Surface where $k_o \cdot b_o = 0$

$\mathbf{B}$ lifted from the surface by $5r$ without getting bent.

Sudar's Criterion for Interchange

Before using the Euler equation to minimize SW, we will examine a couple of cases where we can show that $SW < 0$ for specific admissible trial functions. Though we are not minimizing $SW$, the Energy Principle tells us that unstable modes then exist. The first case addresses highly localized modes at singular points in $r$.

We begin by noting that the expression for $g$ can be manipulated into

$$g = \frac{2\mu_0 h^2}{h_0^2} \rho' + \frac{r(h_0^2-1)}{h_0^2 r^2} F^2 + \frac{2 h^2}{r h_0^2} \left( h h_0^2 - r_2 r_0^2 \right) F$$

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For $r \approx r_s$ (where $r_s$ is a singular point), $F=0$, so
\[ g \approx \frac{3\mu_0 k^2}{\rho_{_0}^2} \rho \bigg|_{r=r_s} \]
we also approximate $f$,
\[ f = \left[ \frac{r}{\rho_{_0}^2} (F')^2 \right] \bigg|_{r=r_s} \]
\[ \alpha \approx r - r_s \]

For localized modes, we can ignore surface terms, and
\[ \delta W \approx \frac{2\mu_0 \rho_{_0}^2}{\mu_0} \left[ \frac{r}{\rho_{_0}^2} F^2 \right] \bigg|_{r=r_s} \int_{-A}^{+A} dx \left( \alpha^2 s'^2 - D_s s^2 \right) \]
where $A \ll \alpha$, and
\[ D_s = -\frac{3\mu_0 k^2}{r F'^2} \rho \bigg|_{r=r_s} = -\frac{3\mu_0 k^2}{r B_z^2} \frac{\beta}{\gamma^{1/2}} \rho \bigg|_{r=r_s} \]

If we absorb the constant in front of the $\int_{-A}^{+A}$ integral into $\delta W$, the Euler equation is
\[ \frac{d}{dx} \left( x^2 \frac{ds}{dx} \right) + D_s s = 0 \]

Trying solutions of the form $|x|^\alpha$, we find (for $x > 0$)
\[ \frac{d}{dx} \left( \alpha x^{\alpha+1} \right) + D_s x^\alpha = 0 \]

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\[ x(x+1)x^n + D_x x^n = 0 , \]

and we would find the same relation for \( x < 0 \) by changing variables to \( y = -x \), \( |x|^\alpha = y^\alpha \). We therefore have solutions of the form \( |x|^\alpha \) when \( \alpha^2 + \alpha + D_x = 0 \)

\[ \alpha = \frac{-1 \pm \sqrt{1 - 4D_x}}{2} \]

For \( 1 - 4D_x \geq 0 \), \( \alpha \) is a real number. For example, a hollow pressure profile with \( \rho' > 0 \) has \( D_x < 0 \), \( \sqrt{1 - 4D_x} > 1 \), so

Hollow \( \rho \) solutions

For \( 1 - 4D_x < 0 \), \( \alpha \) is complex

\[ \alpha = \frac{-1 \pm i\sqrt{4D_x - 1}}{2} \]

and the solutions are

\[ |x|^{\frac{1}{2} + \frac{i\sqrt{4D_x - 1}}{2}} \quad \text{or} \quad \frac{1}{|x|^\frac{1}{2}} \sin \left( \frac{1}{2} \sqrt{4D_x - 1} \phi(x) \right), \quad \frac{1}{|x|^\frac{1}{2}} \cos \left( \frac{1}{2} \sqrt{4D_x - 1} \phi(x) \right) \]
Since \( |x| \) changes rapidly as \( x \to 0 \), \( \sin (2 Ax |x|) \) and \( \cos (2 Ax |x|) \) oscillate very rapidly as \( x \to 0 \).

\[
\begin{align*}
\int_{-\infty}^{\infty} |x| \left[ (x^2 s') s' - 0 s s^2 \right] dx &= \left. x^2 s' \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dx}{x} s \left[ (x^2 s')' + 0 s \right] \\
&= 0 \text{ for Euler eqn.}
\end{align*}
\]

so if we construct a trial function that follows the Euler equations over a region that starts and ends with \( s' = 0 \) or \( s = 0 \), that region does not contribute to \( \delta W \).

Trial Function for \( 1 - 40 \delta s < 0 \)

\[ s' = 0 \]

\[ \text{Euler eqn. and test function coincide here} \]

\[ \text{Euler eqn. solution} \]

\[ \text{no contribution to } \delta W \]

\[ \text{no contribution to } \delta W \]

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The test function for $1 - 4 \rho_0 \leq 0$ in the sketch therefore has no contribution to $\delta W$ in the regions $x_1 \leq x \leq x_2$, $x_3 \leq x \leq x_4$, in addition to $x > x_1$ and $x > x_4$. The only nonzero contribution is in $x_2 < x < x_3$ where $\delta'$ is set to 0, so

$$\delta W \sim \frac{2 \pi \rho_0}{\rho_0} \left[ \frac{r}{\rho_2^2} \right] \left( - \delta_0 \delta'(0) \right) (x_3 - x_2).$$

Since $\delta_0 > 0$, $\delta W < 0$, for this trial function, so the configuration is unstable. For $1 - 4 \rho_0 \geq 0$, the $\delta^2 \delta'$ terms at $x_2$ and $x_3$ would dominate $\delta W$ for a similar trial function, and $\delta W > 0$.

