average $B_z$, and the vacuum condition then keeps $B_z$ uniform and independent of time. The vacuum field is displaced from some regions and fills in others.

What is significant about the magnetic behavior is that its pressure on the fluid doesn't change in this case. If this were not true, $\| \frac{\delta^2}{\delta t^2} r + p \| = 0$ would force a different response in the fluid.

Very Nonlinear $10^{-10}$ Behavior

After studying the basics of small-perturbation dynamics, we can round-out our view of MHD behavior by going to the opposite extreme of very nonlinear conditions. In general, nonlinear PDE systems can exhibit more complex behavior than linear PDE systems. In the limit of large-amplitude perturbations, however, two basic phenomena, shocks and rarefaction waves, tend to underlie the complete dynamics of fluids.
The physics behind shocks and rarefactions is the change in the rate of information propagation as compression or expansion occurs. To simplify the argument supporting this, consider 1-D systems where \( \vec{B} \) is perpendicular to the direction of variation; \( \vec{B} = B(x) \hat{z} \), for example. Here we have

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\dot{V} \times \vec{B})
\]

\[
= -\dot{V} \cdot \vec{B} - \vec{B} \cdot \dot{V} + \frac{\partial \vec{B}}{\partial t}
\]

\[
\frac{d\vec{B}}{dt} = -\vec{B} \cdot \dot{V}
\]

or

\[
\frac{d\vec{B}}{dt} = -\vec{B} \frac{\partial \dot{V}}{\partial x}, \quad \dot{V} = V(x) \hat{x}
\]

Thus, the equation governing the evolution of \( \vec{B} \) is identical to that for \( \rho \), \( \frac{d\rho}{dt} = -\rho \frac{\partial \dot{V}}{\partial x} \), hence

\[
\frac{d\vec{B}}{dt} = \frac{\vec{B}}{\rho} \frac{d\rho}{dt}
\]

or

\[
\frac{d\vec{B}}{dt} = \frac{\vec{B}}{\rho} \frac{d\rho}{dt}
\]

along the trajectory of a fluid element. Since the magnetic pressure is \( \frac{B^2}{2\mu_0} \),

\[
\frac{1}{\rho_0} \frac{d}{dt} \vec{B} \cdot \vec{B} = \dot{\rho} \frac{\vec{B}^2}{2\mu_0} = \frac{3}{\rho} \frac{\vec{B}^2}{2\mu_0} \frac{d\rho}{dt}
\]

[with this restricted system \( \vec{B} \cdot \vec{B} = 0 \), so \( \dot{\vec{B}} \times \vec{B} = -\nabla (\frac{\vec{B}^2}{2\mu_0}) \) only.] We have already seen that an adiabatic pressure...
evolution gives us
\[ \frac{dP}{\rho} = \frac{\delta P}{\rho} \, d\rho \, , \]

so
\[ dP_T = \left[ \frac{\gamma P}{\rho} + \frac{\rho}{\rho_0} \left( \frac{\delta^2}{2\rho_0} \right) \right] d\rho \, , \]

where \( P_T \) is the total pressure, \( P + \frac{\delta^2}{2\rho_0} \), and we can recognize \( \sqrt{dP/d\rho} \) as the magneto-acoustic speed — the rate of information propagation in this system. For convenience, define \( C_T = \sqrt{dP_T/d\rho} \).

To see how \( C_T \) changes with compression,
\[
\frac{1}{2} C_T \frac{dC_T}{d\rho} = \frac{d^2 P_T}{d\rho^2} = \frac{d}{d\rho} \left[ \frac{\gamma P}{\rho} + \frac{\rho}{\rho_0} \left( \frac{\delta^2}{2\rho_0} \right) \right]
\]

\[
= \frac{1}{\rho} \left[ \gamma \frac{dP}{d\rho} + \frac{\rho}{\rho_0} \frac{\delta^2}{d\rho} \right] - \frac{C_T^2}{\rho}
\]

\[
= \frac{1}{\rho} \left[ \frac{\gamma^2 P}{\rho} + \frac{\rho}{\rho_0} \frac{\delta^2}{\rho_0} \right] - \frac{C_T^2}{\rho}
\]

