \frac{cG_s}{\nu_0} (r \delta_r)' + i \frac{2 \delta_r}{r} - \frac{cG_s}{\nu_0} k \nu_0 \phi_{\nu_0} - \frac{cG_s}{\nu_0} \frac{\partial^3 \phi_{\nu_0}}{\partial r^3} + k \partial^3 \phi_{\nu_0} \\
&= i \frac{cG_s}{\nu_0} (r \delta_r)' + i \frac{2 \delta_r}{r} k \nu_0 \phi_{\nu_0}
\end{align*} \]

So the minimizing \( \delta m \) is

\[ \delta m_{\text{min}} = \frac{i}{r \nu_0} \left[ \delta r (r \delta_r)' + 2 \delta_r k \nu_0 \right] \]

Using this \( \delta m \), \( \delta r \) no longer appears in \( SW_F \), so \( \delta r \) can be multiplied by an arbitrary phase factor, \( e^{i\phi} \), without changing \( SW_F \). We can then choose the phase such that \( \delta r \) is real, hence \( \delta r^* = \delta r \). Each term of the integrand then has a factor of \((\delta r)^2\), \(\delta r \delta r^*\), or \(\delta r^2\). We will be able to identify the coefficient for each factor after substituting \( \delta m_{\text{min}} \). Redefining \( \delta \equiv \delta r \), and dropping "0" subscripts on equilibrium \( B \) components and magnitude,

\[ SW_F = \frac{2 \hbar^2 \nu_0}{\rho_0} \int_0^\infty r \, dr \left\{ F^2 \delta^2 + \left( -\frac{F}{\rho_0 \hbar^2} \left[ G_s (\nu_3^* + s) + 2 \delta_3 \delta_0 \right] + \frac{\delta r}{\rho_0} \left[ \delta_B \delta_0' - \delta_B' \delta_0 \right] \right)^2 \right\} \]

\[ + \delta^3 \left( \frac{2 \delta_0^*}{\rho_0} - \frac{G_s}{\rho_0 \hbar^2} \left[ G_s (\nu_3^* + s) + 2 \delta \delta_0 \right] \frac{\delta_0^2}{\rho_0^2} \right)^2 \]

\[ + \frac{1}{\rho_0} \left( \delta_B \delta_0' + \delta_B^* \delta_0 - \delta_B \delta_0' \right) \left( -2 \frac{F}{\rho_0 \hbar^2} \left[ G_s (\nu_3^* + s) + 2 \delta \delta_0 \right] \right) \]

\[ + \frac{\delta^2}{\rho_0} \left( \delta_B \delta_0' - \delta_B \delta_0' + \frac{\delta_0^2}{\rho_0} \right) \]
In the integrand, \( r \{ 3 \}, \) the coefficients are

\[
3': \quad r \left( \frac{F^2}{B^2 k_0^2} + r B^2 \left( 1 - \frac{G}{B^2 k_0^2} \right)^2 \right)
\]

\[
= r \left( \frac{G}{r^2} + 2 \frac{\partial \phi}{\partial r} \frac{h_2}{h_0^2} \right) \left( \frac{B_0^2 \cdot r^2}{r^2} - 2 \frac{m_2 \cdot \phi_2}{r^2} \right) + h^2 b_2^2
\]

\[
+ B^2 \left( 1 - \frac{2 \left( \frac{B_2 \cdot \phi_2}{r^2} - 2 \frac{m_2 \cdot \phi_2}{r^2} + h^2 b_2^2 \right)}{r^2} \right) + \frac{2 \left( m_2^2 \cdot \phi_2^2 - 2 \frac{m_2 \cdot \phi_2}{r^2} + h^2 b_2^2 \right)}{r^2}
\]

\[
= r \left\{ \frac{1}{B^2 k_0^2} \left[ B_0^2 \cdot \phi_2 \left( \frac{m_2^2}{r^2} - \frac{m_2^4}{r^2} \right) + B_2^2 \cdot \phi_2 \left( m_2^2 \cdot \phi_2^2 - 2 \frac{m_2 \cdot \phi_2}{r^2} + h^2 b_2^2 \right) \right] + \frac{m_2^2}{r^2} \right\}
\]