When written as

$$\frac{1}{4} \geq \frac{2 \pi \rho_0 g^2}{\rho_2^2 q^2} \left( - \rho' \right) \left| \rho' \right|_{r=r_0},$$

the stability criterion shows that magnetic shear $\sim \frac{1}{q}$ tends to stabilize a negative pressure gradient. The condition is known as the Suydam Criterion for interchange stability, named after its discoverer.

In toroidal geometry, localized ideal modes are analyzed analytically in the $k_2 \to \infty$ limit (the most unstable modes for ideal MHD) by using an eigenonal approach.
For interchange modes, this leads to the Mercier stability criterion, which is a flux-surface-local condition like Suydam in a cylinder. The general expression involves flux-surface averages of geometric and equilibrium quantities, but in a large-aspect-ratio, circular cross-section tokamak, the Mercier criterion reduces to

\[
\frac{1}{4} > -(1-q^2) \frac{220q^2}{r_q} \left( \frac{r}{R} \right)^2
\]

which mainly differs from the Suydam criterion by the factor of \((1-q^2)\) on the right side. This is significant in that for \(|q| > 1\), the sign changes, and a decreasing pressure profile is stable, regardless of shear. This results from the good curvature on the inboard side of the torus.

The ballooning mode is also a local mode, but it has no cylindrical analog. This mode is concentrated on the outboard side of the torus, where curvature is bad. See Freidberg, Ch. 10 for more information.
Global modes often have eigenfunctions that extend to the plasma surface or to the wall when there is a close-fitting conducting shell. An important exception is the \( m \) mode in a tokamak \( q \)-profile. Like the localized modes, stability is therefore determined by \( \delta W F_j \cdot \delta W S_j = \delta W V = 0 \).

Consider the expression for \( q \) as given by Newcomb,

\[
q = \frac{2p d k^2}{k^2} p' + \frac{r \left( k^2 r^2 - 1 \right)}{k^2 r^2} F^2 + \frac{2 k^3}{r k^4} \left( k B_0 - \frac{n B_0}{r} \right) F
\]

For the periodic cylinder \( \tau \tau \tau \)

\[
k \Rightarrow -\frac{n}{R}
\]

where the \( ' - ' \) is introduced for tokamak convention. Here the resonance condition, \( F = 0 \) is

\[
q_f = + \frac{m}{n}
\]

Unlike the RFP convention. In addition,

\[
F = \frac{m B_0}{r} - \frac{n B_0}{R} = \frac{n B_0}{r} \left( \frac{m}{n} - q \right)
\]

so

\[
q = \frac{2 p d k^2}{k^2} \left( \frac{r}{R} \right)^2 p' + \frac{\left( m^2 + n^2 r^2 / k^2 - 1 \right)}{k^2 r^2} \frac{n^2 B_0^2}{r^2} \left( \frac{m}{n} - q \right)^2
+ \frac{2 n^2}{r^4 k^4} \left( \frac{r}{R} \right)^2 \frac{n^2 B_0^3}{r^3} \left( q^2 - \frac{m^2}{n^2} \right)
\]

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In conventional low aspect ratio tokamaks

\[
\frac{r}{R} \approx \epsilon
\]

\[\rho \rho' \approx \frac{b_0^3}{r}, \quad \text{and} \]

\[h_0^2 p^2 = m^2 + n^2 \epsilon^2\]

so for \(\frac{m}{n} \approx q \approx 1\), \(h_0^2 p^2 \approx m^2\)

Thus, the first and third terms in \(g\) are \(O(\epsilon^4)\), but the second term is \(O(\epsilon^3)\). An exception occurs at \(m=1\), however. The second term becomes

\[
\frac{n^2}{h_0^2 p^2} \left( \frac{r}{R} \right)^2 \frac{\epsilon^3 b_0^3}{r} \left( \frac{m}{n} - q \right)^2 \approx O(\epsilon^4) \quad \text{for} \quad m=1
\]

Since this second term is positive definite, it is a stabilizing term in \(\delta \mathcal{W}_F\)

\[
\delta \mathcal{W}_F \sim \int_0^q d\epsilon \left[ f \tilde{S}^2 + \tilde{g} \tilde{S}^2 \right]
\]

For \(m \neq 1\), it dominates \(g\) and global modes (those not concentrated where \(F > 0\)) are stable.