\[
= \frac{1}{\rho} \left[ (\gamma-1) \frac{\delta^2}{\rho} + \frac{\rho}{\rho_0} \frac{\delta^2}{\rho_0} \right]
\]

\[
\frac{dC_T}{d\rho} = \frac{2}{pc_T} \left[ (\gamma-1) \frac{\delta^2}{\rho} + \frac{\rho}{\rho_0} \frac{\delta^2}{\rho_0} \right]
\]

and \( \frac{dC_T}{d\rho} > 0 \) for \( \gamma > 1 \) or for sufficiently
small $\beta \left( \equiv \frac{2\mu_0 p}{B^2} \right)$ regardless of $\gamma$. The isothermal case ($\gamma=1$), for example, has $\frac{dC_T}{dp} > 0$ for nonvanishing $B$.

With the knowledge that $\frac{dC_T}{dp} > 0$, we can anticipate (but not yet solve) large-amplitude results. A disturbance that compresses the fluid leaves $C_T$ increased in its wake. This tends to allow equilibration behind the front of the disturbance. If conditions force the front to move at speeds approaching $C_T$ in the region ahead of the front, the pressure gradient cannot be spread by magnetoacoustic waves, and the gradient tends to build.

\[ C_T \text{behind} \rightarrow V_{\text{front}} \rightarrow C_T \text{ahead} \]

When the front is forced to move faster than $C_T$ ahead of it, the solution fields develop a discontinuity known as a shock.

The opposite happens during expansion. In this case, information about a disturbance propagates faster than the disturbance, so the profile
of the disturbance grows in time. This behavior is a rarefaction.

The complete ideal nonlinear system that we shall examine is:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = - \rho \frac{\partial \mathbf{u}}{\partial x} \]

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = - \frac{\partial p}{\partial x} - \frac{\partial}{\partial x} \left( \rho + \frac{\mathbf{u}^2}{2\rho} \right) \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u} = - \rho \frac{\partial \mathbf{u}}{\partial x} \]

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{B} = - \mathbf{B} \frac{\partial \mathbf{u}}{\partial x} \quad (\mathbf{B} = \mathbf{B}_0 \text{ only}) \]

**Shocks**

Our thorough treatment of interface conditions earlier covers most of the general analysis needed for shocks, since shocks are (mathematically) just discontinuities in the solution field. Like the plasma/vacuum interface relation, the interface (Rankine-Hugoniot) relations allow us to connect continuous solutions across the discontinuity. Here, the continuous segments are solutions to the MHD equations on both sides of the discontinuity.
\[ \rho_-, v_-, p_-, B_- \]
\[ \rightarrow \]
\[ \rho_+, v_+, p_+, B_+ \]

**Mass conservation:**
\[ \rho_+ (v_+ - v_s) = \rho_- (v_- - v_s) \equiv -m \]

**Momentum conservation:**
\[ \rho_+ (v_+^2 - v_s^2) + p_{T+} = \rho_- (v_-^2 - v_s^2) + p_{T-} \]
\[ -m (v_+ - v_s) + p_{T+} = -m (v_- - v_s) + p_{T-} \]
\[ m (v_- - v_+) = p_{T-} - p_{T+} \]

**Magnetic flux conservation:**
\[ (v_+ - v_s) B_+ = (v_- - v_s) B_- \]
\[ \frac{B_+}{\rho_+} m = \frac{B_-}{\rho_-} m \]
\[ \frac{B_+}{\rho_+} = \frac{B_-}{\rho_-} \]

(or just as a consequence of \( dB = \frac{B}{\rho} d\rho \Rightarrow d\left( \frac{B}{\rho} \right) = 0 \))

We also need a relation for \( \rho \), but the adiabatic equation does not hold through the shock. A physical system will always have some amount of viscous dissipation, and a shock will generate heat from this dissipation, regardless of how small the viscosity is.
What we can use is another interface relation, the Rankine-Hugoniot relation for energy conservation. Starting from the energy equation derived as homework,

$$\frac{\partial (\rho u^2 \frac{\partial u}{\partial t} + \rho v^2 \frac{\partial v}{\partial t} + U)}{\partial t} = -\nabla \cdot \left( \frac{\partial E}{\rho \rho_0} + \frac{\rho u^2}{\nu} + U \right) - \nabla \cdot (\rho \nu)
$$

we use the ideal Ohm's law $\vec{E} = -\nabla \times \vec{B}$,

$$\frac{\partial \vec{B}}{\partial t} = \frac{\vec{B} \times (\nabla \times \vec{B})}{\mu_0} = \frac{B^2}{\mu_0} \vec{v} + \nabla \cdot \vec{B} \cdot \nabla \times \vec{B}
$$