\[
+ B^2 \left( \frac{2}{r^2} \left( \frac{m_2^2}{r^2} - \frac{m_2^4}{r^2} \right) + B_0^2 \cdot \phi_2 \right)
\]

\[
= r \left\{ \frac{1}{B^2 k_0^2} \left[ B_0^2 \cdot \phi_2 \left( m_2^2 \cdot \phi_2^2 - 2 \frac{m_2 \cdot \phi_2}{r^2} + h^2 b_2^2 \right) \right] + \frac{m_2^2}{r^2} \right\}
\]

\[
= \frac{r F^2}{B^2 k_0^2} \equiv A_1 (r)
\]
\[ \frac{2 F^2 \beta_0}{r B_0^2} \left( \frac{m_{\beta z}}{r} + h \beta_0 \right) - \frac{4 F \beta_0}{B_0^2} \left( \frac{n_{\beta z}}{r} \right) \frac{B_0 \beta_0}{r} \]
\[ + 2 \left( \beta_0^2 - \frac{G_{\beta}}{B_0^2} \right) \left[ \frac{1}{r} \left( 1 - \frac{G_{\beta}}{B_0^2} \right) \left( \frac{m_{\beta z}}{r} \right) \right] \]
\[ + 2 \left( \beta_0^2 - \frac{G_{\beta}}{B_0^2} \right) \left( \frac{n_{\beta z}}{r} \right) \left( \frac{2 \beta_0^2}{r} \right) \left( 1 - \frac{2 \beta_0^2}{r} \right) \]
\[ = \frac{2}{B_0^2} \left( \frac{r^2 B_0^2}{r^2 B_0^2} - \frac{m_{\beta z}^2}{r^2 B_0^2} \right) \equiv A_2(r) \]

\[ \frac{2 F^2}{r B_0^2} \left( 1 - \frac{G_{\beta}}{B_0^2} \right) \left( \frac{m_{\beta z}}{r} \right) \left( \frac{2 \beta_0^2}{r} \right) \left( 1 - \frac{2 \beta_0^2}{r} \right) \]
\[ + 2 \left( \beta_0^2 - \frac{G_{\beta}}{B_0^2} \right) \left( \frac{n_{\beta z}}{r} \right) \left( \frac{2 \beta_0^2}{r} \right) \left( 1 - \frac{2 \beta_0^2}{r} \right) \]
\[ = \frac{2 F^2}{r B_0^2} \left( 1 - \frac{G_{\beta}}{B_0^2} \right) \left( \frac{m_{\beta z}}{r} \right) \left( \frac{2 \beta_0^2}{r} \right) \left( 1 - \frac{2 \beta_0^2}{r} \right) \]
\[ = A_2(r) \]

Collecting the \( A_1, A_2, A_3 \) factors,
\[ \delta W_F = \frac{2\pi^2 R_0}{\mu_0} \int_0^a dr \left( A_1 s'^2 + 2A_2 s s' + A_3 s^2 \right) \]

\[ = \frac{2\pi R_0}{\mu_0} \left[ A_2 s^2 \right]_0^a \int_0^a dr \left( A_1 s'^2 + (A_3 - A_0') s^2 \right) \]

\[ A_2 = \frac{\kappa^2 r^2 \beta^2 - \mu_0^2 B_0^2}{\kappa^2 r^2 + m^2} \]

For \( m^2 = 0 \), \( s = s_r \rightarrow 0 \) as \( r \rightarrow 0 \)

For \( m^2 = 1 \), \( A_2(r) \rightarrow 0 \) as \( r \rightarrow 0 \)

So the surface term is nonzero at \( r = 0 \) only.