For \(m=1\),

\[
g = \left\{ 2 \rho_0 n^2 p' + \frac{\epsilon^4 b_0^3}{r} \left[ \left( \frac{1}{n} - q \right)^2 + 2 \left( q^2 - \frac{1}{m^2} \right) \right] \right\} \frac{r^2}{R^2}
\]

\[
\approx \left\{ 2 \rho_0 n^2 p' - \frac{n^4 b_0^3}{r} \left( \frac{1}{n} - q \right) \left( \frac{1}{n} + 3 q \right) \right\} \frac{r^2}{R^2}
\]
When \( q < 1 \) and \( p' < 0 \), \( g < 0 \) for \( n = 1 \) (\( m = 1 \)), which is destabilizing.

Although \( f s^2 \) is positive-definite (stabilizing), its contribution is minimized by perturbations that concentrate \( s' \) where \( q' = 0 \), i.e., where \( F^2 = 0 \). The following trial function does this and has only negative contributions from \( g s^2 \),

\[
\begin{align*}
\text{Trial functions close to this 'top hat' shape have } & \delta W < 0, \\
\text{so the } m = 1, n = 1 & \text{ mode is unstable for conventional tokamak profiles in cylindrical geometry. The 'top hat' } s = s_r \\
\text{distribution modulated by } e^{i \theta + iz}/r & \text{ implies that the inner column of plasma with } g(r) < 1 \\
\text{tends to deform into a helix inside the quiet outer region, where } g(r) > 1. \\
\end{align*}
\]
In the cylinder, this mode can be driven by the pressure gradient or by current. In toroidal geometry the linear coupling among different m-numbers can lead to stability below q = 1 if there is sufficient shear (see Freidberg, Ch. 10). This is not the norm in tokamak experiments, however, and all kink modes typically lead to "sawtooth" signals when \( q(0) < 1 \). [The potentially stabilizing effect of energetic ions is an area of active research, as is the nonlinear dynamics of the sawtooth.]

**External Kinks**

To allow for perturbations where \( \delta a \neq 0 \), we need to consider the surface term in \( SW_{\text{cyl}} \), \( SW_{\Omega} \), and \( SW_{\nu} \), in addition to the radial integral in \( SW_{\text{cyl}} \). Since \( SW_{\nu} \) is always \( \geq 0 \), we can examine \( SW_{\Omega} + \delta SW_{\Omega} \) for unstable displacements and then check if \( SW_{\nu} \) is sufficiently stabilizing to keep \( SW > 0 \).

First note that the surface term in \( SW_{\text{cyl}} \) (p. 203) is

\[
\frac{h^2 \dot{b}_0^2(a) - m^2 b_0^2(a)}{h^2 a^2 \tau m^2} \delta a^2 = \frac{h^2 b_0^2(a)}{n^2 (\frac{a}{\tau} + m^2)} \left( q^\prime(a) - \frac{m^2}{n^2} \right) \delta a^2
\]
For periodic cases. Clearly, this term is destabilizing when \( q(a) < |\frac{\mu_0}{\pi}| \) and \( s_a \neq 0 \).

The surface \( \delta W \) from p. 186, for cylindrical geometry is

\[
\delta W_{\text{cyl}} = \frac{2\pi^2 R_o a}{\mu_0} s_a^2 \left[ \left. \frac{\partial}{\partial r} \left( \frac{1}{r^2} \left( \frac{\partial}{\partial r} + \mu_0 p \right) \right) \right|_{r=a} \right]
\]

which is nonzero whenever \( J \neq 0 \) at \( r = a \). This may be associated with \( p \neq 0 \) at \( r = a \) (on the plasma side), or it may be associated with parallel current density. In a z-pinch with \( B = 0 \) for \( r < a \), for example,

the \( \omega \) jump in \( \frac{\partial}{\partial r} \left( \frac{1}{r^2} \left( \frac{\partial}{\partial r} + \mu_0 p \right) \right) < 0 \), making \( \delta W_s < 0 \). If the pressure and current density go to zero at \( r = a \), the equilibrium matches a vacuum magnetic distribution in the outer part of the plasma region, so there is no discontinuity, and \( \delta W_s = 0 \).

Analyzing the contribution from the radial integral in \( \delta W_{\text{cyl}} \) is very similar to the analysis of internal humps.
The key points are:

1. The positive-definite \( g \propto F^2 \) induces restoring potential energy unless \( g \) is flat between singular points.

