Resistive Magnetic Reconnection

When the resistive term ($\nabla \times J$) dominates the $\nabla \times B$ term in Ohm's law on global scales, we should expect rather uninteresting behavior. Magnetic fields induced by different current sources diffuse together as a consequence of the linear nature of

$$-\nabla \times (\eta \nabla J) = -\nabla \times \frac{\eta}{\mu_0} \nabla \times B.$$  

However, when resistivity is small but non-zero, such that $\nabla \times B$ dominates $\nabla \times J$ in most regions, the resistive diffusion that does occur can alter the dynamics in a fundamental way. Magnetic field lines can reconnect as sketched here:

Before $B$

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After $B$

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Within the context of electromagnetic dynamics alone, reconnection can change global nonlinear dynamics on a fast time scale; it can allow linear instabilities (precluded in ideal MHD) on a longer time scale; and it gives rise to dynamo action, over the resistive-transport time-scale. Beyond the electromagnetic dynamics themselves, topology changes associated
with reconnection directly affect the confinement properties of a magnetic configuration.

There has been a resurgence of interest in magnetic reconnection over the last few years, as theory and experiment delve further into two-fluid and kinetic effects beyond the scope of MHD. Though these non-MHD processes are critical in most plasmas, the MHD model of reconnection is the foundation from which our understanding has grown.

As indicated in the sketch above, reconnection necessarily involves the inter-diffusion of more than one magnetic field component. However a good starting point for studying reconnection is the basic convection-diffusion process that occurs in steady resistive conditions.

Consider the 1D sheared-slab configuration with \( x \) being the direction of variation. We can choose \( \vec{B} = B_y(x) \hat{y} + B_z(x) \hat{z} \) and \( \vec{V} = V_x(x) \hat{x} \) only. We simply assume that transport processes beyond the scope of MHD keep the density profile fixed (regardless of \( \nabla \cdot \vec{V} \)).
Furthermore, we take $\beta \neq 0$. For $V^2 \ll \frac{B^2}{\rho_0}$, the momentum balance equation is

$$\frac{\delta}{\delta t} \mathbf{B} = \frac{1}{\rho_0} \frac{d}{dx} \left( B_y^2 + B_z^2 \right) \frac{\delta}{\delta x} = \mathbf{0}$$

so

$$B_y^2 + B_z^2 = B_0^2 \quad \left[ B_0 \text{ is a constant} = |B|^2 \right]$$

$\nabla \times \mathbf{B} = 0$.

For $\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$, we also need $\nabla \times \mathbf{E} = \mathbf{0}$. In the 1D sheared slab, this condition requires uniform $\mathbf{E}$. If we choose $\mathbf{E} = E_0 \mathbf{z}$, the two nontrivial components of Ohm's law are

$$E_y = 0 = \eta J_y + V_x B_z = -\frac{\eta}{\rho_0} \frac{\partial B_z}{\partial x} + V_x B_z$$

$$E_z = E_0 = \eta J_z - V_x B_y = \frac{\eta}{\rho_0} \frac{\partial B_y}{\partial x} - V_x B_y$$

The $E_y$ equation determines $V_x$:

$$V_x = \frac{1}{B_z} \frac{\partial B_z}{\partial x}$$

Inserting into the $E_z$ equation,

$$E_0 = \frac{\eta}{\rho_0} \left[ \frac{\partial B_z}{\partial x} - \frac{B_y}{B_z} \frac{\partial B_z}{\partial x} \right]$$

and using force balance,
\[ \frac{\mu_0 E_0}{\eta} = \left( 1 + \frac{b_y^2}{b_z^2} \right) \frac{dB_y}{dx} \]

\[ = \left( 1 + \frac{b_y^2}{b_0^2 b_z^2} \right) \frac{dB_y}{dx} \]

\[ \frac{dB_y}{dx} = \frac{\mu_0 E_0}{\eta} \left( 1 - \frac{b_y^2}{b_0^2} \right) \]

We can make this non-dimensional by changing variables: \[ b_y = b_0 b_y', \quad x = Lx' \]

\[ \frac{1}{1-b_y^2} \, \frac{dB_y'}{dx'} = L \lambda_0 \, dx' \quad \lambda_0 = \frac{\mu_0 E_0}{\eta b_0} \]

\[ \tanh^{-1}(b_y') = L \lambda_0 \, (x' - x_0) \quad (-L \lambda_0 x_0 = \text{integration constant}) \]

\[ b_y' = \tanh(L \lambda_0 (x' - x_0)) \]

If \[ b_y' = 0 \quad \text{at} \quad x = 0, \quad x_0 = 0 \]

we can also determine \( V_x \)

\[ V_x = \frac{c_0}{\mu_0} \left( \frac{1}{b_z} \frac{dB_z}{dx} \right) \]

\[ = -\frac{c_0}{\mu_0} \left( \frac{b_y}{b_z^2} \frac{dB_y}{dx} \right) \]
\[ V_x = -\frac{\rho_0}{\mu_0} \left( \frac{\nu}{\beta_0^2 \gamma^2} \frac{\nu_0 \beta_0}{\eta} (1 - \frac{\beta_0^2}{\beta_0^2}) \right) \]

\[ = -\frac{E_0 \beta_0}{\beta_0^2} \frac{\nu_0 \beta_0}{\eta} \]

At large \( x \), this is the \( \frac{E_0}{\beta_0} \) drift velocity, and the Poynting flux, \(-\frac{E_0 \beta_0}{\eta}\), reflects the flux of magnetic energy density (\( V_x \frac{\beta_0^2}{\rho_0} \)) into the dissipation region.