The conventional form for \( \delta W_{F\text{cyl}} \) is then

\[ \delta W_F = \frac{2\pi R_0}{\mu_0} \left[ \int_0^a dr \left( f s'^2 + g s^2 \right) + \frac{\kappa^2 r^2 \beta^2 - \mu_0^2 B_0^2}{\kappa^2 r^2 + m^2} \right] \]

where \( f = \frac{r \beta^2}{\mu_0^2} = A_1 \)

\( g = A_3 - A_0' \)

A few more manipulations achieve Newcomb's form for \( g \), his eqn. (17) \( [\alpha A, \text{Newcomb}, \text{Annals of Physics} 10, 232 (1960)] \)

\[ g = r F^2 - \left( \frac{\kappa^2 r^2 \beta^2 - \mu_0^2 B_0^2}{\kappa^2 r^2 + m^2} \right)' - \frac{2 \beta_0}{r} (r \beta_0)' + \frac{1}{r \beta_0} \left( \frac{\kappa^2 \beta_0^2}{r} - \kappa \beta_z \right)^2 \]
Newcomb points out the fact that if we substitute $k \to mk$ for $m \neq 0$, the only term that depends on $m$ in either $f$ or $g$ is then \( r F^2 \gamma^2 = \frac{m^2}{r} (R_0^2 + 2R_1 R_2) \gamma^2 \) term in $SW$, which is positive definite. Thus, if the cylindrical configuration is stable for $|m| = 1$, $-\infty < k < \infty$, it is stable for $|m| \geq 2$.

For $m = 0$, factors of $k$ cancel, except for the same $r F^2 \gamma^2 = \frac{k^2 R_2^2 \gamma^2}{r}$ term in $SW$, which is also positive definite. Thus, if the $m = 0$ modes are stable in the limit of $k \to 0$, they are stable for all $k$.

If we (finally) vary $SW_F$ with respect to $\delta$ in order to find its minimum value which determines stability, we have

\[
\delta(SW_F) = \frac{4 \pi R_0}{\rho_0} \left[ S_{\theta}^{\alpha} \Delta r (55', \delta \delta + g \delta \delta) + A_2(a) \delta \delta \delta \delta \right]
\]

\[
= \frac{4 \pi R_0}{\rho_0} \left[ S_{\theta}^{\alpha} \Delta r \delta \delta \left( - (55'), g \delta \delta \right) + (A_2(a) \delta \delta + f(a) \delta \delta) \delta \delta \delta \delta \right]
\]

Ignoring displacements at the plasma surface, setting
$S(b \omega_f) = 0$ to find extrema leads to

the Euler equation

$$\frac{d}{dr} \left( f \frac{dS}{d \omega_f} \right) - qG S = 0$$

This second-order ODE for $S = S_0$ is related to the
Sturm-Liouville form found in the example case given
at the start of this Ideal Linear Analysis section.
However, this ODE is not an eigenvalue equation—$\omega^2$
does not appear.

Note that singularities occur where

$$f = \frac{n F^2}{r \phi_0} = \frac{n}{r \phi_0^2} \left( \phi_0, \dot{\phi}_0 \right)^2 = 0$$

The radii where $f = 0$ are the same locations where
we could not ensure $D \cdot \mathbf{g} = 0$, and they occur

where

$$q_1(r) = -\frac{m}{n}$$

Since $\phi_0 \perp \phi_0$ at these locations, nonzero displacements
tend to move entire field lines from these surfaces
without bending them, which avoids positive changes
in potential energy.

205
Surface where $\mathbf{\hat{K}}_0 \cdot \mathbf{\hat{B}}_0 = 0$

\[ \hat{B} \text{ lifted from the surface by } \varepsilon \text{ without getting bent.} \]

**Sudan's Criterion for Interchange**

Before using the Euler equation to minimize $SW$, we will examine a couple of cases where we can show that $SW < 0$ for specific admissible trial functions. Though we are not minimizing $SW$, the Energy Principle tells us that unstable modes then exist. The first case addresses highly localized modes at singular points in $\varepsilon$.

We begin by noting that the expression for $\mathcal{G}$ can be manipulated into

\[
\mathcal{G} = \frac{2\nu_0 \varepsilon^2}{h_0^2} \mathbf{\hat{r}} + \frac{r (h_0^2 r^2 - 1)}{h_0^2 r^2} F^2 + \frac{2 h_0^2}{r h_0^2} \left( h B_0 - \frac{\varepsilon^2}{2} \right) F
\]