2. Selecting \( m=1 \) makes the positive-definite and term in \( g \) small, particularly for large aspect ratio tokamaks, where \( h^2 r^3 \sim \varepsilon^2 \). [Repeating \( g \) from 212 for convenience.]

\[
g \propto \frac{2\pi n^2}{h_0^2} \left( \frac{r}{R} \right)^3 \frac{1}{r} + \frac{4\pi^2 n^2}{h_0^2} \left( \frac{r^3 + n^2 h_0^2 / R^2 - 1}{h_0^2 n^2} \right) \frac{r^2 h_0^2}{r} \left( \frac{r}{R} \right)^2 \frac{1}{r} + \frac{2\pi n^2}{h_0^2} \left( \frac{r^2}{R^2} \right) \frac{r^3 h_0^2}{r} \left( \frac{r}{R} \right)^2 \frac{1}{r^2} \left( \frac{r}{R} \right)^3 \frac{1}{r^2}
\]

3. For cases where \( h^2 r^3 \) is comparable to \( m^2 \), the 3rd term is still a destabilizing influence where \( q^2 < \frac{h^2}{m^2} \), but determining the sign of \( \delta W_F \) requires quantitative analysis.

For the special case of \( m=1 \) and \( h^2 r^3 \sim \varepsilon^2 \), and \( q < 1 \), the flat trial function used for the internal hint can be extended to \( r < a \). This makes \( g \propto c_{\alpha} \) for \( 0 < r < a \). The surface term \( = n^2 b_0^2 (q(a) - \frac{1}{\kappa}) \int n^2 \delta q < 0 \), and if \( \delta W_S \neq 0 \), we can expect \( \delta W_S < 0 \). Thus, unless we place a conducting wall extremely close to the plasma surface to effectively prevent the hint displacement, there will be an unstable mode. The condition

\[ q < 1 \]
for stability to external m=1 modes is known as the Kruskal-Shafranov condition. Note that since \( q(r) \) is independent of the current density profile—just the total current relative to \( B_0 \)—the result is quite general for \( q(r) \) profiles that increase with \( r \).

To complete a numerical evaluation of \( SW \), we also need

\[
SW_v = \frac{1}{2\rho_0} \int_{R_v} d\xi \; |\nabla \hat{A}|^2 = \frac{1}{2\rho_0} \int_{R_v} d\xi \; |\hat{Q}|^2
\]

where \( \hat{Q} \) is the perturbed magnetic induction in the Eulerian frame in the vacuum region. We may solve for \( \hat{Q} \) as

\[
\hat{Q} = \nabla \phi \quad \text{(vacuum is free of current density)}
\]

\[
\nabla \cdot \hat{Q} = \nabla^2 \phi = 0 \quad \text{(no magnetic monopoles)}
\]

and we would solve Laplace's equation for \( \phi \) subject to the Neumann conditions

1. \( \nabla \cdot \nabla \phi = 0 \) along the outer conductor
2. \( \nabla \cdot \nabla \phi \) prescribed by interface conditions along the plasma/vacuum interface
With this solution,

\[ \delta W_V = \frac{1}{2 \rho_0} \int_{\mathcal{V}_0} \delta \Phi \cdot \Phi \, d\mathcal{V} \]

where \( \hat{n}_0 \) is the unit normal to the original plasma/vacuum interface pointing from the plasma region to the vacuum region.

What we have from the interface relations is

\[ \hat{n} \cdot \left[ \nabla \Phi \right] = 0 \]

for the (Lagrangian) \( \hat{\Phi} \) and \( \hat{n} \) at the perturbed interface.

We will proceed by evaluating \( \hat{n} \), \( \hat{\Phi} \), and \( \hat{\mathcal{P}} \) at the perturbed interface to first order in \( \mathcal{E} \) and use this information to find \( \hat{\Phi} \) at the unperturbed interface. We have already determined the first order contributions to \( \hat{\Phi} \) at the perturbed interface

\[ \hat{\mathcal{P}}_L = \hat{\Phi} + \mathcal{E} \cdot \nabla \hat{\Phi} \]  

(fluid side - p. 182)

\[ \hat{\mathcal{P}}_L = \hat{\Phi} + \mathcal{E} \cdot \nabla \hat{\Phi} \]  

(vacuum side - p. 184 with \( \mathcal{E} \times \hat{\mathcal{A}} = \Phi \))

There is also a contribution due to \( \hat{n} \) changing. Consider a scalar field \( \Phi \) that is a flux function within the plasma region and increases smoothly across the plasma/
Vacuum interface. We can then express the unit normal as

\[ \hat{n} = \frac{\partial x}{|\partial x|} \]

The evolution of $X$ such that it remains a flux function is just that of a passive scalar

\[ \frac{\partial x}{\partial t} + \vec{v} \cdot \nabla_X = 0 \]

or for the linear displacement

\[ X = X_0 - \hat{S} \cdot \nabla X_0 \]

What we need, however, is $\Delta X$ at the perturbed interface. The Lagrangian evolution of $X$ itself ($X = X_0$) is not helpful, but we can first find the Eulerian evolution of $\Delta X$ as

\[ \Delta X = \nabla (X_0 - \hat{S} \cdot \nabla X_0) \]

\[ = \nabla X_0 - \nabla \left[ \hat{S} \cdot (\nabla X_0) \right] \]

\[ = \nabla X_0 - \nabla X_0 \times (\nabla \hat{S}) - (\hat{S} \cdot \nabla) \nabla X_0 - (\nabla X_0 \cdot \nabla) \hat{S} \]

\[ = \nabla X_0 - \hat{S} \cdot \nabla X_0 - (\hat{S} \cdot \nabla) \nabla X_0 \]