Here we take $\nabla \times \vec{B} = 0$ and arrive at

$$\frac{\partial E}{\partial t} = -\nabla \cdot (E \nabla) - \nabla \cdot \left( \rho \nabla \left( \frac{B^2}{\partial \rho} \right) \right)
$$

$$= -\nabla \cdot (E \nabla) - \nabla \cdot (\rho + \frac{B^2}{\partial \rho_0})
$$

where $E = \frac{B^2}{\partial \rho_0} + \frac{\rho u^2}{\nu} + U$ is the total energy density.

We proceed in the same way as for the other interface relations by using an infinitely thin control volume at the interface, moving with the interface. Take the primed coordinate system to be moving at $v_i$, $x' = x - v_i t$, then Galilean invariance allows us to write the
energy equation in the primed system in the same form (as long as \( v \) changes slowly).

\[
\frac{dE'}{dt'} = - \mathbf{v}' \cdot (E' \mathbf{v}') - \rho' v' \cdot (p v')
\]

using \( t' = t, \rho'_t = \rho_t \).

A finite change in energy in the infinitesimal volume is not physical, so we end up with

\[
\left[ E' v'_n + p v'_n \right] = 0
\]

To transform to the laboratory reference frame, note that

\[
E' = \frac{B^2}{\rho_0} + U + \frac{\rho \mathbf{v}'^2}{2}
\]

\[
= \frac{B^2}{\rho_0} + U + \rho \frac{(\mathbf{v}' - \mathbf{v})^2}{2}
\]

\[
= \frac{B^2}{\rho_0} + U + \frac{\rho \mathbf{v}^2}{2} + \frac{\rho \mathbf{v}_i^2}{2} - \rho \mathbf{v} \mathbf{v}_i
\]

\[
= E + \frac{\rho \mathbf{v}_i^2}{2} - \rho \mathbf{v} \mathbf{v}_i \quad (\mathbf{v}_i \to \mathbf{v} \text{ only})
\]

so in the laboratory frame,

\[
\left[ (E + p + \frac{\rho \mathbf{v}_i^2}{2} - \rho \mathbf{v} \mathbf{v}_i \cdot \mathbf{v}_n - \mathbf{v}_i) \right] = 0
\]
This expression can be simplified by using the other Rankine-Hugoniot relations.

- \[ \left[ \frac{\rho v_i^2 (v_n - v_i)}{2} \right] = \frac{V_i^2}{2} \left[ \rho (v_n - v_i) \right] = 0 \]

\( \uparrow \) mass relation

- \[ \left[ \rho_T (v_n - v_i) - \rho v_n v_i (v_n - v_i) \right] \]

\[ = \left[ \rho_T v_n \right] - V_i \left[ \rho_T + \rho v_n (v_n - v_i) \right] + V_i^2 \left[ \rho (v_n - v_i) \right] \]

\[ = \left[ \rho_T v_n \right] - V_i \left[ \rho_T + \rho (v_n - v_i)^2 \right] \]

\( \hat{n} \cdot \) momentum relation \( \Rightarrow 0 \)

\[ = \left[ \rho_T v_n \right] \]

\[ \therefore \left[ E (v_n - v_i) + \rho_T v_n \right] = 0 \]

As a physical example of an MHD shock, consider a magnetic flux compression generator that has an ideal plasma (or compressible conducting fluid) as a working fluid. With an infinite linear geometry, it would look
There is a current source that creates an initial magnetic field in the infinite direction. A gas fill is ionized after the initial \( \mathbf{B} \) is established, trapping the magnetic flux in the conducting plasma.

When the movable piston on the left is driven from a large external force, it creates a shock and a current sheet that propagate into the undisturbed plasma. The induced electric field,
$E_y = v_x B_z$ - creates a potential drop that is applied to the load.