If we consider \( \nu_0 \) and \( \beta_0 \) at large \( x \) as the given conditions, the dissipation is driven and forced by the MHD dynamics. The rate of dissipation is set by these boundary conditions, and the length scale of the diffusion, \( 1/\lambda_0 \), adjusts. If \( V_0 \equiv \lim_{|x| \to \infty} |V_x(x)| \)

\[ \frac{1}{\lambda_0} = \frac{\eta}{\nu_0 \nu_0} \]

The Sweet-Parker model of magnetic reconnection [E. N. Parker, JGR 62, 509 (1957)] extends the convection-diffusion process to two dimensions with both 'in-plane' components of \( \beta \) being nonzero. The motivation was the dynamics of magnetic loops extending from the photosphere of the sun.
Into the solar corona, the external drive is the motion of the conducting photosphere, which can force together the magnetic loops above.

The simplified 2D geometry is:

If the dynamics are slow in comparison to sound and magneto-acoustic propagation, we can expect the compressional waves to make the flow nearly incompressible. Then, by geometry the incoming and outgoing flow speeds ($u$ and $v$, respectively)
Satisfy \( v \Delta = u \Delta L \), by a control volume analysis over the dissipation region.

For the reconnection geometry to exist over time scales that are long compared to the wave propagation times, we need to have forces balanced. Along the mid-plane \((y = 0, \text{ where } \frac{\partial}{\partial y} = 0)\), this requires a build-up of \( p \) such that \( p(x = 0, y = 0) = \frac{B_0^2}{2\rho_0} \), where \( B_0 \) is the magnitude of \( B \) at large \( |x| \). This energy density is then available to accelerate the outflux of mass. If the conversion of energy is complete,

\[
\frac{1}{2} \rho v^2 = \frac{B_0^2}{2\rho_0}
\]

\( V \approx \sqrt{\frac{B_0}{\rho_0}} \), the Alfvén speed

so that

\( u \approx \frac{V_{\text{Alf}}}{L} \)

since diffusion must keep up with the flux of magnetic energy (recall \( \frac{\delta}{\delta t} = \frac{\delta}{\delta x} \frac{\delta}{\delta x} \) from the 1D preview)

\[
L = \frac{\delta}{\rho_0 u}
\]

Eliminating \( u \), we find

\[\frac{\delta \Delta \Delta}{\rho_0 L} \approx \frac{V_{\text{Alf}}}{L}\]

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\[ \frac{L}{L} = \sqrt{\frac{\eta}{\rho_0 V_A}} \]

The ratio of global diffusion time-scale
\[ \tau_d = \frac{\kappa L^2}{\eta} \]

to global Alfvén propagation time
\[ \tau_A = \frac{L}{V_A} \]

is the Lundquist number \( S \), hence
\[ \frac{L}{L} \sim S^{-\frac{1}{2}} \]

so the reconnection rate \( \frac{L}{U} \) scales like \( \tau_A S^{-\frac{1}{2}} \).

This is fast in comparison to a global diffusion rate, \( \tau_A S \) for highly conducting plasmas where \( S \gg 1 \), but it's not sufficient to explain observed reconnection rates. [At least this is the usual claim. Recall that the rate is set by boundary conditions; the geometry of the reconnection is the more fundamental issue.]

The physics of the Sweet-Parker model is just this dimensional argument. The Parker paper provides a partial solution to the nonlinear 2D resistive MHD equations in this configuration,
but a complete solution is not tractable.

For a small degree of completeness, it is worth mentioning a couple more analytic efforts to describe nonlinear reconnection in resistive MHD. [See chapter 6 of Biskamp, "Nonlinear Magnetohydrodynamics," for more information and references.] Petscheck proposed a configuration where the flow pattern is characterized by stationary shock fronts.

Here, the shocks are with respect to the slow magnetoacoustic mode, \( w_s = c_s \sqrt{k^2 - \frac{\nabla^2}{\rho}} \), which allows incompressible flow across the shocks. The redirection of velocity occurs from the tangential component of the Rontine-Hyman relation describing momentum conservation.

Magnetic reconnection occurs in a small region with aspect ratio \( \sim 1 \) at the center of the configuration. This produces a reconnection rate with a logarithmic dependence on \( S \), suggesting faster reconnection than the Sweet-Parker configuration.
As described by Biskamp, however, there is an inconsistency. Petschek did not consider the effects of fast magnetoacoustic dynamics, which would also be present. These effects alter the inflow magnetic field, preventing the formation of the small reconnection region.

A geometric approach was offered by Syrovatskii, who suggested that the current is localized in infinitely thin sheets. The magnetic field outside the sheets is then a vacuum distribution.

Syrovatskii analyzed the velocity field required to assemble the configuration, but the result is not a complete solution to the nonlinear MHD equations.

Complete solutions have only been possible through numerical simulation. Biskamp provided a thorough study of resistive MHD reconnection for the simple 2D configuration in his Phys. Fluids 29, 1520 (1986) paper. He found basic agreement with the Sweet–Parker scaling for the geometry.
of the reconnection region, but the numerical solutions also display dynamics not foreseen in the analytic investigation. Magnetic induction increases adjacent to the reconnection region as resistivity is decreased, and a Syrovatskiĭ-like configuration forms.

Application to Plasma Physics

After surveying a variety of MHD behavior, we turn to the question of its applicability as a model for plasma physics. This will proceed in two parts. First, we derive the fluid-like form of low-order moments of the distribution function of plasma particles and compare the result with the MHD equations. We then consider the conditions under which the MHD equations become the dominant part of the complete system.

We start with Boltzmann equations for two oppositely charged species that together form the plasma.

\[
\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \nabla f_i + \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla f_i = C_i(f_i)
\]