We can then add $(\hat{S} \cdot \nabla) \nabla X_0$ to find $\Delta X$ in the Lagrangian frame. Note that the denominator of $\hat{n}$, $|\nabla X_0|$ will modify the magnitude, but not the direction of $\hat{n}$. This will make a first-order correction to $\nabla X_0$, but it cannot lead to a first-order correction to $\hat{n}$, $\hat{S}$ because $\hat{S} \cdot \nabla X_0 = 0$. To first order, $\hat{n}$ at the perturbed interface is then

\[ \hat{n}_0 + \left( \frac{1}{|\nabla X_0|} \right) \Delta X_0 - \hat{S} \cdot \hat{n}_0 \]

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Combining the first order contributions from perturbations in \( \hat{B} \) and \( \hat{n} \),

\[
\hat{n} \cdot \hat{B} = \hat{n} \cdot \hat{B} \\
\hat{n}_0 \cdot \hat{A} + \left( \frac{\mathbf{3}}{\hat{B}_0} \right) \hat{n}_0 - \hat{B}_0 \cdot (\nabla \times \hat{n}) = \hat{n}_0 \cdot \hat{A} + \left( \frac{\mathbf{3}}{\hat{B}_0} \right) \hat{n}_0 - \hat{B}_0 \cdot (\nabla \times \hat{n})
\]

For \( \nabla \times \hat{n} = 0 \), the LHS is

\[
\hat{n}_0 \cdot \left[ \hat{A} - \nabla \times (\hat{B}_0 \times \hat{n}_0) \right] = 0
\]

So

\[
\hat{n}_0 \cdot \hat{A} = \hat{B}_0 \cdot \nabla \times \hat{n}_0 - \hat{B}_0 \cdot \hat{n}_0
\]

In cylindrical geometry, \( \hat{n}_0 = \hat{r} \), and \( \hat{B}_0 \cdot \hat{n}_0 = 0 \), so we find

\[
\hat{n}_0 \cdot \hat{A} = \frac{i m \hat{B}_0}{r} \hat{z} + ik \hat{B}_2 \hat{r} - \frac{\hat{B}_0 \hat{B}_2}{r} = \left( - \frac{\hat{B}_0 \hat{B}_2}{r} \right) \\
= i \hat{r} \left( \frac{m \hat{B}_0}{r} + k \hat{B}_2 \right)
\]

Using this relation to determine \( \hat{n}_0 \cdot \nabla \phi \) along the unperturbed interface then ensures \( \nabla \phi \) satisfies \( \hat{n} \cdot [\hat{B}] = 0 \) along the perturbed interface to first order in \( \hat{B} \).

With the boundary condition related to \( \hat{S}_r \), we can solve the Laplace's equation for the vacuum magnetic potential.
\[ \nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r}(r \frac{\partial \phi}{\partial r}) - \frac{\hbar^2}{r^2} \phi - \hbar^2 \phi = 0 \]
\[ = r^2 \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial \phi}{\partial r} - (\hbar^2 + \frac{\hbar^2 r^2}{R^2}) \phi = 0 \]

Defining \( x = \frac{\ln r}{R} \) we arrive at a modified Bessel equation of order \( m \),
\[ x^2 \frac{\partial^2 \phi}{\partial x^2} + x \frac{\partial \phi}{\partial x} - (x^2 + m^2) \phi = 0 \]

\[ \phi(x) = C_1 I_m(x) + C_2 K_m(x) \]
\[ \phi(r) = C_1 I_m \left( \frac{\ln r}{R} \right) + C_2 K_m \left( \frac{\ln r}{R} \right) \]

Since \( |I_m(x)| \to \infty \) as \( x \to \infty \), we would set \( C_1 = 0 \) if there were no conducting wall enclosing the vacuum region. Otherwise, for a wall at \( r = b \),
\[ \hat{n} \cdot \nabla \phi \bigg|_{r = b} = C_1 \left( \frac{\ln b}{R} I_{m-1} \left( \frac{\ln b}{R} \right) - \frac{m}{b} \ln I_m \left( \frac{\ln b}{R} \right) \right) \]
\[ - C_2 \left( \frac{\ln b}{R} K_{m-1} \left( \frac{\ln b}{R} \right) + \frac{m}{b} \ln K_m \left( \frac{\ln b}{R} \right) \right) = 0 \]

\[ \frac{C_1}{C_2} = \frac{\frac{\ln b}{R} K_{m-1} \left( \frac{\ln b}{R} \right) + \frac{m}{b} \ln K_m \left( \frac{\ln b}{R} \right)}{\frac{\ln b}{R} I_{m-1} \left( \frac{\ln b}{R} \right) - \frac{m}{b} \ln I_m \left( \frac{\ln b}{R} \right)} \]