In this system, the fluid ahead of the shock is at rest, $\dot{v}_+ = 0$, with the shock speed $v_+$ as the interface velocity, the Rankine-Hugoniot relations give us

1) \[ v_+ (\rho_- - \rho_+) = \rho_- v_- \]

2) \[ p_+ v_+^2 - \rho_- (v_- - v_+)^2 + p_{T+} - p_{T-} = 0 \]

\[
\begin{align*}
v_+^2 (p_+ - \rho_-) - \rho_- v_-^2 + \frac{\rho_- v_- v_+}{p_+ - \rho_-} + p_{T+} - p_{T-} &= 0 \\
\rho_- v_- (v_- - v_+) + p_{T+} - p_{T-} &= 0 \quad \text{[using 1]} \\
m v_- + p_{T+} - p_{T-} &= 0 \\
0 &= \frac{\rho_- v_-^2}{p_+ - \rho_+} - \rho_- v_-^2 + \frac{\rho_- v_- v_+}{p_+ - \rho_-} + p_{T+} - p_{T-} = 0 \\
\rho_- v_-^2 + (p_+ - \rho_+) (p_{T+} - p_{T-}) &= 0
\end{align*}
\]

3) \[ E_+ (v_+^2 - v_-^2) + p_{T+} v_-^2 - E_- (v_- - v_+) - p_{T-} v_- = 0 \]

\[
\begin{align*}
\frac{(E_+ - E_+) p_- v_-}{\rho_- - \rho_+} - E_- v_- - p_{T-} v_- &= 0 \quad \text{[using 1]} \\
(E_+ - E_+ \rho_- v_- - (\rho_- - \rho_+) E_- v_- - (\rho_- - \rho_+) p_{T-} v_- &= 0 \\
\frac{E_- - E_+}{\rho_- - \rho_+} = \frac{p_{T-} (\rho_- - \rho_+)}{\rho_- - \rho_+}
\end{align*}
\]
\[ \frac{B_{-}^{2}}{2\rho_{-}p_{-}} + \frac{U_{-}}{p_{-}} - \frac{B_{+}^{2}}{2\rho_{+}p_{+}} - \frac{U_{+}}{p_{+}} = \frac{\rho_{-} - (\rho_{-}p_{+})}{\rho_{+} + \rho_{-}} - \frac{1}{2} V_{-}^{2} \]

use the relation for \( V_{-}^{2} \) from 2)

\[ \frac{B_{-}^{2}}{2\rho_{-}p_{-}} + \frac{U_{-}}{p_{-}} - \frac{B_{+}^{2}}{2\rho_{+}p_{+}} - \frac{U_{+}}{p_{+}} = \frac{\rho_{-} + \rho_{+}}{\rho_{+} - \rho_{-}} (\rho_{+} - \rho_{-}) \]

With the increase in wave propagation speed due to compression, the region behind the shock will tend to a translating uniform equilibrium moving at the same speed as the piston. If we consider the upstream (+) conditions and the piston speed \( (V_{-}) \) to be specified, we can determine the rest of the downstream (-) field values once the shock speed is computed.

The shock speed is determined by the three Rankine-Hugoniot relations. After a page or so of algebra, one finds

\[ (\gamma - 2) \frac{\Delta B^{2}}{2\rho_{0}p_{+}V_{+}^{2}} = \frac{1}{\chi_{-}V_{+}} \left[ \left( \frac{\gamma - 1}{2} \right) V_{+}^{2} + C_{+}^{2} + (\gamma - 2) \frac{\Delta B^{2}}{2\rho_{0}p_{+}} \right] = \frac{\gamma + 1}{\gamma} \frac{V_{s}}{V_{-}} \]

or the quadratic form for \( V_{s} \):

