F. E. Basis Function

Continuity Discussion

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Example: 1D - cubics

\( C^0 \)

(Lagrange or spectral)

\( C^1 \)

(Hermite)

- \( C^0 \) has \( \sim 3 \) DOF/element
- \( C^1 \) has \( \sim 2 \) DOF/element
Is one representation ($C^0$ or $C^1$) better than the other?

→ No obvious answer
→ Likely depends on the application

+ Advantages of $C^1$

1) Second derivatives have finite 'energy' (square integrable),
   • reduced MHD + viscosity
   • Hall s.i. operator without auxiliary field

\[ \text{rhs} = \nabla \times \left( \frac{\mathbf{a}}{\mu_0} \left( \nabla \times \mathbf{b} \right) \times \mathbf{b} \right) - \nabla \times \frac{\mathbf{a}}{\mu_0} \left( \nabla \times \mathbf{b} \right) \times \mathbf{b} \]

2) Continuity of derivatives seems desirable qualitatively.
**Disadvantages of $C^1$**

1) Possibly slower convergence if the solution isn't smooth.
2) Mappings are more restricted.

**Why?**

1) $\Rightarrow$ FE approach stems from variational / Galerkin formulation and the choice of solution space. Physical model $\Rightarrow$ diff. eqns. $\Rightarrow$ strong form

\[ Lu = f \]

- Mapping from space containing $u$
- to space containing $f$
- space containing $u$ has higher continuity.
Strong form (continued)

- If \( \frac{\partial^2 f}{\partial x^2} = 0 \) and

\( L \) is a second (fourth) - order diff op., then \( L \) is a mapping

from \( H^2 \to H^0 \)

(here, superscript indicates degree of deriv. with finite energy) \([H^s]\)

- Can restrict to a space satisfying b.c.s to order \( s-1 \)
  only \( \to H^s \).
- Weak form
  \[(L u, v) = (f, v)\] for all \(v\) in \(\mathcal{H}^s_B\)?
  - Int by parts
    \[a(u, v) = (f, v)\]
  - Energies of derivatives of order \(m\) appear (only)
    \[s = 2m \text{ or } 2m-1\]

  → Enlarge Space to \(\mathcal{H}^{m+1}_B\)?
  - Yes if minimum / stationary 'point' (function) is the same.

\[\frac{du}{dx}\]

[Diagram]

True soln.

Func. in \(\mathcal{H}^1\)

Others in \(\mathcal{H}^2\) (and \(\mathcal{H}\))
FE approximation uses a family of subspaces in $\mathcal{V}_h \rightarrow$ chooses 'best' approximation in each space (parameterized by $h,p$).

$\rightarrow$ Resolve solution by progressing to subspaces that contain more of $\mathcal{V}_h$

$\rightarrow$ $C^1$ vs. $C^0$ is a question of whether it helps to restrict in the process of converging.

[Previous page argues converged solution is the ..]
• If the converged solution has $C^1$ continuity, how can $C^0$ do better for the F.E.A.?

Generic Example

- For this mesh spacing $C^1_h$ solution would overshoot.
2) Mappings are more restricted with $C^1$.

- In F.E., the physically motivated operator becomes a norm of the error, which is used to choose the best soln in $S_h$.
- In this norm, the F.E. soln is better than interpolate functions in the same space.

$\rightarrow$ Makes the connection to Taylor approximation wrt ind. vars. of PDE (not logical or element coordinates).

$\rightarrow$ With the exception of special cases, (reduced quintic on triangles with linear maps), $C^r$ requires a $C^1$ map.

$A(5,m), Z(3,m)$
\( c^0 \) map:

- Lines of constant \( s \)

\( c^1 \) map

→ Map itself requires a global solve of \( \mathbb{R}(s, \theta), \mathbb{Z}(s, \theta) \)

→ Not used for structural shell elements.