At \( r = a \), we use the interface relation
\[ \hat{n}(a) \left( \frac{\nabla \phi_a(a)}{a} - \frac{\nabla \phi_b(a)}{R} \right) = C_1 \left( \frac{\ln a}{R} I_{m-1} \left( \frac{\ln a}{R} \right) - \frac{m}{a} \ln I_m \left( \frac{\ln a}{R} \right) \right) \]
\[ - C_2 \left( \frac{\ln a}{R} K_{m-1} \left( \frac{\ln a}{R} \right) + \frac{m}{a} \ln K_m \left( \frac{\ln a}{R} \right) \right) \]

\[ 222 \]
We can now find $C_1$ and $C_2$ in terms of $S_{\alpha}$, so that $SW_v$ can be computed for specific trial functions. We therefore have everything ($SW_{\psi}$, $SW_{s}$, and $SW_v$) needed for quantitative evaluations of $SW$ for arbitrary $10$ equilibria in cylindrical geometry.

Computations for external modes with increasing $q(r)$ profiles and $q(\alpha) > 1$ show that the stability of $m=2$ modes depends rather sensitively on the edge current density profile. Instabilities can be avoided if both $J_0$ and $J_0'$ go to zero, implying peaked current profiles.

In toroidal geometry, low-$\beta$ results for external modes are similar to cylindrical results. However, as $\beta$ is increased, external modes acquire a ballooning character, and the overall picture of stability is more complicated. [See Freidberg, Ch. 10 for a summary.]

Newcomb Analysis

The necessary and sufficient conditions for determining the ideal MHD stability properties of an arbitrary...
cylindrical configuration were derived by W. M. Newcomb
[Annals of Physics 10, 232 (1960)]. Though now more than
forty years old, the work still stands out as an
elegant example of mathematical physics. Here, we
shall summarize the major findings that have not been
covered earlier.

Since we will be analyzing solutions of the
Euler (or Euler-Lagrange) equation, we need to note
that one cannot integrate through singular points. Thus,
we will need to consider solutions in separate intervals
that are bounded by the geometric origin, singular
points, and/or the outer wall (the analysis does
not address vacuum regions explicitly).

As we approach a singular point, the Euler
solutions obey the limiting behavior that we have
investigated for local interchange modes. If the
Snyder criterion is violated for an \((m, k)\) pair, we
know that the configuration is unstable. If the
Snyder criterion is satisfied, the non-oscillatory solutions
approach the singular point as

\[ S_3(x) \propto |x|^{k + \sqrt{1 - \frac{4m^2}{k^2}}} \]

\[ S_2(x) \propto |x|^{-k - \frac{1}{2}\sqrt{1 - \frac{4m^2}{k^2}}} \]
where the "s" and "L" subscripts indicate "small" and "large" solutions as \( l x l \to 0 \). Since

\[
\frac{1}{a} \sqrt{1-4a} > 0,
\]
we note that \( \int_{-a}^{+a} s dx \) is finite for \( s = s_5 \), but not for \( s = s_4 \) in the vicinity of a singular point. Therefore, we only consider solutions that behave like \( s_5 \) near each singular point.

Away from the singular points, we can expect to find two linearly independent solutions to the Euler equations: \( s_1(x) \) and \( s_2(x) \). According to the general theory of second-order ODEs, we should be able to solve an initial value problem in the vicinity of any nonsingular point. If we want the solution such that \( s(\alpha) = y \) and \( s'(\alpha) = z \), we need to find constants \( c_1 \) and \( c_2 \) such that

\[
\begin{align*}
    c_1 s_1(\alpha) + c_2 s_2(\alpha) &= y \\
    c_1 s_1'(\alpha) + c_2 s_2'(\alpha) &= z
\end{align*}
\]

or, in matrix form

\[
\begin{pmatrix}
    s_1(\alpha) & s_2(\alpha) \\
    s_1'(\alpha) & s_2'(\alpha)
\end{pmatrix}
\begin{pmatrix}
    c_1 \\
    c_2
\end{pmatrix} =
\begin{pmatrix}
    y \\
    z
\end{pmatrix}
\]

Unique solutions can be expected when the Wronskian,

\[
W(s_1, s_2) = s_1 s_2' - s_1' s_2 = \det ( ) \neq 0
\]
at the nonsingular point \( \alpha \). This is equivalent to stating that \( s_1 \) and \( s_2 \) are linearly independent.