\[ V_{s}^{2} + \left[ \frac{(\gamma - 2) \Delta B^{2}}{2\rho_{0}p_{+}V_{+}} - \frac{\gamma + 1}{\gamma} \frac{V_{s}}{V_{+}} \right] V_{s} - \left[ \frac{(\gamma - 1)}{2} V_{+}^{2} + C_{+}^{2} + (\gamma - 2) \frac{\Delta B^{2}}{2\rho_{0}p_{+}} \right] = 0 \]
where $V_{A+}^2$ and $C_s^2$ are the squares of the Alfvén and sound speeds, respectively ($\frac{B_0^2}{\rho_0 p^+}$ and $\frac{\gamma p^+}{\rho^+}$), and $\frac{\Delta B_z^2}{\rho_0}$ is the change in magnetic pressure across the shock ($\frac{B_0^2}{\rho_0} - \frac{B_z^2}{\rho_0}$). We can re-cast the $\Delta B_z^2$ in terms of the current per unit length of the z-dimension by using Ampere's law,

$$\mu_0 I = \oint \mathbf{dl} \cdot \mathbf{B} = L_z \Delta B_z = L_z (B_0 - B_z)$$

$$\Delta B_z^2 = B_0^2 - B_z^2 = (B_0 - B_z)(B_0 + B_z)$$

$$= (B_0 - B_+)(B_0 - B_+ + 2B_+)$$

$$= \frac{\mu_0 I}{L_z} \left[ \frac{\mu_0 I}{L_z} + 2B_+ \right]$$

Therefore, the piston's motion induces a current that is determined by the degree of compression through the shock, and it induces an electric field with $\mathbf{E} = \mathbf{V} \times \mathbf{B}_z$. Complete behavior of the circuit is determined by the impedance of the load.

The flow of power could also be reversed. An external voltage source can drive a shock in a plasma creating kinetic energy.
Rarefactions

The most basic rarefaction problem starts with a static uniform equilibrium on a semi-infinite domain \((0 \leq x < \infty)\) is fine. If the wall at \(x=0\) is put in motion in the negative \(x\)-direction, information about the disturbance will propagate in the positive \(x\)-direction. We can solve the dynamics of the system if the wall (or piston) speed is a step function in time:

\[
U_w(t) = \begin{cases} 
0, & t < 0 \\
U_p, & t \geq 0
\end{cases}
\]

where \(U_p\) is a constant.

Basic rarefaction problem; \(c_m = \sqrt{V_A^2 + c_s^2}\)

\([s_1\text{ and } s_2\text{ are just 'position vectors,' not moving with the fluid.}]

We can expect that the leading edge of the rarefaction (displacement \(s_2\)) moves at the magneto-acoustic speed of the undisturbed fluid \((c_{m_0})\), since that
speed is the rate of information propagation prior to changes by the expansion. Here, this speed is constant, due to the uniform initial conditions. The piston speed is also constant by our design. We can therefore expect the motion of the back edge of the disturbance (displacement $S$) to move at a rate such that $u_p < \frac{dS}{dt} < \frac{dS}{dt} = C_m$, so that it does not overtake either the leading edge or the piston.

If we assume that $\frac{dS}{dt}$ reaches a constant value—a reasonable assumption since the decreasing $C_m$ decreases the springiness of the fluid—then we can anticipate self-similar behavior. This means that the shape of the disturbance is not changed as it expands and propagates.

This reasoning suggests a coordinate system transformation. If this helps us find a solution to the full equations, the self-similar behavior is proven. The transformation we use to simplify the equations is
Clearly, $s$ is just the inverse slope of a line in $\alpha - t$ space, and we can use this transformation as long as $t > 0$. Applying the chain rule to an arbitrary function $F$,

\[
\frac{\partial F}{\partial x} = \frac{\partial F}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial F}{\partial t'} \frac{\partial t'}{\partial x} = \frac{\partial F}{\partial s} \frac{1}{s} = \frac{\partial F}{\partial s} \frac{1}{s t} \frac{1}{t'} \\
\frac{\partial F}{\partial t} = \frac{\partial F}{\partial s} \frac{\partial s}{\partial t} + \frac{\partial F}{\partial t'} \frac{\partial t'}{\partial t} = \frac{\partial F}{\partial s} \left(-\frac{x}{s^2}\right) + \frac{\partial F}{\partial t} \frac{x}{s} \frac{1}{s t} \\
\frac{\partial F}{\partial s} = \frac{\partial F}{\partial s} \frac{1}{s t} \frac{1}{t'}
\]