For the Euler solutions

\[
\begin{align*}
\dot{s}_1 \left[ (f s_1)' - g s_1 \right] &= 0 \\
\dot{s}_1 \left[ (f s_2)' - g s_2 \right] &= 0 \\
\dot{s}_1 (f s_2)' - \dot{s}_2 (f s_1)' &= 0 \\
f' W(s_1, s_2) + f \left( s_1 s_2'' - s_2 s_1'' \right) &= 0 \\
f' W + f W' &= 0 \\
(f W)' &= 0 \implies f W = \text{const}
\end{align*}
\]

Since \( f \) does not change sign in an interval, \( W \) also does not change sign. We therefore have a separation theorem,

\[
\begin{align*}
\dot{s}_1 > 0, & \quad \dot{s}_2 < 0 \\
\dot{s}_1 > 0 \\
\therefore \dot{s}_2 \text{ must be } < 0 \text{ to keep } W < 0.
\end{align*}
\]

The zeros of the two linearly independent solutions must alternate. Similarly, the zeros of the first derivatives...
must alternate, too.

Let us now choose two nonsingular endpoints, \( r_1 \) and \( r_2 \), within an interval and find Euler solutions such that

\[
\begin{align*}
S_1(r_1) &= 0 & S_1'(r_1) &= 1 \\
S_2(r_1) &= 1 & S_2'(r_1) &= 0
\end{align*}
\]

We may then satisfy boundary conditions

\[
S_1(r_0) = c_1, \quad S_2(r_0) = c_2
\]

with the unique solution

\[
S_0(r) = \frac{c_2 - c_1 S_2(r_0)}{S_1(r_0)} S_1(r) + c_1 S_2(r)
\]

We can then find a family of characteristic curves,

\[
S_A(r) = A S_1(r) + S_0(r)
\]

which have a different form depending on whether \( S_1(r) \) vanishes for \( r_1 < r < r_2 \).

\( S_1(r) \) does not vanish at \( r_1 < r_0 < r_2 \).
Note that

1) When \( s_1(r) \) does not vanish in \( r, c_1 < r < r_2 \), the
region \( r, c_1 < r < r_2, -\infty < s < \infty \) is covered
by points on characteristic curves.

2) When \( s_1(r) \) vanishes at \( r = r_0 \), \( r_1 < r_0 < r_2 \),
points along \( r = r_0, s > s_0(r_0) \) and \( r = r_0, s < s_0(r_0) \)
in the region \( r_1 < r < r_2, -\infty < s < \infty \) are not
covered in any characteristic curve.

If we seek \( SWF \) for an arbitrary trial function
that has \( s_1(r_1) = c_1 \) and \( s_1(r_2) = c_2 \), we compute

\[
W(r_1, r_2, s) = \int_{r_1}^{r_2} f(s') dr (s'^2 + g(s'^2))
\]

like \( SWF \) (absorbing constants), except \( r_1 \leq r \leq r_2 \). From here, we
should think of \( W(r,s,s) \) as a path integral through the
\( r-s \) plane. Define \( p(r,s) = \left. \frac{dS}{dr} \right|_{r,s} \) as the slope
of the characteristic curve at \( r, s \):

\[\text{arbitrary trial function}\]

\[\begin{align*}
\text{slope} = p & \quad \text{of } S_p(r,s) \\
\text{slope} = s' & \quad \text{for trial function}
\end{align*}\]
Then
\[ W(x, y, z) = \int_{r_1}^{r_2} dr \left( f s'^2 + g s^2 \right) \geq \int_{r_1}^{r_2} dr \left[ (f s'^2 + g s^2) - f(s' - p)^2 \right] > 0 \]
\[ \geq \int_{r_1}^{r_2} dr \left[ f p^2 + g s^2 - 2fp( s' - p) \right] \]
\[ \geq \int_{r_1}^{r_2} dr \left[ f p^2 + g s^2 + 2fp s' \right] \]

When the trial function coincides with \( s_0(x) \), \( s' = p \), and the equality holds. For other trial functions satisfying \( s_1(x) = c_1 \), \( s_2(x) = c_2 \), note that
\[ \int_{r_1}^{r_2} dr \left[ f p^2 + g s^2 + 2fp (s' - p) \right] = \int_{r_1}^{r_2} dr \left[ g s^2 - f p^2 + 2fp \frac{ds}{dr} \right] \]
\[ = \int_{C=\text{trail}} \left[ (g s^2 - f p^2) ds + 2fp ds \right] \]

writing the integral as a contour integral in the \( r-s \) plane.

We shall now show that this contour integral is path-independent (in the \( r-s \) plane). To do this, we observe that since \( p = \frac{ds}{dr} \) along a characteristic curve (that are solutions to the Euler eqn)

\[ \frac{dr}{dt} (fp) - gs = 0 \]

\[ d(fp) - g s dr = 0 \]

\[ p \frac{dr}{dt} + f \frac{ds}{dt} dr + f \frac{df}{dt} ds - g s dr = 0 \]

\[ \frac{dr}{dt} = f \frac{ds}{dt} \]
\[
\rho \frac{\partial \phi}{\partial t} + \rho \frac{2\phi}{\partial r} + \rho \frac{2\phi}{\partial s} \frac{ds}{dr} - \gamma s = 0
\]

\[
\frac{2}{\partial r} \left( \phi \rho \right) + \phi \rho \frac{2\phi}{\partial s} - \gamma s = 0
\]

\[
\frac{2}{\partial r} \left( \phi \rho \right) + \frac{1}{2} \frac{2}{\partial s} \left( \phi \rho^2 \right) - \gamma s = 0
\]