We can use this to transform our 1D PDE system,

\[
\rho \frac{\partial \rho}{\partial t} + \rho u \frac{\partial \rho}{\partial x} = -\frac{\partial}{\partial x} \rho T = -\frac{\partial}{\partial \rho} \frac{\partial p}{\partial x} \\
(\text{using } p_T(\rho) \text{ only for ideal, Ohm's law and adiabatic } P)
\]
\[
\rho \left( \frac{\partial V}{\partial t} - \frac{V}{c^2} \frac{\partial V}{\partial s} \right) + \rho v \left( \frac{1}{c^2} \frac{\partial V}{\partial s} \right) = -\frac{\partial \mathbf{F}_T}{\partial p} \frac{1}{c^2} \frac{\partial \rho}{\partial s}
\]

\[
t' \rho \frac{\partial v}{\partial t'} + \rho (v-s) \frac{\partial v}{\partial s} = -\frac{\partial \mathbf{F}_T}{\partial p} \frac{\partial \rho}{\partial s} \quad \text{(Velocity eqn.)}
\]

\[
\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = -\rho \frac{\partial v}{\partial x}
\]

\[
\frac{\partial \rho}{\partial t'} - s \frac{\partial \rho}{\partial s} + \frac{v}{c^2} \frac{\partial \rho}{\partial s} = -\frac{\rho}{s} \frac{\partial v}{\partial s} \quad \text{(Transformed)}
\]

\[
t' \frac{\partial \rho}{\partial t'} + (v-s) \frac{\partial \rho}{\partial s} = -\rho \frac{\partial v}{\partial s} \quad \text{(Continuity)}
\]

If it exists, self-similar behavior is a steady-state solution in the transformed variables.

\[
\frac{V}{c} = \frac{dS}{dt} = \text{constant}
\]

Sketch of self-similar behavior

The steady-state equations in the transformed coordinates \( \frac{\partial}{\partial t'} = 0 \) are

\[
\rho (v-s) \frac{\partial v}{\partial s} = -\frac{\partial \mathbf{F}_T}{\partial p} \frac{\partial \rho}{\partial s}
\]

\[
(v-s) \frac{\partial \rho}{\partial s} = -\rho \frac{\partial v}{\partial s}
\]
Eliminating \( p \) using \( \frac{1}{p} \frac{\partial p}{\partial s} = -\frac{1}{v-s} \frac{\partial v}{\partial s} \)

\[(v-s)^2 \frac{\partial v}{\partial s} = \frac{d\rho}{dp} \frac{\partial v}{\partial s}\]

\[\left[ (v-s)^2 - \frac{d\rho}{dp} \right] \frac{\partial v}{\partial s} = 0\]

This relation is satisfied when \( \frac{\partial v}{\partial s} = 0 \)

(ahead of \( s_2 \) or behind \( s_1 \)) or when

\[(v-s)^2 - \frac{d\rho}{dp} = (v-s)^2 - [C_m(s)]^2 = 0\]

At the leading edge, \( s = \frac{d\rho}{dt} \Rightarrow C_m(s) = C_m0 \) and the fluid is just beginning to move, \( v=0 \).

\[\frac{d\rho}{dt} = C_m0 \]

using \( v-s = -C_m \) for a piston moving to the left, confirming the physical intuition that the leading edge moves at the magneto-acoustic speed of the undisturbed fluid. The region adjacent to the piston satisfies \( \frac{\partial v}{\partial s} = 0 \), so \( v = U_p \) for \( s \leq \frac{d\rho}{dt} \). Matching the nonuniform part of the solution at \( s = \frac{d\rho}{dt} \) gives (writing \( v = -c_m \))

\[U_p - \frac{d\rho}{dt} = -C_m0 \]

\[\frac{d\rho}{dt} = C_m0 |U_p| \quad (U_p < 0)\]
where $c_{mp}$ is the magneto-acoustic speed in the region adjacent to the piston.