Therefore, the coefficients of \(dr\) and \(ds\) in the contour integral satisfy

\[
\frac{2}{\partial s} \left( \gamma s^2 + \phi \rho^2 \right) = 2\gamma s - \frac{2}{\partial s} \left( \phi \rho^2 \right)
\]

\[
= 2\gamma s + \frac{2}{\partial r} \left( \phi \rho \right) - 2\gamma s
\]

\[
= \frac{2}{\partial r} \left( 2\phi \rho \right)
\]

which means that the contour integral acts like the integrated work due to a conservative potential, \(-\oint \mathbf{A} \cdot d\mathbf{x}\), i.e., it is path-independent. The path-independent contour integral in the \(r-s\) plane is an example of a Hilbert invariant integral.

For the \(s_0(r)\) characteristic curve connecting \((r_1, c_1)\) and \((r_2, c_2)\), \(s' = \rho\), so

\[
W(r_1, r_2, s_0) = \int_{r_1}^{r_2} ds \left( \gamma s^2 + \phi \rho^2 \right) = \int_{r_1}^{r_2} ds \left[ \frac{2}{\partial s} \left( \phi \rho^2 \right) + 2\gamma s \right]
\]

All other paths therefore have \(W(r_1, r_2, s) > W(r_1, r_2, s_0)\), so when \(s_i(r)\) does not vanish, the Euler solution minimizes \(W(r_1, r_2, s)\), subject to the endpoint constraints. If we have
$c_1 = c_2 = 0$, $s_0(r) = 0$ identically, and $W(r, r_2, s_0) = 0$.

Then $W(r_1, r_2, s)$ for any nontrivial $s(r)$ is positive.

If we now consider the situation where $s_1(r)$ vanishes at $r_0$ ($r_1 < r_0 < r_2$), we may apply the Hilbert integral in the region $r_1 < r < r_0$, $-\infty < s < 0$, but not across the $r = r_0$ line. In our sketch of characteristic curves

![Characteristic curves diagram]

We know that $W(r_1, r_0, s_{13}) = W(r_1, r_0, s_{167})$, since both lie along characteristic curves in a region where $s_1(r)$ does not vanish (except at end points). Therefore

$W(r_1, r_2, s_0) = W(r_1, r_2, s_{1345}) = W(r_1, r_2, s_{167345})$

since we may break the integral into separate pieces.

If we now consider a smaller region, $r_1 < r < r_2$, we can start over and find new solutions to the
Euler equation as an initial value problem from $r_2$. If $r_7$ and $r_4$ are sufficiently close, the $s$ solution that is zero at $r_7$ will not cross $s = 0$ before $r_4$. Therefore, the characteristic curve connecting points 7 and 4 minimizes $W(r_7, r_4, s)$. This implies

$$W(r_7, r_4, s_{r_7}) < W(r_7, r_4, s_{r_7})$$

and

$$W(r_1, r_2, s_{167045}) < W(r_1, r_2, s_{167345}) = W(r_1, r_2, s_{18345})$$

Thus, we have found a curve such that $W(r_1, r_2, s)$ is less than $W(r_1, r_2, s_0)$ for the Euler solution connecting $(r_1, c_1)$ and $(r_2, c_2)$. Here, when $s$ vanishes in $r < r_2$, the Euler solution makes $W$ stationary, but not minimal.

If we again take $c_1 = c_2 = 0$, $W(r_1, r_2, s_0) = 0$, but with $s(r)$ crossing 0 in $r < r_2$, we know there is a $W(r_1, r_2, s) < W(r_1, r_2, s_0) = 0$.

If there is a center conductor, and we pick an $(m, k)$ pair such that $f$ does not vanish, we can apply these findings directly. The boundary
conditions are \( c_1 = c_2 = 0 \) (no vacuum region) at the locations of the walls, \( r_1 \) and \( r_2 \), and

\[
W(r_1, r_2, s) = \delta W \rho
\]

for this \((m, k)\) pair. The procedure is then to compute \( \delta_1(r) \). If it vanishes between \( r_1 \) and \( r_2 \), the configuration is unstable for this \((m, k)\).

If \( \delta_1(r) \) does not vanish, it is stable for this \((m, k)\).

There are also proofs in Newcomb's paper that show that for \((m, k)\) with singular points, \( \delta_1 \) becomes the small solution, \( \delta_5(r) \) as \( r \) approaches a singular point. Therefore, when there are singularities, we treat each interval separately and check if the solution that is small on one side vanishes before the end of the interval.

Note that if the Suydam condition were violated, the oscillator behavior guarantees that \( \delta_5 \) crosses zero, so the Suydam condition is a subset of the general necessary condition for stability.