To determine the field profiles, substitute $v - s = -c_m$ in the steady, transformed velocity equation,

$$(v - s)\frac{\partial v}{\partial s} = -\frac{1}{\rho} C_m^2 \frac{\partial s}{\partial s}$$

$$-c_m (1 - \frac{\partial c_m}{\partial s}) = -\frac{1}{\rho} C_m^2 \frac{\partial s}{\partial s}$$

$$\frac{\partial c_m}{\partial s} = 1 - \frac{c_m}{\rho} \frac{\partial s}{\partial s}$$

With the self-similarity, this equation is an ODE (independent of time) in the transformed system,

$$\frac{dC_m}{ds} = 1 - \frac{C_m}{\rho} \frac{dp}{ds}.$$

Since we chose $C_m = C_m(\rho)$,

$$\left(\frac{dC_m}{dp} + \frac{C_m}{\rho}\right) \frac{dp}{ds} = 1$$

$$\left(\frac{dC_m}{dp} + \frac{C_m}{\rho}\right) dp = ds$$

$$C_m = s - \int \frac{c_m dp}{\rho} + \text{Constant}$$
\[ C_m = S - \int \frac{\sqrt{V_x^2 + c^2}}{\rho} \, dp + \text{const} \]

\[ = S - \int \sqrt{\frac{\beta_0^2 \rho^2}{\rho_0^2} + \frac{\gamma \rho_0^2}{\rho^2}} \, dp + \text{const} \]

For adiabatic pressure, \( \frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} \), and for ideal Ohm's law, \( \frac{B_0}{\rho} = \frac{B_0}{\rho_0} \),

\[ C_m = S - \int \sqrt{\frac{\beta_0^2 \rho_0^2}{\rho_0^2} + \frac{\gamma \rho_0^2}{\rho^2}} \, dp + \text{const} \]

which can be integrated to find \( C_m(s) \). In the large \( \beta \) \( (\frac{\beta v_0 p_0^2}{\beta_0^2} \gg 1) \) limit, the expression simplifies to

\[
C_m \approx S - \int \frac{\sqrt{\rho_0^2}}{\rho_0^2} \rho^{\frac{\gamma-3}{\gamma-1}} dp + \text{const}
\]

\[ = S - \frac{2}{\gamma-1} \int \frac{\gamma \rho_0^2}{\rho^2} \rho^{2-1} \, dp + \text{const} \]

\[ = S - \frac{2}{\gamma-1} \int \frac{\gamma \rho_0^2}{\rho} \, dp + \text{const} \]

\[ \left(\frac{\gamma+1}{\gamma-1}\right) C_m = S + \text{const} \]

We know that \( C_m(s) = C_m(0) \), \( \alpha^+ S = \frac{d \frac{s^2}{a}}{dt} = C_m(0) \), so

\[ \text{const} = \left(\frac{\gamma+1}{\gamma-1}\right) C_m - C_m(0) = \frac{2}{\gamma-1} \cdot C_m(0) \]
For this large $-\beta$ limit then
\[
C_m = \frac{\beta - 1}{\beta + 1} \left( s + \frac{2}{\beta - 1} C_{m0} \right) \quad \text{or} \quad C_m(x) = \frac{\beta - 1}{\beta + 1} \left( \frac{x}{\varepsilon} + \frac{2}{\beta - 1} C_{m0} \right)
\]

and
\[
V - s = -C_m = - \frac{\beta - 1}{\beta + 1} \left( s + \frac{2}{\beta - 1} C_{m0} \right)
\]
\[
V = \frac{2}{\beta + 1} s - \frac{2}{\beta + 1} C_{m0}
\]
\[
V = \frac{2}{\beta + 1} (s - C_{m0}) \quad \text{or} \quad V(x) = \frac{2}{\beta + 1} \left( \frac{x}{\varepsilon} - C_{m0} \right)
\]

We can also use these expressions for the small $-\beta$ limit by using the analogy of a $\vartheta = 2$ gas for these perpendicular-to-$\beta$ motions.

Small $-\beta$ \((\beta < \frac{\beta_{mo}}{\alpha_{mo}} \text{ only})\) limit

\[
C_m = \frac{1}{3} (s + 2 C_{m0})
\]
\[
V = \frac{2}{3} (s - C_{m0})
\]

Besides propagating rarefactions, these solutions are also relevant to thruster applications. If a source supplied a plasma with a velocity of $-C_{m0}$ from the right side, the expansion would convert magnetic and internal energy into thrust. Meeting a useful steady solution in \((x, t)\) would take a more complicated arrangement, however.